Optimal Design of a Beam under Uncertain Loads

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Abstract—**Optimal design of beams subject to a combination of uncertain and deterministic transverse loads is presented using a min-max approach. The compliance of the beam is maximized to compute the worst case loading and minimized to determine the optimal cross-sectional shape. The uncertain component of the transverse load acting on the beam is not known a priori resulting in load uncertainty subject only to the constraint that its norm is finite. The min-max approach leads to robust optimal designs which are not susceptible to unexpected load variations as it occurs under operational conditions. The optimality conditions in the form of coupled differential equations are derived with respect to load and the shape functions. The resulting equations are solved analytically and the results are given for several cases to illustrate the method and to study the behavior of the optimal shapes and the worst case loadings. The efficiency of the optimal designs is computed with respect to a uniform beam under worst case loading taking the maximum deflection as the quantity for comparison.**

Keywords: Optimal design; Uncertain loading; Min-max approach; Optimal beams; Robust design

1 Introduction

Under operational conditions, a structure is usually subjected to uncertainties may arise from fluctuation and scatter of external loads, environmental conditions, boundary conditions, and geometrical and material properties. Design uncertainties can also arise from incomplete knowledge of the load and the material.

In the present study only the load uncertainties are considered such that the load applied on the beam consists of unknown and known parts with the norm of the unknown load specified a priori. In conventional design, it is common practice to neglect the load uncertainties when analyzing a structure and assess the structural performance on the basis of a deterministic model. To compensate for performance variability caused by load variations, a safety factor is introduced magnitude of which correlates with the level of uncertainty with higher levels leading to larger safety factors. However, the safety factors specified may be either too conservative or too small to compensate for the lack of knowledge of operational loads. Efficiency and reliability of the structure can be improved by taking the load uncertainties into consideration in the design process leading to a design which is robust under load variations. This approach is equivalent to optimizing the design under worst case loading, thereby reducing the sensitivity of the beam to load variations. This is accomplished by maximizing its compliance over loading while minimizing it with respect to its cross-sectional shape resulting in a min-max optimal design problem. This formulation is suitable for designing structures to carry loads that are not known in advance as discussed in papers [6, 7, 8, 15] where the authors proposed a minimax formulation to maximize the design compliance under the most unfavorable loading condition. The method is also known as anti-optimization where the objective is to compute the 'best' design under 'worst' case loading. Examples of anti-optimization applied to uncertain loading problems can be found in Refs [1, 9, 10, 16] where optimization under uncertain bending and buckling loads is studied.

An alternative strategy to treat the uncertainties is convex modeling in which the uncertainties belong to a convex set [2, 3, 4, 5, 14]. This approach allows the designer to use not the averaged results but extremal properties of the system being modeled, according to the convex set chosen. The limitation of the convex modeling is that only small variations around a nominal value of the uncertain quantity can be considered and the model becomes less accurate as the variations become larger. Other methods of taking load uncertainties in the design process can be found in [11, 17]. The main objective of these techniques is to achieve robust designs which are not susceptible to failure under unexpected conditions [13, 18].

In the present work, the cross-sectional shape of a beam is optimized under a combination of deterministic and uncertain transverse loads. The optimization method involves a minimax formulation where the objective is to minimize the compliance with respect to the cross-sectional shape and maximize it with respect to load function. The formulation ensures that the optimal designs found correspond to the most unfavorable loading configuration and, therefore, these designs are conservative for

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any other loading.

2 Design Problem Formulation

We consider a simply supported beam subject to an uncertain load, $F(x), 0 \leq x \leq 1$, which may be acting on part of the beam and may have an upper limit as shown in Figure 1 where $g(x)$ is the deterministic component of the transverse load. The uncertain load can be defined as

$$F(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq s_1, \\ f(x), & \text{if } s_1 \leq x \leq s_2, \\ 0, & \text{if } s_2 \leq x \leq 1, \end{cases} \tag{1}$$

where s_1 and s_2 are given parameters and $f(x) \in C^0[s_1, s_2]$ is an unknown continuous function. In addition the beam is subjected to uncertain moments $\overline{m}_0$ and $\overline{m}_1$ at the boundaries $x = 0$ and $x = 1$, respectively (Figure 1).

Figure 1: Beam Diagram with external forces.

The equation governing the deflection of the beam can be expressed in non-dimensional form as

$$(a(x)y(x)'')'' = F(x) + G(x) \tag{2}$$

where $a(x) \in C^0[0, 1]$ is the cross-sectional area of the beam, $y(x)$ is the deflection, $F(x)$ is the uncertain load and $G(x) \in C^0[0, 1]$ is the continuous deterministic load. The primes denote the derivative with respect to $x \in [0, 1]$. The area and load functions are subject to the constraints

$$\begin{gathered} \int_0^1 a(x)dx = 1, \quad \|F(x)\|_{L_2}^2 \equiv \int_0^1 F(x)^2 dx = 1, \\ \max_{0 \leq x \leq 1} F(x) \leq f_{max}, \quad \overline{m}_0^2 + \overline{m}_1^2 = \eta, \end{gathered} \tag{3}$$

where the integrals represent constraints on the volume of the beam and the L_2 norm of the undeterministic load, $\overline{m}_0$ and $\overline{m}_1$ are uncertain moments, and $\eta \geq 0$ is a given constant. For a simply supported beam the boundary conditions are given by

$$\begin{aligned} y(0) = 0, \quad m(0) = a(0)y''(0) = \overline{m}_0, \\ y(1) = 0, \quad m(1) = a(1)y''(1) = -\overline{m}_1, \end{aligned} \tag{4}$$

where $m(0)$ and $m(1)$ are moments at the boundary points $x = 0$ and $x = 1$. The design problem involves the minimization of the potential energy of the beam under worst case of loading and as such involves optimization with respect to the area function $a(x)$ and anti-optimization with respect to the loading functions $F(x)$, $\overline{m}_0$ and $\overline{m}_1$ subject to the constraints (3). This problem can be expressed as a minimax problem, viz.

$$\min_{a(x)} \max_{F(x), \overline{m}} P_I(a(x), F(x), \overline{m}; y), \tag{5}$$

where P_I is the performance index (potential energy) given by

$$\begin{aligned} P_I(a, F, \overline{m}; y) &= P_I(a(x), F(x), \overline{m}; y) \\ &= \frac{1}{2}\int_0^1 a(x)(y'')^2 dx - \int_0^1 (F(x) + G(x))y dx \\ &\quad + \overline{m}_0 y'(0) - \overline{m}_1 y'(1), \end{aligned} \tag{6}$$

and $\overline{m}$ denotes the vector $\overline{m} = (\overline{m}_0, \overline{m}_1)$. In Eqn. (6), the first term is the strain energy and the second, third and fourth terms make up the potential energy of the external loadings.

3 Method of Solution

In computing the optimal area function $a(x)$ and the worst case loading $F(x), \overline{m}_0$ and $\overline{m}_1$ subject to the constraints (3), method of Lagrange multipliers is employed. Let the Lagrangian be defined as

$$\begin{aligned} L(a, F, \overline{m}; y) &= P_I(a, F, \overline{m}; y) + \mu_1 \int_0^1 F(x)^2 dx \\ &+ \mu_2 \int_0^1 a(x)dx + \mu_3(\overline{m}_0^2 + \overline{m}_1^2 - \eta), \end{aligned} \tag{7}$$

where $\mu_i, i = 1, 2, 3$ are Lagrange multipliers. The variation of $L(a, F, \overline{m}; y)$ with respect to y gives the differential equation (2) and the boundary conditions (4). The variation of $L(a, F, \overline{m}; y)$ with respect to $a(x)$ yields

$$\int_0^1 (y'')^2 \delta a \; dx + \mu_2 \int_0^1 \delta a \; dx = 0, \tag{8}$$

where δa is arbitrary. Thus, from the fundamental theorem of calculus of variations, it follows that

$$(y'')^2 + \mu_2 = 0. \tag{9}$$

Similarly, the variations of $L(a, F, \overline{m}; y)$ with respect to F and $\overline{m}$ yield

$$-y + 2\mu_1 f(x) = 0 \quad \text{for } x \in [s_1, s_2], \tag{10}$$

$$y'(0) + 2\mu_3 \overline{m}_0 = 0 \quad -y'(1) + 2\mu_3 \overline{m}_1 = 0. \tag{11}$$

Thus, the optimality condition of the problem is given by

$$y'' = constant = \beta. \tag{12}$$

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Similarly, the anti-optimization conditions can be expressed as

$$f(x) = \begin{cases} \dfrac{y}{2\mu_1}, & \text{for } y > 2\mu_1 f_{max} \\ f_{max}, & \text{for } y < 2\mu_1 f_{max} \end{cases} \quad \text{where } s_1 \leq x \leq s_2, \tag{13a}$$

$$\overline{m}_0 = -\frac{y'(0)}{2\mu_3}, \text{ and } \overline{m}_0 = \frac{y'(1)}{2\mu_3}. \tag{13b}$$

Substituting the optimality and anti-optimality conditions into the differential equation (10), we obtain

$$a''(x) = \begin{cases} G(x)/\beta & \text{for } 0 \leq x \leq s_1, \\ (f(x) + G(x))/\beta & \text{for } s_1 \leq x \leq s_2, \\ G(x)/\beta & \text{for } s_2 \leq x \leq 1, \end{cases} \tag{14}$$

where $f(x)$ is given by (13a). A system of linear differential equations in $y(x)$ and $a(x)$ given by (12), (13a) and (14) can be solved simultaneously. In the present case it is possible to find an analytical solution for $y(x)$ satisfying the boundary conditions, *viz.*

$$y = \frac{\beta}{2} x(x-1) \quad \text{for } 0 \leq x \leq 1. \tag{15}$$

Similarly, the optimal area function is given by

$$a_{opt}(x) = \begin{cases} g(x)/\beta + c_1 x + c_2, & \text{for } 0 \leq x \leq s_1, \\ \dfrac{x^3}{48\mu_1}(x-2) + \dfrac{1}{\beta} g(x) + c_3 x + c_4, & \text{for } s_1 \leq x \leq s_2, \\ G(x)\beta + c_5 x + c_6, & \text{for } s_2 \leq x \leq 1, \end{cases} \tag{16}$$

when $y > 2\mu_1 f_{max}$ where $g(x)$ is the second indefinite integral of $G(x)$ and $c_i, i = 1, \ldots, 6$ are integration constants to be determined from the boundary conditions (4) and continuity conditions

$$a_-(s_1) = a_+(s_1), a_-(s_2) = a_+(s_2), \tag{17}$$

where a_- and a_+ denote the area function to the left and right of the points s_1 and s_2, respectively. Furthermore in the absence of concentrated loads as required by the continuity of the uncertain and deterministic loads, the shear force $V(x) = (a(x)y''(x))'$ on the beam will also be continuous. From the optimality condition (12), it follows that $V(x) = \beta a'(x)$. Thus we have the further continuity conditions

$$\begin{aligned} V(s_1) = & a'_-(s_1) = a'_+(s_1), \\ & V(s_2) = a'_-(s_2) = a'_+(s_2), \end{aligned} \tag{18}$$

where a'_- and a'_+ denote the derivatives of the area function to the left and right of the points s_1 and s_2 , respectively. The case when $y(x) < 2\mu_1 f_{max}$ for x in a finite interval will be solved in the example problems. The uncertain functions $f(x), \overline{m}_0$ and $\overline{m}_1$ can be computed from equations (3), (10), (11) and (14). In particular the uncertain loading $f(x), s_1 \leq s \leq s_2$ is given by

$$f(x) = \begin{cases} \dfrac{\beta}{4\mu_1} x(x-1), & \text{for } s_1 \leq x \leq d_1, \\ f_{max}, & \text{for } d_1 \leq x \leq d_2, \\ \dfrac{\beta}{4\mu_1} x(x-1), & \text{for } d_2 \leq x \leq s_2, \end{cases} \tag{19}$$

where d_1 and d_2 are unknown locations to be determined from the continuity conditions

$$\begin{aligned} f_-(d_1) = f_+(d_1) = f_{max}, \\ f_-(d_2) = f_+(d_2) = f_{max}, \end{aligned} \tag{20}$$

where $f(x)_-$ and $f(x)_+$ denote the uncertain load functions to the left and right of the points d_1 and d_2, respectively. From equations (3), (11) and (14), it follows that $\overline{m}_0 = \overline{m}_1 = \eta/\sqrt{2}$.

It is noted that the number of unknowns equals the number of equations resulting in unique solutions. This aspect the method of solution will be illustrated in the next section by applying the technique to several problems of practical interest.

To assess the efficiency of the optimal designs, comparisons are made with uniform beams under uncertain loads for which $a(x) = 1$ for $0 \leq x \leq 1$. The anti-optimality condition (10) applies to this case also and consequently the differential equation for a uniform beam under worst case loading becomes

$$\frac{d^4 y}{dx^4} = \begin{cases} G(x), & \text{for } 0 \leq x \leq s_1, \\ \dfrac{y}{2\mu_1} + G(x), & \text{for } s_1 \leq x \leq d_1, \\ f_{max} + G(x), & \text{for } d_1 \leq x \leq d_2, \\ \dfrac{y}{2\mu_1} + G(x), & \text{for } d_2 \leq x \leq s_2, \\ G(x), & \text{for } s_2 \leq x \leq 1. \end{cases} \tag{21}$$

The solution of the differential equation (21) subject to the boundary conditions (4) and the constraints (3) gives the deflection $y_{un}(x)$ of a uniform beam under worst case loading. The efficiency of the design can be determined by comparing the maximum deflections of the uniform and optimal beams, *viz.*

$$I_{eff} = \frac{y_{max}}{y_{un}} \times 100\%, \tag{22}$$

where I_{eff} is the efficiency index in percentage, y_{un} and y_{max} are the maximum deflections of the uniform and optimal beams under worst case of loadings.

4 Applications of Method

4.1 Example 1: Unconstrained $F(x)$ with $0 < s_1 < s_2 < 1$.

Let the beam be subjected to only the uncertain transverse load $F(x)$ given by equation (1) with $0 < s_1 <$

$s_2 < 1$, *i.e.*, no uncertain moments are applied on the boundaries so that $\overline{m}_0 = \overline{m}_1 = \eta = 0$ and there is no deterministic load applied, *i.e.*, $g(x) = 0$. Moreover is set equal to $f_{max} = \infty$. For this case the optimal area function satisfying the moment boundary conditions in equation (4) can be computed from equations (16) as

$$a(x) = \begin{cases} c_1 x, & \text{for } 0 \le x \le s_1 \\ \frac{1}{48\mu_1} x^3(x-2) + c_2 x + c_3, & \text{for } s_1 \le x \le s_2, \\ c_4(1-x), & \text{for } s_2 \le x \le 1. \end{cases} \tag{23}$$

Equation (20) for $a(x)$ contains six unknowns $\beta, \mu_1, c_1, c_2, c_3$ and c_4 which are computed from six equations for the volume and L2 norm constraints (3), and the continuity conditions (17) and (18). These constants in terms of s_1 and s_2 are given by

$$\begin{aligned}
\mu_1 &= \frac{1}{40}s_2^5 - \frac{1}{40}s_1^5 - \frac{1}{16}s_2^4 + \frac{1}{16}s_1^4 + \frac{1}{24}s_2^3 - \frac{1}{24}s_1^3, \\
c_1 &= \frac{5}{L}\begin{pmatrix} 3s_1^3 - 8s_1^2 + 3s_2 s_1^2 + 6s_1 - 8s_2 s_1 + 3s_2^2 s_1 + \\ 6s_2 - 8s_2^2 + 3s_2^3 \end{pmatrix}, \\
c_2 &= \frac{5(3s_1^4 - 4s_1^3 + 8s_2^3 - 6s_2^2 - 3s_2^4)}{K}, \\
c_3 &= \frac{-5s_1^3(3s_1 - 4)}{K}, \\
c_4 &= \frac{-5(3s_1^3 - 4s_1^2 + 3s_2 s_1^2 - 4s_2 s_1 + 3s_2^2 s_1 - 4s_2^2 + 3s_2^3)}{L}, \\
\beta &= \pm\frac{\sqrt{P}}{2},
\end{aligned}$$

where

$$\begin{aligned} L = {} & 6s_1^4 - 15s_1^3 + 6s_1^3 s_2 + 10s_1^2 - 15s_2 s_1^2 + 6s_2^2 s_1^2 + 10s_2 s_1 \\ & - 15s_2^2 s_1 + 6s_1 s_2^3 + 10s_2^2 - 15s_2^3 + 6s_2^4 \end{aligned}$$

and

$$\begin{aligned}
K &= -6s_2^5 + 6s_1^5 + 15s_2^4 - 15s_1^4 - 10s_2^3 + 10s_1^3, \\
P &= \frac{1}{5}s_2^5 - \frac{1}{5}s_1^5 - \frac{1}{2}s_2^4 + \frac{1}{2}s_1^4 + \frac{1}{3}s_2^3 - \frac{1}{3}s_1^3.
\end{aligned}$$

A numerical example is given for the case $s_1 = 0.2$ and $s_2 = 0.8$ for which $\beta = -0.08584$, $\mu_1 = 0.003684$, $c_1 = 4.479$, $c_2 = 5.655$, $c_3 = -0.1538$, and $c_4 = 4.479$. The optimal area function $a(x)$ and the anti-optimal $F(x)$ (worst case loading) are shown in Figure 2.

In the case of a uniform beam, the worst case loading is given by

$$f(x) = 5.825x(1-x)$$

for $s_1 \le x \le s_2$ and the corresponding deflection is $y = 0.04292x(1-x)$. In this case $y_{un} = 0.01452$, $y_{max} = 0.01073$ and the efficiency is 74% as determined by the efficiency index given by equation (22).

(a)

(b)

Figure 2: Curves of optimal $a(x)$ and worst case loading $F(x)$ are plotted against x for $s_1 = 0.2$ and $s_2 = 0.8$ (Example 1).

4.2 Example 2: Unconstrained $F(x)$ with $s_1 = 0, s_2 = 1$.

Let the beam subject to only the uncertain transverse load $F(x)$ given by equation (1) with $s_1 = 0, s_2 = 1$, and subject to the constraint

$$\max_{0 \le x \le 1} f(x) \le f_{max}.$$

No uncertain moments or deterministic load are applied so that $\overline{m}_0 = \overline{m}_1 = \eta = 0$ and $g(x) = 0$. Thus

$$F(x) = \begin{cases} f(x), & \text{for } 0 \le x \le d_1, \\ f_{max}, & \text{for } d_1 \le x \le d_2, \\ f(x), & \text{for } d_2 \le x \le 1, \end{cases} \tag{25}$$

where d_1 and d_2 are unknown locations, but due to the symmetry of the loading we have $d_2 = 1 - d_1$. For this case the uncertain load function is given by equation (19) which can be substituted into equation (14) to obtain the differential equation for the optimal area function, *viz.*

$$a''(x) = \begin{cases} \frac{1}{4\mu_1} x(x-1), & \text{if } 0 \le x \le d_1, \\ \frac{f_{max}}{\beta}, & \text{if } d_1 \le x \le 1 - d_1, \\ \frac{1}{4\mu_1} x(x-1), & \text{if } \ 1 - d_1 \le x \le 1. \end{cases} \quad (26)$$

The solution of equation (26) gives the optimal area function

$$a(x) = \begin{cases} \frac{1}{48\mu_1} x^3(x-2) + c_1 x, & \text{if } 0 \le x \le d_1, \\ \frac{f_{max}}{2\beta} x^2 + c_2 x + c_3, & \text{if } d_1 \le x \le 1 - d_1, \\ \frac{1}{48\mu_1} (x^3(x-2)+1) \\ \quad + c_4(x-1), & \text{if } \ 1 - d_1 \le x \le 1, \end{cases} \quad (27)$$

which satisfies the boundary conditions (4). Equation (27) contains seven unknowns β, μ_1, d_1, c_1, c_2, c_3 and c_4 which can be computed seven equations for the volume and L_2-norm constraints (3), and the continuity conditions (17), (18) and (20). The optimal area function $a_{opt}(x)$ and the anti-optimal $F(x)$ (worst case loading) are shown in Figure 3.

In the case of a uniform beam, the worst case loading is given by

$f(x) = 7.305x(1-x)$, when $0 \le x \le d_1$ and $1-d_1 \le x \le 1$,

and the corresponding deflection is $y = 0.045x(1-x)$. In this case $y_{un} = 0.01452$, $y_{max} = 0.01125$ and the efficiency is 78% for $f_{max} = 1.15$ as determined by the efficiency index given by equation (22).

(a)

(b)

Figure 3: Curves of optimal $a(x)$ and the worst case loading $F(x)$ are plotted against x for $f_{max} = 1.15$ (solid line), $f_{max} = 1.25$ (broken-line) and $f_{max} = 1.35$ (dots) (Example 2).

5 Conclusions

The problem of optimal shape design of a simply supported beam was solved under a combination of uncertain and deterministic transverse loads. The compliance of the beam was maximized with respect to load to determine the least favorable loading which can be considered as an anti-optimization problem. Optimality conditions were derived as coupled differential equations involving the shape and load functions and the deflection of the beam. Analytical solutions of these equations were obtained subject to boundary and continuity conditions using symbolic computation software.

Two cases are presented for different loadings, and the graphs of the optimal beam shapes and worst-case loadings are given. The efficiencies of the optimal designs are computed in terms of the maximum deflections of the optimal beam and the uniform beam under least favorable loading.

Load uncertainties arise due to the unpredictable conditions occurring under operational conditions. This situation indicates the importance of a robust design since an optimized design is strong for the given load conditions, but weak if these conditions happen to change. The extension of this paper along with different cases is also under the authors' consideration [12].

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