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Transient and Asymptotic Analysis of Synchronization Processes in Assembly-Like Queues

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Abstract—**We analyze the transient and asymptotic behavior of a simple assembly-like queueing system often found as a component within a larger network. This system consists of two distinct types of items/customers arriving at separate buffers, according to independent Poisson processes, so as to be synchronized into pairs at a synchronization node. Once synchronized, a pair then queues up for service from a single server on a first-pair-in-first-pair-out basis. Service times of synchronized pairs are independently and exponentially distributed. We obtain explicit expressions for the transient and limiting values of the mean and variance of the cumulative number of synchronized pairs. When the two arrival rates are different, the process of synchronized pairs is asymptotically a Poisson process, enabling the use of an M/M/1 approximation. When the two arrival rates are equal, the synchronized process is not asymptotically a Poisson process, contradicting a result in [13]. However, the queue length process of synchronized pairs is still reasonably well approximated by an M/M/1 queueing system for low to moderately high traffic intensities. Most interestingly, by choosing equal arrival rates, both a transient and a long run benefit are obtained: the variance of the queue length process is approximately $\frac{1}{3}$ lower than that with unequal arrival rates.** ***Keywords: assembly-like queues, synchronization, Poisson processes, transient analysis, asymptotic approximations.***

1 Introduction

Consider an assembly system in which items/customers from two distinct infinite populations arrive at buffers dedicated to their respective types for synchronization into pairs. The two types of items are referred to as A-type and B-type and their dedicated, infinite capacity buffers as the A-buffer and B-buffer. The instant an item of each type is present in the buffers, they are immediately synchronized into a pair and sent to an infinite capacity queue for service by a single server on a *first-pair-in-first-pair-out* basis. Once a pair has finished receiving service, the pair leaves the system. If another pair is waiting, it then immediately enters service. If no pair is waiting, the server remains idle until the next pair is synchronized. The schematic diagram in Figure 1 illustrates this queueing system.

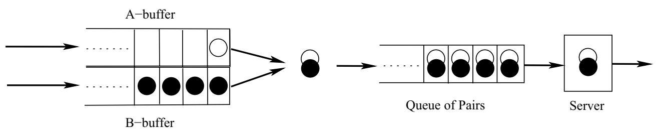

Figure 1: Synchronization and service of pairs.

Denote by A_t and B_t the cumulative number of A- and B-type items that have entered the A- and B-buffers during the time interval $[0, t]$. We assume $A_0 = B_0 = 0$. Let S_t denote the cumulative number of pairs that have been synchronized during $[0, t]$. Since we are assuming the synchronization and queueing of pairs for service is instantaneous, S_t is therefore determined by the minimum number of arrivals of the two types, *i.e.*,

$$S_t = A_t \wedge B_t := \min(A_t, B_t). \tag{1}$$

Another consequence of assuming instantaneous synchronization and queueing of pairs is that at all times, there is at least one buffer empty, possibly both but it is never the case that both buffers have items waiting to be synchronized. Although A_t and B_t are independent Poisson processes, S_t is not a Poisson process; this fact is confirmed in Lemma 1. The idea behind our approximation of the queueing and servicing of pairs is that if the arrival of synchronized pairs S_t to the service queue is sufficiently similar to a Poisson process then, together with the exponential service times, our system may be sufficiently similar to an M/M/1 queueing model to approximate its behavior by that of an appropriately chosen M/M/1 model. To justify this approximation and to

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determine which M/M/1 approximation would be appropriate, we study the behavior of the synchronized process S_t.

2 The Synchronized Process S_t

Using the independence of the two Poisson arrival processes A_t and B_t and their Poisson distributions, one easily derives the distribution of S_t:

Lemma 1 *Let A_t and B_t be independent Poisson processes with rates $a, b > 0$, respectively. For any $t > 0$ and any $p \in \mathbb{N}$, the probability $\mathbb{P}(S_t = p)$ is given by*

$$e^{-(a+b)t}\left\{\frac{(bt)^p}{p!}\sum_{m=p}^{\infty}\frac{(at)^m}{m!}+\frac{(at)^p}{p!}\sum_{n=p+1}^{\infty}\frac{(bt)^n}{n!}\right\} \quad (2)$$

It is clear that S_t does not have a Poisson distribution; see Section 3. Using (2), one can obtain fairly explicit expressions for the mean and variance of S_t:

Proposition 1 *For any $t > 0$ and for any $a, b > 0$ the mean of S_t is given by*

$$\mathbb{E}[S_t] \quad = \quad bt\,\mathbb{P}(A_t \geq B_t + 1) + at\,\mathbb{P}(B_t \geq A_t + 2) \quad (3)$$

Expression (3) for the mean is quite intuitive: When $A_t \geq B_t + 1$, there is nothing in the B-buffer but an excess in the A-buffer, and hence whenever a B-type arrives in the B-buffer, it is immediately paired with a waiting A-type and sent to the service queue. Thus, the rate at which synchronized pairs enter the queue for service is b, the Poisson arrival rate of the type fewest in number. If $B_t \geq A_t + 2$, then the A-type is fewest in number and the synchronizing rate is a. These two rates are then, roughly speaking, weighted by the probabilities of these two events occurring. If A lags more often than B then the synchronization rate will be weighted more toward a. On the other hand, if B lags more often than A, the synchronization rate will be weighted more toward b. Note the absence in (3) of the two events $(A_t = B_t)$ and $(B_t = A_t + 1)$.

Proposition 2 *In the symmetric case when $a = b$, the mean of S_t given by (3) further reduces to*

$$\mathbb{E}[S_t] \quad = \quad at\Big[1 - e^{-2at}\Big(I_0(2at) + I_1(2at)\Big)\Big] \quad (4)$$

where $I_m(\,\cdot\,)$ is the modified Bessel function of the first kind of order $m \in \mathbb{N}$.

The absence of $(A_t = B_t)$ and $(B_t = A_t + 1)$ in (3) is also seen in (4) as the subtraction of the two modified Bessel functions $I_0(2at)$ and $I_1(2at)$. For finite t, we see that S_t has an arrival rate that is slightly less than the common rate $a = b$ of the Poisson arrival processes.

One can also obtain a fairly explicit expression for the variance:

Proposition 3 *For any $a, b > 0$ and $t > 0$, the variance of S_t is given by*

$$\begin{aligned}\mathbb{V}[S_t] \quad &= \quad (bt)^2\mathbb{P}(A_t \geq B_t + 2) + (at)^2\mathbb{P}(B_t \geq A_t + 3) \\ &\quad +\mathbb{E}[S_t] - (\mathbb{E}[S_t])^2\end{aligned} \quad (5)$$

Expression (5) for the variance is not quite as intuitive as (3) is for the mean. The first three terms of (5) correspond to $\mathbb{E}[S_t^2]$. One can see in (5) a few missing events which is reflected in the symmetric case (6) again by a subtraction of modified Bessel functions.

Proposition 4 *In the symmetric case when $a = b$, the variance of S_t given by (5) further reduces to*

$$\begin{aligned}\mathbb{V}[S_t] \quad &= \quad at - (at)e^{-2at}I_0(2at) \\ &\quad -(at)^2e^{-4at}\Big(I_0(2at) + I_1(2at)\Big)^2\end{aligned} \quad (6)$$

Propositions 1 to 4 give the transient mean and variance. To obtain the asymptotic behavior, we let t increase to infinity. The key result needed in this direction is:

Theorem 1 *For any $M, N \in \mathbb{Z}$, we have*

$$\lim_{t\to\infty} t^N\mathbb{P}(A_t \leq B_t + M) = 0 \quad (7)$$

when $a > b$. When $a < b$ we have

$$\lim_{t\to\infty} t^N\mathbb{P}(B_t \leq A_t + M) = 0 \quad (8)$$

And, when $a = b$ we obtain

$$\lim_{t\to\infty}\mathbb{P}(B_t \leq A_t + M) = \lim_{t\to\infty}\mathbb{P}(A_t \leq B_t + M) = \frac{1}{2} \quad (9)$$

Limits (7) and (8) reflect the fact that the probabilities converge to zero exponentially. In the symmetric case, we obtain (9) but we no longer have exponential convergence; the probabilities approach their limits like $1/\sqrt{t}$. Theorem 1 applied to S_t then yields:

Corollary 1 *When $a > b$ we have*

$$\lim_{t\to\infty}\mathbb{P}(S_t = B_t) = 1 \quad (10)$$

When $a < b$ we have

$$\lim_{t\to\infty}\mathbb{P}(S_t = A_t) = 1 \quad (11)$$

And, when $a = b$ we obtain

$$\lim_{t\to\infty}\mathbb{P}(S_t = A_t) = \lim_{t\to\infty}\mathbb{P}(S_t = B_t) = \frac{1}{2} \quad (12)$$

When $a > b$, the sample paths of S_t are increasingly likely to coincide with those of B_t as t increases to infinity. Similarly, for $a < b$, S_t becomes more like A_t. Thus, in the non-symmetric case of $a \neq b$, S_t is asymptotically the same as the Poisson arrival process having the slower arrival rate. In the symmetric case of $a = b$, we see that the paths of S_t become equally likely to coincide with those of A_t and B_t, indicating that the symmetric case is fundamentally different; it is some kind of mixture of two independent Poisson processes with equal rates. One might think intuitively that S_t then becomes Poisson with the common rate $a = b$, which is the result claimed in [13] proved using more abstract techniques. We use much simpler techniques and show that this intuition is incorrect, as highlighted by our main result:

Theorem 2 *In the non-symmetric case of $a \neq b$, we have*

$$\lim_{t\to\infty} \frac{\mathbb{E}[S_t]}{t} = \lim_{t\to\infty} \frac{\mathbb{V}[S_t]}{t} = a \wedge b \tag{13}$$

In the symmetric case of $a = b$, we have

$$\lim_{t\to\infty} \frac{\mathbb{E}[S_t]}{t} = a \quad \textit{and} \quad \lim_{t\to\infty} \frac{\mathbb{V}[S_t]}{t} = a\left(1 - \frac{1}{\pi}\right) \tag{14}$$

In the non-symmetric case, the mean and variance rates converge to that of the more slowly arriving Poisson process. In the symmetric case, we see that the mean rate converges to that of the Poisson arrival process but the variance rate converges to a value less than one associated with a Poisson process: the average mean is a but the average variance is $a(1 - \frac{1}{\pi})$ which is approximately 68% of a, a surprising result.

3 Some Intuition

To provide intuition behind our results, note that for large t the sums in (2) are approximated by e^{at} and e^{bt}, respectively. Thus we expect the large t behavior of the probability density of S_t to be like

$$e^{-(a+b)t}\left\{\frac{(bt)^p}{p!}e^{at} + \frac{(at)^p}{p!}e^{bt}\right\} \;=\; e^{-bt}\frac{(bt)^p}{p!} + e^{-at}\frac{(at)^p}{p!}$$

In the non-symmetric case when $a > b > 0$, e^{-at} goes to zero exponentially faster than e^{-bt} and so we further expect the density to behave like $e^{-bt}(bt)^p/p!$ for large t. A similar observation holds for $b > a > 0$. Corollary 1 confirms this intuition: in the non-symmetric case, S_t becomes Poisson with the smaller rate. And, as a consequence, result (13) of Theorem 2 is fully consistent with well known results for Poisson processes. More simply, when $a > b$, the number of A-type in the A-buffer grows without bound while the number of B-type in the B-buffer drops to zero. With an unbounded supply of A-type items waiting, the only limitation on the synchronization of pairs is the rate at which B-type arrive.

However, when $a = b$ the same intuitive reasoning leads to the approximation

$$e^{-(a+a)t}\left\{\frac{(at)^p}{p!}e^{at} + \frac{(at)^p}{p!}e^{at}\right\} \;=\; 2e^{-at}\frac{(at)^p}{p!}$$

which cannot be correct since it sums to a value greater than 1. One might then argue intuitively that from Corollary 1, S_t becomes the same as A_t and B_t, each with probability $\frac{1}{2}$, so that the factor of 2 in the above approximation somehow cancels in the mixing and S_t becomes Poisson. From [2], pg. 98, we know that when $a = b$ we have $\limsup_{t\to\infty}(A_t - B_t) = +\infty$ and $\limsup_{t\to\infty}(A_t - B_t) = -\infty$ which also seems to support the idea of mixing. If this heuristic reasoning about mixing was accurate, then we could explain the asymptotic mean in (14) of Theorem 2 but we would remain unable to explain the asymptotic variance in (14) of Theorem 2. Such intuitive arguments easily mislead us. Corollary 1 and Theorem 2 show that we do have stable behavior in the $a = b$ case but not one that is obvious at an intuitive level.

4 Numerical Results

First, we numerically confirm the theoretical results of Theorem 2 by computing the mean and variance directly from the probability distribution (2). The point of this exercise is to obtain numerical results which do not depend on any of the theoretically derived results. These results are presented in Figures 2,3,4 and 5 below. Figures 2 and 3 plot the numerically computed values of $\mathbb{E}[S_t]/t$ and $\mathbb{V}[S_t]/t$ for a relatively short time period ($t \in [0, 5]$) to reveal the transient behavior. Figures 4 and 5 plots these same values over a longer time range ($t \in [0, 200]$) to reveal long run behavior. Figures 2 and 4 demonstrate that the means converge to $a = 1$ in all cases of the parameter $b = 1, 2, 10$, including the symmetric case. However, one can see that when $b = 2, 10$ we have a rapid, exponential convergence whereas with $b = 1$ we have a slower $1/\sqrt{t}$ convergence. This same qualitative difference in convergence behavior is seen in the variance graphs of Figures 3 and 5. The important difference with variance is that in the non-symmetric cases it converge to $a = 1$ whereas in the symmetric case the variance converges to $a(1 - \frac{1}{\pi})$.

Next, we evaluate the accuracy of an M/M/1 approximation which uses the slowest rate for the Poisson arrival process and the same service rate at which synchronized pairs are served. We perform a Monte Carlo simulation of S_t and estimate the mean number of pairs in the system, the mean time a pair is in the system, the mean number of pairs waiting in the system, and the mean time pairs wait in the system. The Monte Carlo simulation is run until 780,000 synchronized pairs have passed through the system. Then, using well known analytical formulae for M/M/1 queueing models, we compute the corresponding M/M/1 performance measures which are to serve as our approximation. In all these simulations, we set the service rate to $\mu = 1.0$. When we set $a = 0.9$ and vary b from 0.1 to 0.8 in increments of 0.1, the range of percentage differences between the Monte Carlo estimated performance measures and the analytically computed M/M/1 performance measures is -1.0054% and $+1.4882\%$. This experiment demonstrates the approximation for a near b. To check the approximation for large a we set $a = 10$ and repeated the simulation. In this case all percentage differences are in the smaller range -0.5354% to $+0.5039\%$. To check the approximation for a only a slightly larger than b we set $a = 1.5$ and repeated the simulation. The range of percentage differences is -0.6212% to $+0.7674\%$. Finally, we performed the simulation for $a = b$ for all common values from 0.1 to 0.9 in increments of 0.1. The range of percentage differences for $a = b \leq 0.8$ is -0.5787% to $+0.8840\%$. However, when $a = b = 0.9$, the percentage differences range from $+2.8288\%$ to $+3.2973\%$, indicating that under heavy traffic conditions, the symmetric case is less well approximated by an M/M/1 model than it is in all other cases.

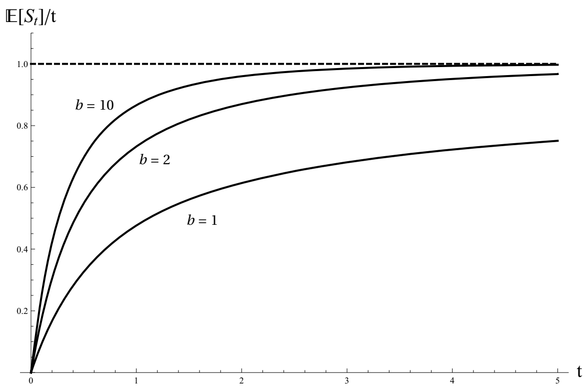

Figure 2: Transient behavior of $\mathbb{E}[S_t]/t$ for $a = 1$ and $b = 1, 2, 10$ and $t \in [0, 5]$.

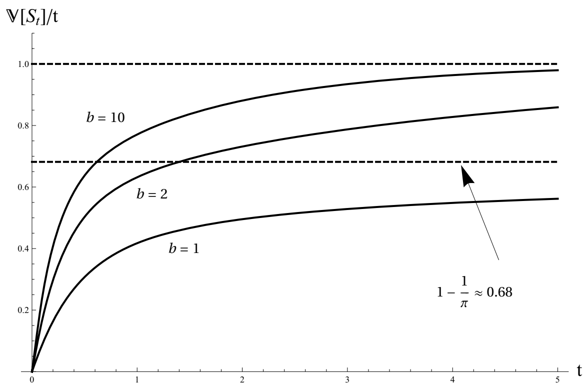

Figure 3: Transient behavior of $\mathbb{V}[S_t]/t$ for $a = 1$ and $b = 1, 2, 10$ and $t \in [0, 5]$.

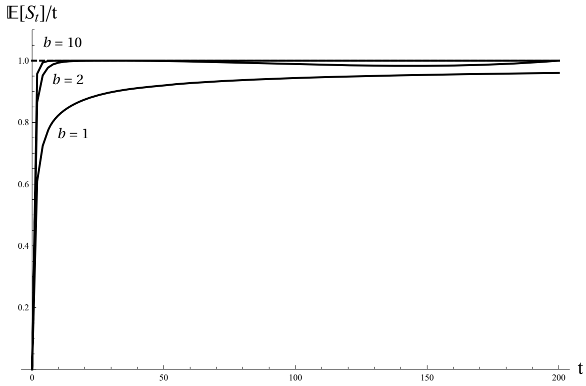

Figure 4: Transient behavior of $\mathbb{E}[S_t]/t$ for $a = 1$ and $b = 1, 2, 10$ and $t \in [0, 200]$.

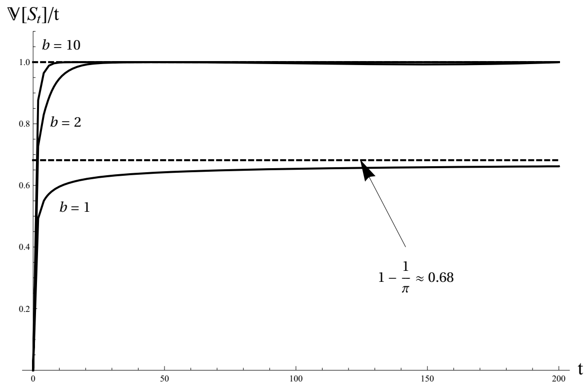

Figure 5: Transient behavior of $\mathbb{V}[S_t]/t$ for $a = 1$ and $b = 1, 2, 10$ and $t \in [0, 200]$.

5 Conclusions and Future Work

Through very simple probability arguments, we derive the transient and asymptotic behavior of the mean and variance of the synchronized process S_t. We demonstrate the correctness of our theoretical results with simple and direct numerical computations. We also evaluate the accuracy of using an M/M/1 model to approximate the behavior of our queueing system. The first conclusion we reach is that there is a discrepancy between our results and those in [13]. The second conclusion is that the M/M/1 approximation is very good for all cases except the symmetric case under conditions of heavy traffic. The third conclusion is that by selecting the arrival processes to have the same rate, we achieve a 32% reduction in the variance of the queue length, even throughout a good portion of the transient phase. However, this reduction of variance comes at the cost of a slower convergence of the mean, as clearly seen in Figure 4.

Future work consists of identifying precisely where the discrepancy lies between our results and those in [13]. A natural extension of this work is to consider more than two independent arrival processes. Also, we have assumed an infinite synchronization rate in that pairs are instantaneously matched and sent to the service queue. It would be interesting to explore the effect of adding a random synchronization time, which would result in a modification of (1) giving S_t in terms of the arrival processes. Finally, since the high traffic intensity case is not very well approximated by an M/M/1 model, one could try a G/M/1 model or perhaps a diffusion approximation with infinitesimal mean and variance given by a and $a(1 - \frac{1}{\pi})$, respectively.

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