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# New Vistas in Inventory Optimization under Uncertainty

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*Abstract*—**In this paper we present an application of robust optimization to inventory optimization under uncertainty. We represent uncertainty in a constraint based framework derived from basic economic principles. This approach offers the ability to use information theoretic concepts to quantify the amount of information used in the optimization. The results are shown to correspond to classical models such as EOQ in simple cases. Not only this, the presented model easily incorporates more realistic constraints, which are complicated and are not easily incorporated by the classical models.**

***Keywords***— Inventory optimization, Optimization under uncertainty, Robust programming

## I. Introduction

A major challenge in supply chain inventory optimization is handling uncertainty, as not all the data required for making decisions are available with certainty at the time of making the decision This problem of design/analysis/optimization under uncertainty is central to decision support systems, and extensive research has been carried out in both Probabilistic (Stochastic) Optimization and Robust Optimization (constraints) frameworks. However, these techniques have not been widely adopted in practice, due to difficulties in conveniently estimating the data they require. Probability distributions of demand necessary for the stochastic optimization framework are generally not available. The constraint based approach of the Robust Optimization School has been limited in its ability to incorporate many criteria meaningful to supply chains. At best, the "price of robustness" of Bertsimas et al [4], [5] is able to incorporate symmetric variations around a nominal point. However, many real life supply chain constraints are not of this form.

Our approach modifies the robust optimization approach and makes it more intuitive and meaningful in the context of supply chains, while coupling optimization with information theory [12]. We represent uncertainty in a constraint based framework naturally derived from basic economic principles [11]. The uncertainty sets (constraint sets) form a convex polytope, built from simple and intuitive linear constraints (simple sums and differences of supplies, demands etc) those are derivable from historical time series data. With this specification, many kinds of future uncertainty can be specified. This specification avoids deterministic but ad-hoc gravity models and their variants, as well as ad-hoc probability distributions. The optimization problems are computationally tractable – LP's or ILP's. Answers are globally valid over the entire range of parameter variations, which can be correlated.

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In essence our work is the first which enables design of supply chains using exactly the information most designers are comfortable with, without introducing any new assumptions, either in deterministic demand or probability distribution functions.

Not only this, our approach offers the unique ability to quantify the amount of information used in the optimization based on information theoretic concepts. Finally, our approach is able to qualitatively compare different sets of uncertainty scenarios, using the relational algebra of polytopes (this is outside the scope of this paper).

*Compared to our earlier work [11], [12], where we discussed static capacity planning, in this paper we discuss aspects of inventory optimization, and present algorithms for min-max policy optimization. We discuss and illustrate how close-to-optimal heuristic techniques can be designed, and compare their performance relative to worst case bounds.*

In the rest of this paper we discuss these areas. In sections III we describe the specification of uncertainty; in section IV we present algorithms for finding close to optimal solutions for the min-max optimizations and in section V we illustrate the ideas with a small but detailed example.

## II. Related work

For inventory optimization, the classical technique is the EOQ model proposed by Harris [9] in 1913. In the 1950's Arrow, Harris and Marschak [2], Dvoretzky, Kiefer and Wolfowitz, [8] and Whitin [13] began work on stochastic inventory control. In 1960, Clark and Scarf [7] proved the optimality of base stock policies using dynamic programming. , These results minimally make some assumptions about the stochastics of the demand. The distribution independent robust optimization approach is typified by the work of Bertsimas, Sim and Thiele [4], [5] where they have proposed uncertainty models using robust optimization that also allow the level of conservatism to be controlled for each constraint. However their work is limited to symmetric polyhedral uncertainty sets with $2^N$ faces, and is not directly related to economically meaningful parameters. This symmetric nature does not distinguish between a positive and a negative deviation, which can be important in evaluating system dynamics (for example poles in the left versus right half plane).

## III. Representation of Uncertainty

Compared to Bertsimas-Sim-Thiele, our work is more expressive, with the ability to describe uncertainty sets described by arbitrary convex polytopes, e.g. with an arbitrary number of faces and orientations.

The polytopes are built from simple and intuitive linear constraints (simple sums and differences of supplies, demands etc) that are derivable from historical time series data. With this specification, many kinds of future uncertainty can be specified. Since the constraints are linear, most optimization problems (with linear metrics) can be modeled as linear, quadratic (in some cases) or integer linear programs. This specification avoids deterministic models using ad-hoc gravity models and their variants, as well as stochastic optimization based on ad-hoc probability distributions. This approach gives bounds on the performance parameters that are globally valid.

In the context of inventory optimization, the flexibility of our formulation allows us to describe absolute bounds on inventories, correlations between inventories and supplies/demands, correlations between inventories at different points, etc. Some examples of these constraints are:

i. Constraints on inventory: The total inventory at a node for a particular product at a particular time period may be limited,

$$Min_{ijk} \leq \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq Max_{ijk} \;;$$
$$\forall\ \mathrm{i} \in \mathrm{nodes},\ \mathrm{j} \in \mathrm{products\ and\ k} \in \mathrm{t}$$

where $Inv_{ijk}$ is the inventory at node i for product j in time period k.

The total inventory for a particular product at a particular node over all the time periods may be within limits,

$$Min_{ij} \leq \sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq Max_{ij} \;;$$
$$\forall\ \mathrm{i} \in \mathrm{nodes\ and\ j} \in \mathrm{products}$$

The total inventory for all the products at a particular node over all the time periods may be within limits,

$$Min_i \leq \sum_j \sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq Max_i \;; \forall\ \mathrm{i} \in \mathrm{nodes}$$

The total inventory for all the products at all the nodes that may ever be stored may be bounded,

$$Min \leq \sum_i \sum_j \sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq Max$$

The total inventory for all the products at a particular node at a particular time period may be within bounds,

$$Min_{ik} \leq \sum_j \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq Max_{ik} \;;$$
$$\forall\ \mathrm{i} \in \mathrm{products\ and\ k} \in \mathrm{t}$$

Etc.

ii. Inventory tracking demand: The total inventory may be limited by total purchases. E.g., the total inventory for a particular product that can be stored over all the nodes, over all time periods may be no more than 50% of the total purchases and no less than 30 % of the total purchases.

$$\sum_i \sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq 0.5\left(d_j\right) \;; \forall\ \mathrm{j} \in \mathrm{products}, \text{ and}$$
$$\sum_i \sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \geq 0.3\left(d_j\right) \;; \forall\ \mathrm{j} \in \mathrm{products}$$

where $d_j$ is the demand for product j.

iii. Inventory tracking supplies: The total inventory may be limited by the total supplies. E.g., the total inventory for a particular product that can be stored at a particular node over all time periods may be no more than 50% of the total supply to that node and no less than 30 % of the total supply to that node.

$$\sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \leq 0.5\left(\sum_{m \in \mathrm{Pr}ed(i)} \sum_k S_{mjk}\right) \;;$$
$$\forall\ \mathrm{i} \in \mathrm{nodes\ and\ j} \in \mathrm{products}$$

and,

$$\sum_k \left(\mathrm{Inv}_{\mathrm{ijk}}\right) \geq 0.3\left(\sum_{m \in \mathrm{Pr}ed(i)} \sum_k S_{mjk}\right);$$
$$\forall\ \mathrm{i} \in \mathrm{nodes\ and\ j} \in \mathrm{products}$$

iv. Inventories tracking each other: We may want to ensure that inventories stored at two different factories are close to each other in amount. One reason why we may want this may be because if a factory fails, the other may be used to satisfy the demand that the failed factory was supposed to satisfy.

$$Min \leq Inv_{xjk} - Inv_{yjk} \leq Max \;;$$
$$\forall\ \mathrm{j} \in \mathrm{products\ and\ k} \in \mathrm{t}$$

where x is the first factory and y is the second factory and the difference in their inventories for a product in a particular time period is bounded.

Similar to above, we can have constraints on demands, supplies, correlations among demands, supplies, inventories etc. In this way sums, differences, and weighted sums of demands, supplies, inventory variables, etc, indexed by commodity, time and location can all be intermixed to create various types of constraints on future behavior. We shall illustrate a detailed example in Section V.

## IV. Optimization Algorithms

Our formulation results in tractable models. Firstly, the classical multi-commodity flow model [1] is a natural formulation for supply chain problems. Supply chains can be viewed as networks where there is flow conservation at the nodes. This flow conservation can be written as linear flow equations, under the influence of which any optimization in the supply chain can be solved using network optimization techniques. The fundamental inventory conservation equation is (the subscript t is the time index)

$$Inventory_{t+1} = Inventory_t + Supply_t - Demand_t$$

Let $\Phi_S$ be the flow vector from the suppliers, $\Phi_D$ the (variable) demand, and $\Phi_I$ the inventory. Define $\Phi$ as the flow vector $[\Phi_S, \Phi_D, \Phi_I]$, indexed by node, commodity and time. Then the flow conservation equations can be written in the matrix form $A\Phi \leq B$, where A is the unimodular flow conservation matrix, and B the source/sink values. All linear metrics are of the form $C^T\Phi$. An optimal inventory policy is one which selects the flows to optimize the metric (minimize the cost /maximize the profit). Hence a generic supply chain optimization is of the form

$$\text{Min } C^T\Phi \quad .$$
$$A\Phi \leq B \qquad \text{(Equation 1.1)}$$

However, a realistic supply chain is subject to non-convex constraints such as cost/price breakpoints, 0/1 facility location decisions etc and in this case the problem has to be modeled as an ILP with associated computational difficulties. Quadratic terms can also appear in both the constraints and the metric.

When we introduce uncertainty, the right hand side B becomes a variable (and moves to the l.h.s), and additional constraints $D^T B \leq E$ for these variables are introduced, yielding the LP

$$\text{Min } C^T\Phi$$
$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Phi \\ 1 \end{bmatrix} \leq 0$$
$$D^T B \leq E$$

Here $D^T B \leq E$ represents all the (linear) uncertainty constraints described in Section III.

The optimal policy finds the correct ordering policy ($\Phi$), which *minimizes* the cost in the *worst case* of the uncertain parameters. This is a *min-max* optimization, and is not an LP. Duality can be employed to convert the max to a min, but the presence of non-convexities precludes strong-duality from being achieved [6]. Heuristics have to be used in general.

Our approach is statistical. First, the performance is bounded using the absolute minimum and absolute maximum possible costs, corresponding to the best policy under the best conditions and the worst policy under the worst conditions respectively (these can be found directly by max/min the ILP).

The above bounds serve as input to a statistical policy sampling process (Fig. 1), which generates a number of different policies, each of which is optimal for a different randomly chosen demand. The one having the lowest worst-case cost is selected. Determining the worst-case cost for a chosen policy can be shown to be an LP (details omitted). The best policy for a given deterministic demand is given by solving the LP/ILP corresponding to Equation 1.1, for increasingly longer time horizons, and the steady state solution picked. For the example in Section IV, for a single time step our LP's had 84 variables (21 integer), and 131 constraints, and the average computation time was just 60 milliseconds.

While the convergence of this process to the Min-max solution is still an open problem, note that our contribution is the complete framework, and the tightest bound is not necessarily required in an uncertain setting.

```
Begin

for i = 1 to maxIteration
{
    parameterSample = getParameterSample(i, constraint Set)
    bestPolicy = getBestPolicy(i, parameterSample)
    findCostBounds(i, betPolicy)
}
chooseBestSolution()

End
```

Fig. 1: The steps of the sampling approach

Details are in Section V.

## V. ILLUSTRATIVE EXAMPLE

In this Section, we illustrate the richness and tractability of our formulation to handle sophisticated constrained inventory optimizations, typical of realistic applications, which are only approximated using classical methods. The classical EOQ methods do not in general incorporate uncertainty, but can be extended to doing so. However, the correlations between different parts of the supply chain are not modellable using this framework. Our methods can very easily handle both, are intuitive, and are also computationally tractable – a unique feature compared to alternative approaches (stochastic programming, traditional robust optimization)

Our methods reduce to EOQ basestock type solutions [9], in simple unconstrained cases. With constraints, we find that the optimal policy is not EOQ in general, and EOQ may not even be feasible. We are unaware of any other work offering these facilities. All of our results were produced on an Intel Celeron 1.60 GHz processor, with a 512 MB RAM.

We consider an example from the automotive sector. Consider a store that deals in cars, tyres, petrol and drivers. There are three kinds of cars and two kinds of tyres, thus there are seven different products that the store provides. This example, while small, is sufficiently illustrative of the capabilities of our approach. To relate our methods to classical EOQ, all the products have a fixed ordering cost and a linear holding cost given in Table 1.

TABLE 1
ORDERING AND HOLDING COSTS OF THE PRODUCTS

| Product | Ordering Cost in Rs. (per order) | Holding Cost in Rs. (per unit) |
|---|---|---|
| Car Type I | 1000 | 50 |
| Car Type II | 1000 | 80 |
| Car Type III | 1000 | 10 |
| Tyre Type I | 250 | 0.5 |
| Tyre Type II | 500 (intl shipment) | 0.5 |
| Petrol | 600 | 1 |
| Drivers | 750 | 300 |

Car type I is a comfort class car which has a monthly demand that is on the high side of the scale and there is an average holding cost per car due to the warehouse space occupied. Car type II is a luxury car with much lower demands than the type I car and a holding cost that is larger due to higher protection that it requires and bigger size. Demands are low due to high costs, high maintenance and poor mileage. Car type III is an economy class car, with high demand and considerably lower holding costs due to its small size. Demands are high because of excellent performance, low maintenance, low costs. Tyre type I is a local made tyre, with high demand due to its low prices. Tyre type II is an imported brand, thus the higher ordering cost, and has lower demand due to its higher prices. Both brands have same holding cost, as they occupy same amount of warehouse space. Petrol is ordered from a supplier in another city and stored in an underground storage tank. The store also provides services of professional drivers.

***(a) Exactly Known Demands, no uncertainty***

If the demands for all these products are known exactly then the optimal order quantities and order frequencies can be calculated using the EOQ model [9] as follows:

$$Q^* = \sqrt{\frac{2CD}{h}} \text{ and } f^* = \sqrt{\frac{Dh}{2C}}$$

where C is the fixed ordering cost per order, h is the per unit holding cost and D is the demand rate.

We see that the solution given by constrained optimization matches exactly with this solution as given in Table 2.

TABLE 2
EOQ AND CONSTRAINED OPTIMIZATION SOLUTION FOR KNOWN DEMANDS

| Product | Demand per month | EOQ Solution | | | Constrained Optimization Solution | | |
|---|---|---|---|---|---|---|---|
| | | Order Freq | Order Quant | Cost | Order Freq | Order Quant | Cost |
| Car Type I | 40 | 1 | 40 | 2000 | 1 | 40 | 2000 |
| Car Type II | 25 | 1 | 25 | 2000 | 1 | 25 | 2000 |
| Car Type III | 50 | 0.5 | 100 | 1000 | 0.5 | 100 | 1000 |
| Tyre Type I | 250 | 0.5 | 500 | 250 | 0.5 | 500 | 250 |
| Tyre Type II | 125 | 0.25 | 500 | 250 | 0.25 | 500 | 250 |
| Petrol | 300 | 0.5 | 600 | 600 | 0.5 | 600 | 600 |
| Drivers | 5 | 1 | 5 | 1500 | 1 | 5 | 1500 |
| Total | | | | 7600 | | | 7600 |

Car type I, car type II, and drivers must be ordered in every month; car type III, tyre type I and petrol every alternate month; and tyre type II every fourth month.

***(b) Bounded Uncorrelated Uncertainty***

Unfortunately, we cannot know the future demands accurately. If we represent this uncertainty as bounds on the individual demands, we can still get min and max bounds from the EOQ model. When the demands are bounded as given by Table 3, the EOQ bounds and bounds from the constrained optimization solution are also almost the same as shown in Table 4. The only difference is in the ordering of "tyre type I" and "drivers" as the EOQ solution specifies an optimal order quantity of 248.99 tyres/order and 2.24 drivers/order which is clearly not realizable, so the constrained optimization rounds this to 248 tyres/order and 2 drivers/order.

For this kind of uncertainty, we need to order car type I, car type II, and drivers at least every alternate month and at most every month; tyre type I, tyre type II and petrol at least every fourth month and at most every alternate month; and car type II at least never and at most every month.

TABLE 3
UPPER AND LOWER BOUND ON DEMANDS

| Product | Min Demand | Max Demand |
|---|---|---|
| Car Type I | 10 | 40 |
| Car Type II | 0 | 25 |
| Car Type III | 50 | 200 |
| Tyre Type I | 62 | 250 |
| Tyre Type II | 125 | 500 |
| Petrol | 75 | 300 |
| Drivers | 1 | 5 |

TABLE 4
EOQ AND CONSTRAINED OPTIMIZATION SOLUTION FOR BOUNDED DEMANDS

| Product | EOQ solution | | | | Constrained Optimization | | | |
|---|---|---|---|---|---|---|---|---|
| | Order Frequency | | Order Quantity | | Order Frequency | | Order Quantity | |
| | Min | Max | Min | Max | Min | Max | Min | Max |
| Car Type I | 0.5 | 1 | 20 | 40 | 0.5 | 1 | 20 | 40 |
| Car Type II | 0 | 1 | 0 | 25 | 0 | 1 | 0 | 25 |
| Car Type III | 0.5 | 1 | 100 | 200 | 0.5 | 1 | 100 | 200 |
| Tyre Type I | 0.25 | 0.5 | 248.99 | 500 | 0.25 | 0.5 | 248 | 500 |
| Tyre Type II | 0.25 | 0.5 | 500 | 1000 | 0.25 | 0.5 | 500 | 1000 |
| Petrol | 0.25 | 0.5 | 300 | 600 | 0.25 | 0.5 | 300 | 600 |
| Drivers | 0.45 | 1 | 2.24 | 5 | 0.5 | 1 | 2 | 5 |

***(c) Beyond EOQ: Correlated Uncertainty in Demand***

So far, the constrained optimization model has incorporated the simple mechanics of the EOQ model perfectly. Now we will show how the model can be used to incorporate more complicated behavior, which EOQ cannot represent. Compared to general constrained optimization approaches (e.g. SAP APO) used in supply chain optimizers, we shall see that our *approach is based on very intuitive information*, which are conveniently available to planners.

Considering that the three types of cars are substitutive and the two types of tyres are also substitutive, this can be represented by the following equations:

**200 ≤ dem_tyre_1 + dem_tyre_2 ≤ 700**
**65 ≤ dem_car_1 + dem_car_2 + dem_car_3 ≤ 250**

The first inequality means that if the demand for one type of tyre increases, the demand for the other type of tyre should go down and vice versa. This constraint takes into account the fact that the demands for the two brands of tyres are correlated and coexist. The lower bound here is greater than the sum of lower bounds on demands of individual types and the upper bound is smaller than the sum of upper bounds on individual demands, thus creating the substitutive effect.

Similarly the second inequality means that the demand for all types of cars cannot increase or decrease simultaneously. Here also we have substitutive effect as the lower bound on the sum is greater that sum of lower bounds on demands of individual types and the upper bound on the sum is smaller than the sum of upper bounds on individual demands.

Complementary effect between different products can also be easily expressed as bounds on differences, for example, consider the following inequality:

**5 ≤ (dem_car_1 + dem_car_2 + dem_car_3) – dem_petrol ≤ 20**

This constraint represents the assumption that the demand of petrol tracks the demand of cars. If there is an increase in the demand for cars, the demand for petrol will simultaneously rise and vice versa.

Let us suppose that people who buy luxury cars (car type II) are more likely to hire drivers too and the drivers provided by the store are almost always for luxury car owners. Then the demand for drivers must track the demand for luxury cars and this is represented by the following constraint:

**5 ≤ dem_car_2 – dem_drivers ≤ 20**

The results for optimization in the best case for different scenario sets are shown in Table 5. The solution in each of these cases is very different from the EOQ solution and demonstrates the capability of our formulation to easily incorporate complicated co-relations amongst different parameters.

TABLE 5
BEST CASE SOLUTIONS FOR DIFFERENT SCENARIO SETS

| Products | With Substitutive Constraints | | With Complementary Constraints | | With both Substitutive and Complementary constraints | |
|---|---|---|---|---|---|---|
| | Order Freq | Order Quant | Order Freq | Order Quant | Order Freq | Order Quant |
| Car Type I | 0.5 | 20 | 0.5 | 20 | 0.5 | 20 |
| Car Type II | 0 | 0 | 0.5 | 12 | 0.5 | 12 |
| Car Type III | 0.5 | 110 | 0.5 | 128 | 0.5 | 128 |
| Tyre Type I | 0.25 | 274 | 0.25 | 248 | 0.25 | 261 |
| Tyre Type II | 0.25 | 526 | 0.25 | 552 | 0.25 | 539 |
| Petrol | 0.25 | 300 | 0.25 | 300 | 0.25 | 300 |
| Drivers | 0.5 | 2 | 0.5 | 2 | 0.5 | 2 |

Comparing the solution in Table 5, when both substitutive and complementary constraints are valid with the EOQ solution of Table 4, we see that the EOQ solution is not even valid for this case. The lower bound on the cost by the EOQ solution is Rs. 3348.248, whereas in the substitutive-complementary constrained case the lower bound on the cost is Rs. 4482.5. With just substitutive constraints car type II need not be ordered at all, but when complementary constraints are considered it must be ordered at least every alternate month.

***(d) Correlated Inventory Constraints***

Let us now consider that our inventory holding capacities are constrained. Let us suppose that the store in the example has a warehouse where it stores cars and tyres. Taking the scenario set when all the constraints are acting the total inventory of cars will begin with 120 cars and the inventory of tyres with 700 tyres. Now since we have limited storing capacity, let us suppose that we cannot store more than 160 tyres at any given time and no more than 68 cars. These limitations can be represented by the following constraints:

**Inv_tyre_1 + Inv_tyre_2 ≤ 120**
**Inv_car_1 + Inv_car_2 + Inv_car_3 ≤ 68**

Since we cannot store more inventories now, we will have to reduce our order quantities. In order to fulfill the demand, now we will have to place more frequent orders than before. This is exactly what the solution from our formulation gives us and is given in Table 6.

TABLE 6
BEST CASE SOLUTION WHEN INVENTORIES ARE CONSTRAINED

| Product | Order Frequency | Order Quantity |
|---|---|---|
| Car Type I | 0.5 | 20 |
| Car Type II | 0.25 | 24 |
| Car Type III | 1 | 64 |
| Tyre Type I | 0.5 | 124 |
| Tyre Type II | 1 | 138 |
| Petrol | 0.25 | 300 |
| Drivers | 0.5 | 2 |

The total cost for this policy is Rs. 5195.5, Rs. 713 greater than when there are no inventory constraints.

***(e) Computational Procedure***

As stated in Section IV, the min-max inventory policy (best decisions for the worst case demands, inventories) for the case when only bounds and substitutive are acting is found by taking 1500 inventory policy samples shown by the scatter plot in Fig. 2. The lower bound on cost (Min-Min solution) is Rs. 3412.5, and the upper bound is Rs. 9100. From this scatter plot, the Min-Max solution has a cost not exceeding Rs. 5775, which is about 70% higher than the (unrealistic) min-min bound. *The important point to note is that these are global bounds, and are valid over the entire (infinite) range of parameter (demand, supply, …) variations, including inventory constraints*. Most alternative approaches either take deterministic demands or a few scenarios (low/average/high). The stochastic programming framework typically makes uncorrelated assumptions about probability distributions. The traditional robust programming approach does not have the rich correlated behavior we can handle.

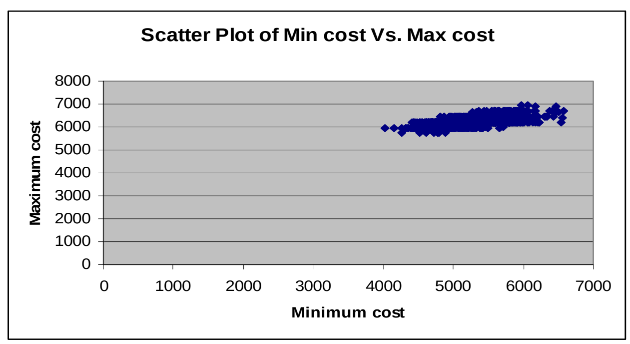


Fig. 2: Scatter plot of min/max cost bounds through demand sampling

While we have described a simple example, we have successfully run examples with up to 500 products, multiple locations, and tens of timesteps. With improvements in our computational methods, we expect to be able to handle industrial scale problems. Table 7 summarizes statistics for some of the examples that we have run and have obtained optimal solution, the largest having 78000 variables (18000 integer) and 126000 constraints.

TABLE 7
DETAILS OF DIFFERENT PROBLEMS SOLVED USING THE NEW FORMULATION

| Locations | Products | Time steps |
|---|---|---|
| 1 | 1 | 100 |
| 1 | 1 | 200 |
| 1 | 25 | 24 |
| 15 | 7 | 12 |
| 60 | 2 | 12 |
| 16 | 10 | 6 |
| 1 | 500 | 12 |
| 100 | 1 | 12 |

Our formulation offers additional capabilities. Using techniques described in [12], we can estimate the amount of information in each one of the scenarios, by estimating the volume of the polyhedron [10] enclosed by the constraints composing the scenario. Table 8 summarizes the bounds on output cost and the amount of information encompassed by the constraints in each of the scenario sets. We can see that as we add more and more constraints, we are adding very less information to the system, as the information content is increased by less than 1 bit of information, but there is considerable change in the bounds. The absolute amount of information (around 55 bits) is based on a normalization volume – this reflects the apriori knowledge in the absence of any constraint information.

TABLE 8
RELATIVE INFORMATION CONTENT OF DIFFERENT SCENARIO SETS

| Scenario sets | Absolute Minimum Cost | Absolute Maximum Cost | Information Content (Number of bits) |
|---|---|---|---|
| Bounds only | 3349.5 | 9187.5 | 55.96 |
| Bounds and Substitutive constraints | 3412.5 | 9100 | 56.10 |
| Bounds and Complementary constraints | 4469.5 | 8972.5 | 56.42 |
| Bounds, Substitutive and Complementary constraints | 4482.5 | 8910 | 56.52 |

As we add more and more constraints, the uncertainty in the input data reduces. It is expected that the uncertainty in the output should also reduce simultaneously. Indeed this is true. When the substitutive and complementary effects are not considered, then the total minimum investment required is Rs. 3349.5. In this particular example, the substitutive effects do not affect the cost very much, but as soon as we consider the complementary effects between different products, the lower bound on cost shoots up to Rs. 4469.5, while the upper bound goes down from Rs. 9187.5 to Rs. 8972.5. When we consider both the substitutive and complementary effects, the lower bound further increases while the upper bound reduces to Rs. 8910.

As we increase the information about the input data, reducing the number of possibilities, the possible range of the output data also reduces. The graph in Fig. 3 shows how the range of output varies with the information content for all the different scenarios.

We can also analyze the relationship between scenarios using the relational algebra of polytopes. This is outside the scope of this paper.

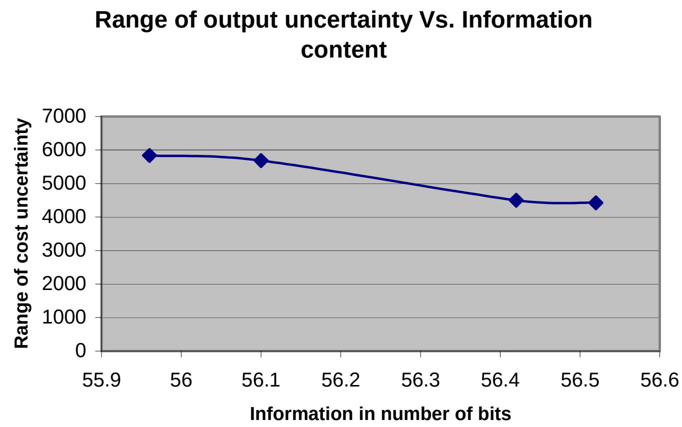


Fig. 3: Information content vs range of output uncertainty

## VI. Conclusion

Our approach offers a convenient and intuitive specification to handle uncertainty in supply chains. Most other formulations handling uncertainty make ad-hoc assumptions about demand variations, independence between different goods, etc. Our approach does away with these in a simple and elegant fashion, using intuitive information meaningful in economic terms, while retaining computational tractability. It has shown considerable promise by being able to solve problems with realistic costs with many breakpoints and simultaneous complicated constraints. It has successfully analyzed semi-industrial scale problems. We believe that our work extends the state-of-art in a theoretically and practically useful direction.

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