Adding Relations in Multi-levels to an Organization Structure of a Complete Binary Tree Maximizing Total Shortening Path Length

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Abstract—**This paper proposes a model of adding relations in multi-levels to an organization structure which is a complete binary tree such that the communication of information between every member in the organization becomes the most efficient. When edges between every pair of nodes with the same depth in $L\,(L = 1, 2, \cdots, H)$ levels are added to a complete binary tree of height H, an optimal set of depths $\{N_1, N_2, \cdots, N_L\}$ $(H \geq N_1 > N_2 > \cdots > N_L \geq 1)$ is obtained by maximizing the total shortening path length which is the sum of shortening lengths of shortest paths between every pair of all nodes in the complete binary tree. It is shown that $\{N_1, N_2, \cdots, N_L\}^* = \{H, H-1, \cdots, H-L+1\}$.**

Keywords: graph theory, organization structure, complete binary tree, shortest path length

1 Introduction

In a formal organization based on the principle of unity of command [2] there exist relations only between each superior and his subordinates. However, it is desirable to have formed additional relations other than that between each superior and his subordinates in case they need communication with other departments in the organization. In companies, the relations with other departments are built by meetings, group training, internal projects, and so on.

The formal organization structure can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively [3, 6]. Then the path between each node in the rooted tree is equivalent to the route of communication of information between each member in the organization. Moreover, adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his subordinates. The purpose of our study is to obtain an optimal set of additional relations to the formal organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the rooted tree minimizing the sum of lengths of shortest paths between every pair of all nodes.

We have obtained an optimal depth for each of the following three models of adding relations in one level to the organization structure which is a complete K-ary tree of height H: (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth and (iii) a model of adding edges between every pair of siblings with the same depth [4]. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have $K\,(K = 2, 3, \cdots)$ children [1]. A complete K-ary tree of $K = 2$ is a complete binary tree.

The above model (ii) corresponds to the formation of additional relations between every pair of all members in the same level such as section chief training. This model gives us an optimal level when we add relations in one level to the organization structure which is a complete binary tree of height H, but this model cannot be applied to adding relations in two or more levels. This paper expands the above model (ii) into the model of adding relations in multi-levels to the organization structure, which is that of adding edges between every pair of nodes with the same depth in $L\,(L = 1, 2, \cdots, H)$ levels to a complete binary tree of height $H\,(H = 1, 2, \cdots)$.

If $l_{i,j} (= l_{j,i})$ denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j $(i, j = 1, 2, \cdots, 2^{H+1} - 1)$ in the complete binary tree of height H, then $\sum_{i<j} l_{i,j}$ is the total path length. Furthermore, if $l'_{i,j}$ denotes the path length from v_i to v_j after adding edges in this model, $l_{i,j} - l'_{i,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called the *total shortening path length.*

In Section 2 we show formulation of the total shortening path length and the optimal depth which maximizes the total shortening path length in the model of adding edges between every pair of nodes with the same depth in one level to a complete binary tree of height H. In Sec-

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tion 3 we formulate the total shortening path length in the expanded model of adding edges between every pair of nodes with the same depth in L levels to a complete binary tree of height H. In Section 4 we obtain an optimal set of depths $\{N_1, N_2, \cdots, N_L\}$ $(H \geq N_1 > N_2 > \cdots > N_L \geq 1)$ by maximizing the total shortening path length in the expanded model.

2 Single Level Adding Model

This section shows formulation of the total shortening path length and the optimal depth which maximizes the total shortening path length in the model of adding edges between every pair of nodes with the same depth $N\,(N = 1, 2, \cdots, H\,)$ to a complete binary tree of height $H\,(H = 1, 2, \cdots\,)$ [4].

The sum of shortening path lengths between every pair of nodes whose depths are equal to or greater than N is given by

$$\alpha_H(N) = \{W(H-N)\}^2 2^{N-1} \sum_{i=1}^{N} (2i-1)2^{i-1}, \quad (1)$$

where $W(h)$ denotes the number of nodes of a complete binary tree of height $h\,(h = 0, 1, 2, \cdots\,)$. The sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or greater than N is given by

$$\beta_H(N) = W(H-N)2^N \sum_{i=1}^{N-1} (2i-1)(N-i)2^{i-1}, \quad (2)$$

and the sum of shortening path lengths between every pair of nodes whose depths are less than N is given by

$$\gamma(N) = 2^{N-1} \sum_{i=1}^{N-2} \sum_{j=1}^{i} (2j-1)(i-j+1)2^{j-1}, \quad (3)$$

where we define $\sum_{i=1}^{0} \cdot = 0$, $\sum_{i=1}^{-1} \cdot = 0$.

From these equations, the total shortening path length $\sigma_H(N)$ is given by

$$\begin{aligned} \sigma_H(N) &= \alpha_H(N) + \beta_H(N) + \gamma(N) \\ &= \{W(H-N)\}^2 2^{N-1} \sum_{i=1}^{N} (2i-1)2^{i-1} \\ &\quad + W(H-N)2^N \sum_{i=1}^{N-1} (2i-1)(N-i)2^{i-1} \\ &\quad + 2^{N-1} \sum_{i=1}^{N-2} \sum_{j=1}^{i} (2j-1)(i-j+1)2^{j-1}. \quad (4) \end{aligned}$$

Since

$$\sigma_H(N+1) - \sigma_H(N) > 0 \quad (5)$$

for $N = 1, 2, \cdots, H-1$, the optimal depth N^* can be obtained and is given in Theorem 1.

Theorem 1. *$N^* = H$ maximizes $\sigma_H(N)$.*

3 Formulation of Total Shortening Path Length of Multi-levels Adding Model

This section formulates the total shortening path length when edges between every pair of nodes with the same depth in $L\,(L = 1, 2, \cdots, H\,)$ levels are added to a complete binary tree of height $H\,(H = 1, 2, \cdots)$. Let $S_H(N_1, N_2, \cdots, N_L)$ denote the total shortening path length when a set of depths is $\{N_1, N_2, \cdots, N_L\}$ $(H \geq N_1 > N_2 > \cdots > N_L \geq 1)$.

Let

$$\begin{aligned} &T(N_{l-1}, N_l) \\ &\quad \equiv \; S_H(N_1, N_2, \cdots, N_{l-1}, N_l) - S_H(N_1, N_2, \cdots, N_{l-1}) \end{aligned} \quad (6)$$

for $l = 2, 3, \cdots, L$ because $S_H(N_1, N_2, \cdots, N_{l-1}, N_l) - S_H(N_1, N_2, \cdots, N_{l-1})$ is independent of $N_1, N_2, \cdots, N_{l-2}$. Then we have

$$S_H(N_1, N_2, \cdots, N_L) = \sigma_H(N_1) + \sum_{l=2}^{L} T(N_{l-1}, N_l), \quad (7)$$

where $\sigma_H(N)$ is as Equation (4) and we define $\sum_{i=2}^{1} \cdot = 0$.

Let $\hat{s}(N_{l-1}, N_l)$ denote the sum of shortening path lengths between every pair of nodes whose depths are less than N_{l-1} when a set of depths is $\{N_1, N_2, \cdots, N_{l-1}, N_l\}$ and let $\bar{s}(N_{l-1})$ denote the sum of shortening path lengths between every pair of nodes whose depths are less than N_{l-1} when a set of depths is $\{N_1, N_2, \cdots, N_{l-1}\}$, then we have

$$T(N_{l-1}, N_l) = \hat{s}(N_{l-1}, N_l) - \bar{s}(N_{l-1}). \quad (8)$$

$\bar{s}(N_{l-1})$ is given by

$$\bar{s}(N_{l-1}) = \gamma(N_{l-1}), \quad (9)$$

where $\gamma(N)$ is as Equation (3). The following formulates $\hat{s}(N_{l-1}, N_l)$.

Let V_1 denote the set of nodes whose depths are less than N_l. Let V_2 denote the set of nodes whose depths are equal to or greater than N_l and are less than N_{l-1}. The sum of shortening path lengths between every pair of nodes in V_1 is given by

$$A(N_l) = \gamma(N_l) \quad (10)$$

from Equation (3). The sum of shortening path lengths between every pair of nodes in V_1 and nodes in V_2 is given by

$$B(N_{l-1}, N_l) = \beta_{N_{l-1}-1}(N_l), \quad (11)$$

where $\beta_H(N)$ is as Equation (2). The sum of shortening path lengths between every pair of nodes in V_2 is formulated as follows.

The sum of shortening path lengths between every pair of nodes in each subtree whose root is a node with depth N_l is given by

$$C(N_{l-1}, N_l) = \gamma(N_{l-1} - N_l)2^{N_l} \tag{12}$$

from Equation (3). The sum of shortening path lengths between every pair of nodes in two different subtrees whose roots are nodes with depth N_l is given by summing up $D(N_{l-1}, N_l)$ and $E(N_{l-1}, N_l)$. $D(N_{l-1}, N_l)$ which is the sum of shortening path lengths by adding edges only between nodes with depth N_l is given by

$$D(N_{l-1}, N_l) = \alpha_{N_{l-1}-1}(N_l), \tag{13}$$

where $\alpha_H(N)$ is as Equation (1). $E(N_{l-1}, N_l)$ which is the sum of additional shortening path lengths by adding edges between nodes with depth N_{l-1} after adding edges between nodes with depth N_l is expressed by

$$\begin{aligned} &E(N_{l-1}, N_l) \\ &= (2^{N_l} - 1) \sum_{i=1}^{N_{l-1}-N_l-2} 2^{N_{l-1}-i} \\ &\quad \times \sum_{j=1}^{N_{l-1}-N_l-i-1} 2^{N_{l-1}-N_l-j}(N_{l-1} - N_l - i - j), \end{aligned} \tag{14}$$

where we define $\sum_{i=1}^{0} \cdot = 0$, $\sum_{i=1}^{-1} \cdot = 0$.

From these equations, $\hat{s}(N_{l-1}, N_l)$ is given by

$$\begin{aligned} &\hat{s}(N_{l-1}, N_l) \\ &= A(N_l) + B(N_{l-1}, N_l) + C(N_{l-1}, N_l) \\ &\quad + D(N_{l-1}, N_l) + E(N_{l-1}, N_l) \\ &= \gamma(N_l) + \beta_{N_{l-1}-1}(N_l) + \gamma(N_{l-1} - N_l)2^{N_l} \\ &\quad + \alpha_{N_{l-1}-1}(N_l) \\ &\quad + (2^{N_l} - 1) \sum_{i=1}^{N_{l-1}-N_l-2} 2^{N_{l-1}-i} \\ &\quad \times \sum_{j=1}^{N_{l-1}-N_l-i-1} 2^{N_{l-1}-N_l-j}(N_{l-1} - N_l - i - j). \end{aligned} \tag{15}$$

From Equations (8), (9) and (15), $T(N_{l-1}, N_l)$ is given by

$$\begin{aligned} &T(N_{l-1}, N_l) \\ &= \hat{s}(N_{l-1}, N_l) - \bar{s}(N_{l-1}) \\ &= \gamma(N_l) + \beta_{N_{l-1}-1}(N_l) + \gamma(N_{l-1} - N_l)2^{N_l} \\ &\quad + \alpha_{N_{l-1}-1}(N_l) - \gamma(N_{l-1}) \\ &\quad + (2^{N_l} - 1) \sum_{i=1}^{N_{l-1}-N_l-2} 2^{N_{l-1}-i} \\ &\quad \times \sum_{j=1}^{N_{l-1}-N_l-i-1} 2^{N_{l-1}-N_l-j}(N_{l-1} - N_l - i - j) \\ &= 2^{N_l-1} \sum_{i=1}^{N_l-2} \sum_{j=1}^{i} (2j-1)(i-j+1)2^{j-1} \\ &\quad + W(N_{l-1} - N_l - 1)2^{N_l} \sum_{i=1}^{N_l-1} (2i-1)(N_l - i)2^{i-1} \\ &\quad + 2^{N_{l-1}-1} \sum_{i=1}^{N_{l-1}-N_l-2} \sum_{j=1}^{i} (2j-1)(i-j+1)2^{j-1} \\ &\quad + \{W(N_{l-1} - N_l - 1)\}^2 2^{N_l-1} \sum_{i=1}^{N_l} (2i-1)2^{i-1} \\ &\quad - 2^{N_{l-1}-1} \sum_{i=1}^{N_{l-1}-2} \sum_{j=1}^{i} (2j-1)(i-j+1)2^{j-1} \\ &\quad + (2^{N_l} - 1) \sum_{i=1}^{N_{l-1}-N_l-2} 2^{N_{l-1}-i} \\ &\quad \times \sum_{j=1}^{N_{l-1}-N_l-i-1} 2^{N_{l-1}-N_l-j}(N_{l-1} - N_l - i - j). \end{aligned} \tag{16}$$

Since the number of nodes of a complete binary tree of height h is

$$W(h) = 2^{h+1} - 1\,, \tag{17}$$

$T(N_{l-1}, N_l)$ of Equation (16) becomes

$$\begin{aligned} &T(N_{l-1}, N_l) \\ &= (N_{l-1} - N_l)2^{N_{l-1}+N_l+1} + N_l(3N_l - 1)2^{N_l-2} \\ &\quad + (3{N_l}^2 - 6N_{l-1}N_l + 9N_l - 8N_{l-1})2^{N_{l-1}-2}. \end{aligned} \tag{18}$$

4 Optimal Set of Depths of Multi-levels Adding Model

This section obtains $\{N_{l-1}, N_l\}^*$ which maximizes $T(N_{l-1}, N_l)$ in Equation (18) and then obtains an optimal set of depths $\{N_1, N_2, \cdots, N_L\}^*$ which maximizes $S_H(N_1, N_2, \cdots, N_L)$ in Equation (7).

We have the following lemma about the depth N_l^* which maximizes $T(N_{l-1}, N_l)$ in Equation (18) for each N_{l-1}.

Lemma 2. *$N_l^* = N_{l-1} - 1$ maximizes $T(N_{l-1}, N_l)$ for each N_{l-1}.*

Proof. Since

$$T(N_{l-1}, N_l + 1) - T(N_{l-1}, N_l)$$

$$= (N_{l-1} - N_l - 2)\left(2^{N_l+2} - 3\right)2^{N_{l-1}-1} + \left(3{N_l}^2 + 11N_l + 4\right)2^{N_l-2} > 0 \quad (19)$$

for $N_l = 1, 2, \cdots, N_{l-1} - 2$, we have the following:
(i) If $N_{l-1} = 2$, then $N_l^* = 1 = N_{l-1} - 1$ trivially.
(ii) If $N_{l-1} \geq 3$, then $N_l^* = N_{l-1} - 1$ since $T(N_{l-1}, N_l + 1) - T(N_{l-1}, N_l) > 0$ for $N_l = 1, 2, \cdots, N_{l-1} - 2$ from Equation (19). □

Let $R(N_{l-1})$ denote $T(N_{l-1}, N_l)$ when $N_l = N_{l-1} - 1$, so that we have

$$\begin{aligned} &R(N_{l-1}) \\ &\quad\equiv T(N_{l-1}, N_{l-1} - 1) \\ &\quad= 2^{2N_{l-1}} - \left(3{N_{l-1}}^2 + 5N_{l-1} + 8\right)2^{N_{l-1}-3} \end{aligned} \quad (20)$$

for $N_{l-1} = 2, 3, \cdots, H - l + 2$. The depth N_{l-1}^* which maximizes $R(N_{l-1})$ in Equation (20) can be obtained and is given in Lemma 3.

Lemma 3. $N_{l-1}^* = H - l + 2$ *maximizes* $R(N_{l-1})$, *where* $\{N_1, N_2, \cdots, N_{l-2}\} = \{H, H-1, \cdots, H-l+3\}$.

Proof. Since

$$\begin{aligned} &R(N_{l-1} + 1) - R(N_{l-1}) \\ &\quad= \left(3 \cdot 2^{N_{l-1}+3} - 3{N_{l-1}}^2 - 17N_{l-1} - 24\right)2^{N_{l-1}-3} > 0 \end{aligned} \quad (21)$$

for $N_{l-1} = 2, 3, \cdots, H - l + 1$, we have the following:
(i) If $l = H$, then $N_{l-1}^* = 2 = H - l + 2$ trivially.
(ii) If $l \leq H - 1$, then $N_{l-1}^* = H - l + 2$ since $R(N_{l-1} + 1) - R(N_{l-1}) > 0$ for $N_{l-1} = 2, 3, \cdots, H - l + 1$ from Equation (21). □

From Lemmas 2 and 3, the pair of depths $\{N_{l-1}, N_l\}^*$ which maximizes $T(N_{l-1}, N_l)$ in Equation (18) can be obtained and is given in Theorem 4.

Theorem 4. $\{N_{l-1}, N_l\}^* = \{H - l + 2, H - l + 1\}$ *maximizes* $T(N_{l-1}, N_l)$.

From Theorems 1 and 4, the set of depths $\{N_1, N_2, \cdots, N_L\}$ which maximizes $S_H(N_1, N_2, \cdots, N_L)$ in Equation (7) can be obtained and is given in Theorem 5.

Theorem 5. $\{N_1, N_2, \cdots, N_L\}^* = \{H, H-1, \cdots, H - L + 1\}$ *maximizes* $S_H(N_1, N_2, \cdots, N_L)$.

Proof.
(i) If $L = 1$, then $\{N_1\}^* = \{H\}$ since $N_1^* = H$ maximizes $S_H(N_1) = \sigma_H(N_1)$ from Theorem 1.
(ii) Suppose that $\{N_1, N_2, \cdots, N_m\}^* = \{H, H-1, \cdots, H - m + 1\}$ maximizes $S_H(N_1, N_2, \cdots, N_m)$. Then $\{N_1, N_2, \cdots, N_m, N_{m+1}\}^* = \{H, H-1, \cdots, H-m+1, H-m\}$ maximizes $S_H(N_1, N_2, \cdots, N_m, N_{m+1}) = S_H(N_1, N_2, \cdots, N_m) + T(N_m, N_{m+1})$ from the above assumption and that $\{N_m, N_{m+1}\}^* = \{H - m + 1, H - m\}$ maximizes $T(N_m, N_{m+1})$ in Theorem 4.

From (i) and (ii), $\{N_1, N_2, \cdots, N_L\}^* = \{H, H-1, \cdots, H - L + 1\}$ for $L = 1, 2, \cdots, H$. □

5 Conclusions

This study considered obtaining optimal depths of adding edges to a complete binary tree maximizing the total shortening path length which is the sum of shortening lengths of shortest paths between every pair of all nodes in the complete binary tree. This means to obtain optimal levels of adding relations to the basic type of a formal organization such that the communication of information between every member in the organization becomes the most efficient.

For the model of adding edges between every pair of nodes with the same depth N in one level to a complete binary tree of height H, we had already obtained an optimal depth $N^* = H$ in our paper [4]. This result shows that the most efficient way of adding relations between every pair of all members at one level such as section chief training is to add relations at the lowest level, irrespective of the number of levels in the organization structure.

This paper expanded the above model into the model of adding relations in multi-levels to the organization structure, which is that of adding edges between every pair of nodes with the same depth in L levels to a complete binary tree of height H. We obtained an optimal set of depths $\{N_1, N_2, \cdots, N_L\}^* = \{H, H-1, \cdots, H-L+1\}$ which maximizes the total shortening path length. This result means that the most efficient manner of adding relations between every pair of all members at each level of L levels is to add relations at the L lowest levels, irrespective of the number of levels in the organization structure.

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