Real Time Study of a $n-out-of-n$ System: n Identical Repairable Elements with Constant Failure and Repair Rates

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Abstract- Reliability models based on Markov chain (Except in queuing systems) have extensive applications in electrical and electronically devices. In this paper we consider a system with n parallel and identical repairable elements with constant failure and repair rates (failure and repair rates are exponentially distributed. The failure rates increase when some elements are failed. The system works until at least 1 elements work. The system of equations are established and the exact equations are sought for the parameters like MTTF and the probability that system working at the time t. A numerical example has been solved to demonstrate the procedure which clarifies the theoretical development. It seems that this model can tackle more realistic situations.

Keywords:
Markov chain, $k-out-of-n$ models, Exponential distribution, Repairable elements.

Nomenclature

The notations used in this paper are as follows:

n : Number of elements,

λ_i : Failure rate of the elements in i st category,

μ : Repair rate of each elements in each category,

$P_i(t)$: Probability that the system is in state i at the time t ,

$R(t)$: Probability that system works at time t ,

$MTTF$: Mean time to failure of the system,

1. *Introduction*

$k-out-of-n$ models, are one of the most useful models to calculate the reliability of electrical and electronically devices and systems. In the literature there are many studies in this area. We try to categorically classify them. At the first glance, they may be classified into two main groups namely $k-out-of-n:F$ and $k-out-of-n:G$ systems. If the failure of a system with n components is abandoned to the failure of at least k components $(k \le n)$, then the system is called $k-out-of-n:F$. On the other hand, if the working of this system is abandoned to the working of at least k components $(k \le n)$, then the system is called $k-out-of-n:G$. Both these systems can be considered in steady state and real time situations. The elements in both situations may be repairable or non repairable. The failure rates of the elements may be constant, increasing or decreasing whereas the repair rate is constant. Each component can be expressed in binary or multiple states

Binary models: Boland and Papastavridis [2], study a situation where there are k distinct components with failure probabilities q_i for $i=1,2,...,k$ where the failure probability of the j th component $(j=mk+i(1 \le i \le k))$ is q_i . They obtained exact expressions for the failure probability of a r consecutive $k-out-of-n:F$ system.

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Gera [3], study the reliability of a consecutive $k-out-of-n:G$ system is and the problem has been solved via a matrix formulation using state space method. Lam and Tony N.G [5] introduce a general model for consecutive $k-out-of-n:F$ repairable system with exponential distribution and $(k-1)$-step Markov dependence. Sarhan and Abouammoh [6], investigate the reliability of non repairable $k-out-of-n$ system with non identical elements subject to independent and common shock. Dutuit and Rauzy [7], study the performance of binary decision diagram for all $k-out-of-n$ system and proposed a new approximation scheme. Krishnamoorthy and Ushakumarti [8], obtain the system state distribution, system reliability and several other measures of performance for a $k-out-of-n:G$ system with repair under D-policy. Gupta [9], calculates reliability function and the failure rate of the $k-out-of-n$ system, with and without incorporating the environmental effect. Cui [10], presents a bound for n_k for which the system does not preserve IFR when $n > n_k$. Arulmozhi [11], propose an expression for reliability of $k-out-of-n:G$ system and developed an algorithm for computing reliability of $k-out-of-n$ system. Yam and Zuo [12], derive the state transition probabilities of the repairable circular consecutive $k-out-of-n:F$ system with one repairman. Milczek [13], presents a class of limit reliability functions of homogeneous series $k-out-of-n$ systems. Chen [14], develop a method for analyzing component reliability and system reliability of $k-out-of-n$ systems with independent and identically distributed components based on system lifetime data. [15], Koucky, deals with reliability of general $k-out-of-n$ systems whose component failures need not be independent and identically distributed and author gives the approximations for the system reliability. Arulmozhi [16], gives simple and efficient computational method for determining the system reliability of $k-out-of-n$ systems having unequal and equal reliabilities for components. Smith-Destombes [17], presents both an exact and an approximate approach to analyze a $k-out-of-n$ system with identical, repairable components. Flynn and Chung [18], present a heuristic algorithm for determining replacement policies in a discrete-time, infinite horizon, dynamic programming model of a binary coherent system with n statistically independent components, and then specialized the algorithm to consecutive $k-out-of-n$ systems. Hsieh and Chen [19], present a simple formula for the reliability lower bound of the two-dimensional consecutive $k-out-of-n:F$ system. Jalali and Hawkes [20], prove two theorems for the optimal consecutive $k-out-of-n:G$ line for $n \le 2k$. Da Costa Bueno [21], define minimal standby redundancy and use the reverse rule of order $2(RR_2)$ property between compensator processes to investigate the problem of where to allocate a spare in a $k-out-of-n:F$ system of dependent components through minimal standby redundancy. Li and Zuo [23], derived formulas for various reliability indices of the $k-out-of-n$ system with independent exponential components including mean time between failures and some other parameters. Guan and Wu [24], study a repairable consecutive $k-out-of-n:F$ system with fuzzy states.

Multi state models: Moustafa [1], using the markov method, develope a closed form availability solution for two $k-out-of-n$ systems with M failure modes. Huang and Zuo [4] investigate two types of multi-state $k-out-of-n:G$ systems (i.e. increasing and decreasing systems). The authors developed an analytical model on the properties of the binary-state $k-out-of-n$ system. Jenab and Dhillon [22] present a flow-graph-based approach to analyze a multi-state $k-out-of-n:G/F/$load-sharing systems. The multi-state $k-out-of-n:G/F/$load-sharing systems comprise ‘n’ identical units, that are under state monitoring and recovery function.

In this paper we work on a system with n parallel and identical repairable binary elements with constant failure and repair rates for real time conditions and the number of repairman is unlimited. The system works until all elements fail (at least 1 element works). The paper is divided into four parts. The second part explores the models. Numerical examples are presented in the third part and the final section deals with the conclusion.

2. Modeling

Assume a system with n parallel and identical repairable elements. The system works until all elements fail. Therefore each element has two states and consequently the system will have 2^n states. Let $A_1 A_2 ... A_k$ be the state where the elements A_1, A_2, … A_k are working and other $(n-k)$ elements are failed. Also $A_1 A_2 ... A_k \eta_j$ is the state that the element η_j works in addition to other k elements. The state O indicates that all the elements are failed. There is C_i^n state that $i, i = 1, 2, ..., n,$ elements are working and the probability of all states in category i are equal. The state structure of the system is shown in figure 1.

In this figure, the category i, $i = 0,1,2,...,n$, indicates that the states in this category are with i elements working and $(n-i)$ elements are failed. Each state is closely related to the states in the antecedent and precedent category. i.e., if an element is failed in any state, then the state is transferred to the next category and if an element repairs in any state, then the state is transferred to the past category. In other words, if the system is in any states in category k, with the failure of one element, the state will be in category $(k-1)$ and with the repair of one element, the state will be in category $(k+1)$.

When some elements are failed, other elements work with more loads and therefore the failure rate is increased for the remaining elements. Then we have $\lambda_n < \lambda_{(n-1)} < ... < \lambda_1$. The system works if at least 1 elements work. Therefore we have:

$$R(t) = \sum_{\substack{i=1 \\ 1 \le A_1 < A_2 < ... < A_i \le n}}^{n} P_{A_1 A_2 ... A_i}(t) \qquad (01)$$

We know that:

$$\sum_{\substack{i=1 \\ 1 \le A_1 < A_2 < ... < A_i \le n}}^{n} P_{A_1 A_2 ... A_i}(t) + P_0(t) = 1 \qquad (02)$$

The first part of equation (02), is related to states with at least 1 elements are working and the second part deals with the state with 0 element is working. In order to find $R(t)$ from equation (02), we must calculate $P_i(t)$ for each states. From the state $A_1 A_2 ... A_n$ through O in figure 1 we have:

$$\begin{aligned} P_{A_1 A_2 ... A_k}(t+\Delta t) &= P_{A_1 A_2 ... A_k}(t) - \\ &\sum_{i=1}^{k} \lambda_k \times \Delta t \times P_{A_1 A_2 ... A_k}(t) - \\ &\sum_{\substack{j=1 \\ \eta_j \ne A_i \quad i=1,2,...,k}}^{n} \mu \times \Delta t \times P_{A_1 A_2 ... A_k}(t) \\ &+ \sum_{\substack{j=1 \\ \eta_j \ne A_i \quad i=1,2,...,k}}^{n} \lambda_{(k+1)} \times \Delta t \times P_{A_1 A_2 ... A_k \eta_j}(t) + \\ &\sum_{i=1}^{k} \mu \times \Delta t \times P_{\left(\frac{A_1 A_2 ... A_k}{A_i}\right)}(t) \end{aligned} \qquad (03)$$

$P_{A_1 A_2 ... A_k}(t)$ is the probability that the system be in the state $A_1 A_2 ... A_k$ at time t. Also $\sum_{\substack{j=1 \\ \eta_j \ne A_i \quad i=1,2,...,k}}^{n} \lambda_{(k+1)} \times \Delta t \times P_{A_1 A_2 ... A_k \eta_j}(t)$ is the transfer rate from states of category $(k+1)$ to this state and $\sum_{i=1}^{k} \lambda_k \times \Delta t \times P_{A_1 A_2 ... A_k}(t)$ is the transfer rate from this state to the states of category $(k-1)$, $\sum_{i=1}^{k} \mu \times \Delta t \times P_{\left(\frac{A_1 A_2 ... A_k}{A_i}\right)}(t)$ is the transfer rate from states of category $(k-1)$ to this state and $\sum_{\substack{j=1 \\ \eta_j \ne A_i \quad i=1,2,...,k}}^{n} \mu \times \Delta t \times P_{A_1 A_2 ... A_k}(t)$ is the transfer rate from this state to the states of category $(k+1)$ at time Δt.

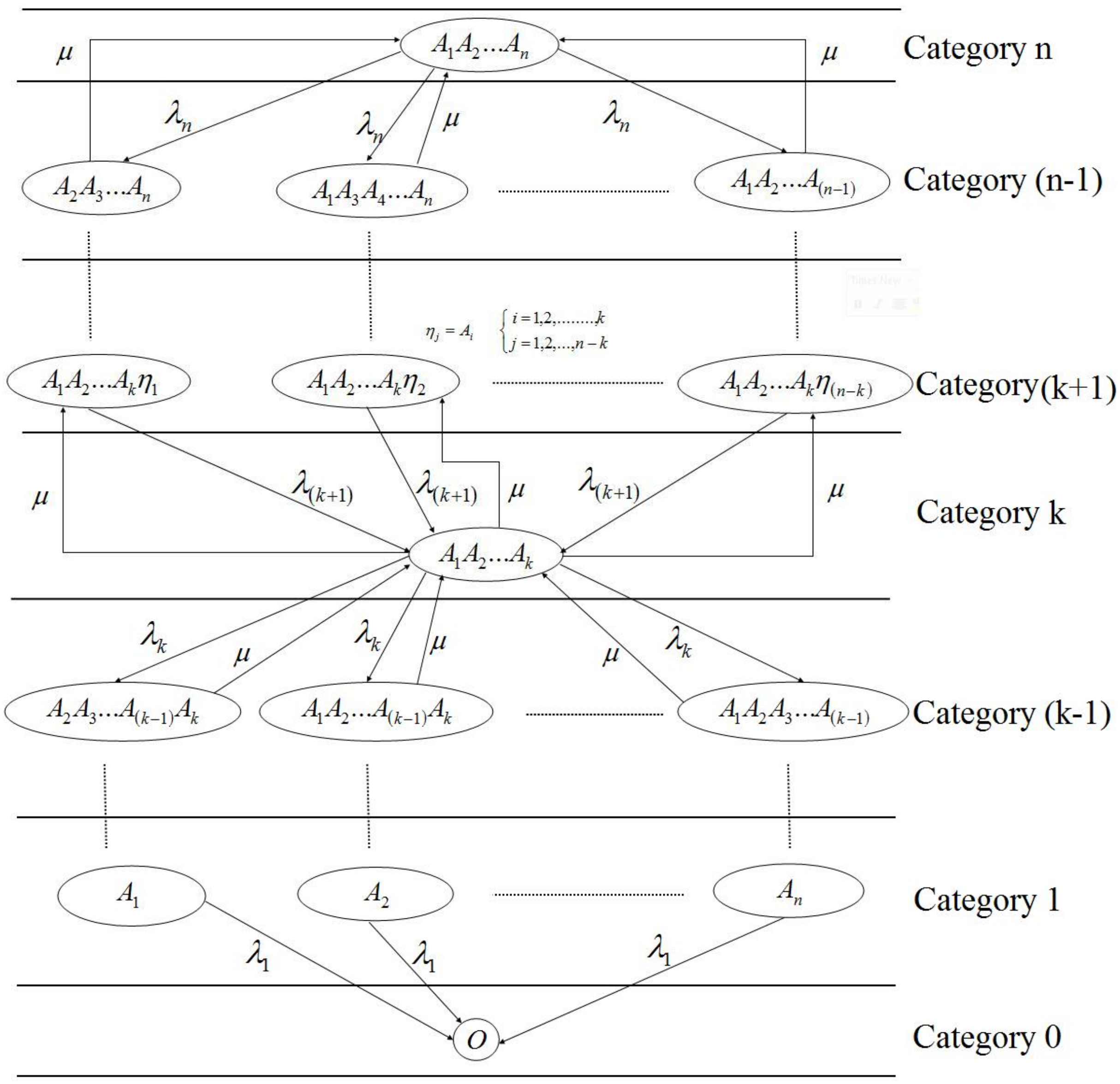

Figure 1: The structure of the model

The differential equation of equation (03) is as follow:

$$\begin{cases} P_1'(t)+\{\lambda_1+(n-1)\mu\}P_1(t)= \\ (n-1)\lambda_2 P_2(t) \qquad k=1 \\ P_k'(t)+\{k\lambda_k+(n-k)\mu\}P_k(t)= \\ (n-k)\lambda_{(k+1)}P_{(k+1)}(t)+ \\ k\mu P_{(k-1)}(t) \qquad k=2,3,\ldots,n \end{cases} \tag{04}$$

And:

$$R(t)=\sum_{i=1}^{n} C_i^n P_i(t) \tag{05}$$

The $MTTF$ of the system is also calculated as follows:

$$MTTF=\int_{t=0}^{+\infty} R(t)dt \tag{06}$$

When a component fails in each category, the other component must work harder and the failure rate of these components increase. We can calculate the failure rate of each category as follows:

$$\lambda_k=\frac{n}{n-\gamma(n-k)}\lambda_n \tag{07}$$

where $0 \le \gamma \le 1$. If $\gamma=0$, then the failure rates are equal and constant and if $\gamma=1$ then $\lambda_k=\frac{n}{k}\lambda_n$.

3. *Numerical example*

In this Example, we consider a system with 2 identical repairable elements. Let $\gamma=0.5$ and based on an independent sample, the failure rate of component in category 2 is $\lambda_2 = 0.001$ and then $\lambda_1 = \frac{2}{2-0.5(2-1)}\lambda_2 = \frac{4}{3}\lambda_2$ and also repair rate is $\mu = 0.0005$. The figure 2, Show the states of this system.

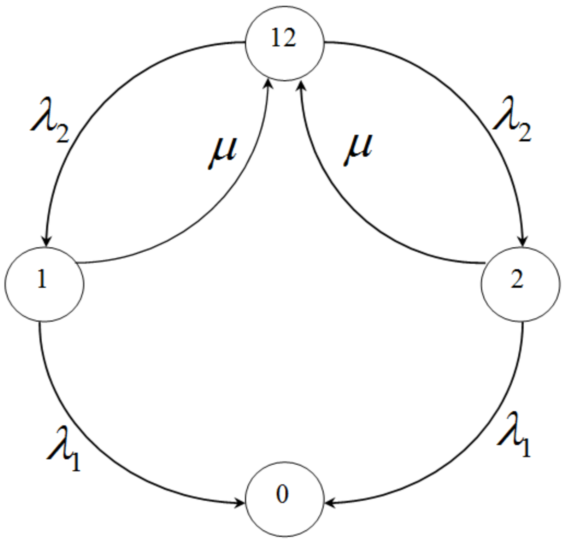

Figure 2: the states of the system in example 3

The system of differential equation is as follow:

$$\begin{cases} P_1'(t) + (\lambda_1 + \mu)P_1(t) = \lambda_2 P_2(t) \\ P_2'(t) + 2\lambda_2 P_2(t) = 2\mu P_1(t) \end{cases} \tag{08}$$

Using Laplace transformation to solving the system of differential equation and using *invlaplace* command in Maple 10, we have:

$$\begin{cases} P_1(t) = \\ 0.00689655 e^{-0.001916t} \begin{Bmatrix} 145 Cosh(0.001003466t) \\ +108.374351 Sinh(0.001003466t) \end{Bmatrix} \\ P_2(t) = 0.996546 e^{-0.001916t} Sinh(0.001003466t) \end{cases} \tag{09}$$

And:

$$R(t) = \sum_{i=1}^{2} C_i^2 P_i(t) = P_2(t) + 2P_1(t) \tag{10}$$

Also the MTTF calculated as follow:

$$MTTF = \int_{t=0}^{+\infty} R(t)dt = \int_{t=0}^{+\infty} \{P_2(t) + 2P_1(t)\} dt = 2.664062499 \times 10^6 \tag{11}$$

4. Conclusion

A real time $n-out-of-n$ system has been studied in which the failure rate of the elements is constant and the elements are repairable. That is by failure of one element, the load on the remaining elements, increases the chance of failure of the remaining elements. Necessary relations have been developed for the failure rates. The ambiguity of the data for failure rates is demonstrated by fuzzy triangular numbers. It seems that this model provides more reliable solutions.

This model can be applied to a wide variety of complex industrial problems like the engines of an airplane. As an extension to this work, the failure rates may be considered different for each element. In that case either a system of $k-out-of-n$ model should be taken into consideration or a certain policy should be developed when whole system fails. Hence, the cost for procurement, repair of the elements and the failure of the whole system may be taken into account in such a way that the optimum number of elements with minimum cost and maximum reliability will be determined. These variations could be studied for the systems with increasing failure rate of the elements too.

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