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The Impact of Stochastic Lead Time: the Mean or the Variance

Xin James He, Member, IAENG

***Abstract*—In this paper we try to shed light on whether and where it is σ^2 or μ that is more important in supply chain management. We first show that the best reorder point, z, may depend on the order quantity, Q, by examining the (z, Q) inventory system whether the demand and lead time are constant or stochastic. Then we illustrate that for a predetermined Q, reducing reorder point z may increase the total inventory cost. Finally we demonstrate by means of a truncated lead time (z, Q) model that it is lead time variability, not average lead time, which affects the inventory policy and the total relevant cost..**

***Index Terms*—Lead Time Variability, Stochastic Inventory Systems**

I. INTRODUCTION

This research investigates whether it is the lead time variance, σ^2, or the mean lead time, μ, that is more important in supply chain management. According to Chopra *et al.* [1], given some threshold service levels, "decreasing (mean) lead time is the right lever if they (the firms) want to cut inventories (safety stocks), not reducing lead time variability." That is, reducing the lead time variability is more important than reducing the mean lead time under certain conditions and the reverse is true otherwise. But this contradicts to the main stream research results in the area where the effect of the mean lead time, μ, on total inventory cost is usually insignificant, as compared with the variance, σ^2. For example, He *et al.* [2] find that "it is σ^2 and not μ that affects the inventory policy and the total relevant cost in stochastic lead-time inventory systems," and also that the cost of lead-time variability is approximately linear in σ, with nary a mention of μ. And in examining the deterministic backlogging model, Kim *et al.* [3] find that neither the optimal order quantity nor the optimal inventory cost is affected by the length of the lead time and go on to show (pp.909-910) in an upper-bound analysis that the increase in expected inventory cost due to stochastic lead time is linear in σ.

While recognizing that the reorder point, z, is not independent from the order quantity, Q, Chopra *et al.* assume that safety stock is a function of the service level, demand uncertainty, and lead time and lead time variability. These authors use a predetermined Q to analyze the reorder point s in order to reduce the safety stock for a given service level. But in the present paper, we first show that the two decision variables in a (z, Q) inventory system even with constant demand and instantaneous lead time are not independent. That is, the best reorder point, z, does depend on the order quantity, Q. Then we extend this model into a stochastic lead time (z, Q) inventory system and show that for a predetermined Q, reducing the reorder point z would in fact increase the total inventory cost. Finally we use a model developed by He *et al.* to demonstrate that it is lead time variability, not the average lead time, which affects the inventory policy and the total relevant cost.

The rest of the paper is organized as follows. Section 2 reviews the existing literature. Section 3 analyzes the models. Section 4 analyzes the impact of lead time variability. Section 5 summarizes the findings.

Xin James He is with the Information Systems & Operations Management Department at Dolan School of Business, Fairfield University, Fairfield, CT 06824 USA (phone: 203-254-4000 ext. 2835; e-mail: xhe@mail.fairfield.edu).

II. LITERATURE REVIEW

After examining the standard procedure of the (z, Q) inventory model in the literature, Eppen and Martin [4] believe that using the economic order quantity (EOQ) formula as the order quantity Q is an acceptable common practice to calculate the reorder point z when the demand and lead time are both stochastic. Based on a normal approximation to estimate the relationship between safety stock and lead time uncertainty, these authors report that reducing lead time variability decreases the reorder point and safety stock. Furthermore, they believe reducing lead time variability is more effective than reducing lead times because it decreases the safety stock by a larger amount. However they caution that using a normal approximation for the lead time distribution to compute the reorder point z for a given cycle service level, which measures the proportion of replenishment cycles in which a stockout does not occur, would lead to certain computational error. Thus they propose two correction procedures for inventory systems with both known and unknown parameters, respectively.

Based on Eppen and Martine research, Chopra *el al.* suggest that for a given order quantity Q and service level in an inventory system with variable demand and stochastic lead time, reducing the lead time variability may or may not reduce the inventory (safety stock). Of the two most commonly used service measures, cycle service level (CSL) and fill rate (P), CSL is in general easier to analyze for the inventory system since it does not have obvious relationship with either holding cost or shortage cost; whereas P, which measures the proportion of demand that is met from stock, is more difficult to analyze but it can be readily derived with given holding and shortage costs. According to Chopra and Meindl [5], "the two measures are very closely related because raising the cycle service level will also raise the fill rate for a firm." For a given order quantity, Q, in a (z, Q) inventory system, Chopra *el al.* [1] try to analyze the reorder point $z = \mu_L + k\sigma_L$, where μ_L is the mean, σ_L the standard deviation of demand during lead time, and $k\sigma_L$ the safety stock. They find that for CSL > 0.5 but below a threshold, reducing σ increases z, whereas reducing lead time decreases z. In addition, Chopra *et al.* suggest that the impact of lead time variability and mean lead time on inventory policy (safety stock) and the total inventory cost varies, depending on the cycle service levels. That is for certain cycle service levels, reducing lead time variability increases the safety stock and total inventory cost; whereas reducing mean lead time decreases the safety stock and total inventory cost.

Research in stochastic inventory systems often focuses on the impact of lead-time variability on inventory cost by examining its effect on a (z, Q) policy with planned shortages, otherwise known as the *EOQ* model with backlogging. When demand is constant and lead time is instantaneous, a (z, Q) system reduces to an (s, Q) where an order of size *Q* is placed each time the inventory position reaches the reorder point, z=*s* for s < 0. However, when both the lead time and the demand rate are stochastic, such an inventory model becomes extremely complicated. Nonetheless, while they believe that the effects of the two decision variables, z and Q, are not independent (i.e., the best value of Q depends on the z, and vice versa), Silver and Peterson [6] suggest that it is a practical assumption for a predetermined Q without knowledge of z. However, this approximation is never intended to derive an exact solution for the reorder point z.

The main reason for assuming a predetermined Q when the lead time is stochastic is the computational complexity in searching for optimal inventory policies for the (z, Q) model where a closed-from expression cannot be derived. Liberatore [7] developed such a model under the assumption that orders will not cross, i.e., orders will be received in the same order sequencing in which they are placed. In examining a (z, Q) model with constant demand rate and truncated stochastic lead times, He *et al.* [2] conclude that to achieve zero inventory (ZI) in a stochastic lead-time setting, both the setup cost and lead-time variability would have to be eliminated. Furthermore, it is the lead-time variance and not the mean lead time that affects the total relevant cost in a stochastic lead-time model.

III. MODEL ANALYSIS

We investigate the impact of lead-time variability on inventory cost by examining its effect on an (s, Q) policy with planned shortages, or the *EOQ* model with backlogging, as our base model. Let

b = shortage cost per unit per unit time,
D = constant demand per unit time,
h = holding cost per unit per unit time,
K = ordering cost per order,
P = fill rate, which is equivalent to b/(b+h).

The total relevant cost is [6]

$$TRC(s, Q) = \frac{KD}{Q} + \frac{(Q+s)^2 h}{2Q} + \frac{s^2 b}{2Q}. \quad (1)$$

Figure 1 describes the (*s, Q*) inventory system with instantaneous lead time and backlogging, with *s* < 0 the planned shortage. Solving Eq.(1) for the optimal decision variables *Q* and *s*, yields

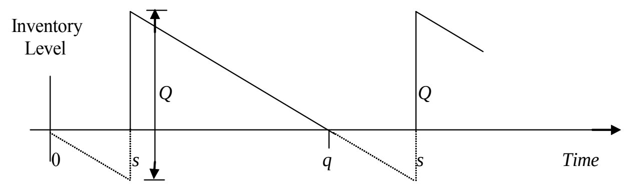
$$Q = \sqrt{\frac{2DK(h+b)}{hb}} = \sqrt{\frac{2Dk}{hP}} \quad (2)$$

$$s = -\sqrt{\frac{2DKh}{b(h+b)}} = -Q\frac{h}{h+b} = -Q(1-P) \quad (3)$$

The minimum cost would be

$$TRC(s,Q) = \sqrt{\frac{2DKhb}{h+b}} = Q\frac{hb}{h+b} = QhP \quad (4)$$

Figure 1: An (*s, Q*) System with Instantaneous Lead Time and Backlogging



LEGEND:

q = *Q/D*, the cycle time to consume Q units of inventory
Q = Order Quantity
s = Reorder Point

Note that Eq.(4) convergences to the traditional *EOQ* model without shortage when $b \to \infty$ or $P \to 1$, with $s \to 0$, $Q \to EOQ = \sqrt{2KD/h}$, and $TRC \to \sqrt{2KDh}$.

This model can be extended to include deterministic lead time, L, by adjusting *s* upward with respect to the demand during the lead time, or

$$z = LD + s. \quad (5)$$

Now (*z, Q*) is the inventory policy for the inventory system with deterministic lead time, where z is the new reorder point, L*D* is the demand during lead time, and s is again the planned shortage. In other words, (1)-(4) can be used for deterministic lead time inventory systems, so long as we relabel *TRC(s, Q)* as *TRC(z, Q)* and use *z* = L*D*+*s* as the adjusted reorder point.

When lead times are stochastic, the (*z, Q*) inventory system is much more difficult to analyze. Liberatore [7] developed a stochastic lead time model with constant demand if demands are assumed noninterchangeable:

$$E(r,q) = \frac{K}{q} + hD\int_0^r[(r-x)+\frac{1}{2}q]f(x)dx + \frac{D}{2q}\int_r^{r+q}[b(x-r)^2 + h(q+r-x)^2]f(x)dx + bD\int_{r+q}^{\infty}(x-r-\frac{1}{2}q)f(x)dx$$

(6)

where

E(r, q) = expected inventory cost per unit time,
f(x) = probability density function (*pdf*) of the lead time,
x = independently and identically distributed (*iid*) lead time variable,
q = *Q/D*, cycle time to consume Q units of inventory,

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r = z/D

In this paper we apply the model in Eq. (6) to evaluate the impact of a predetermined Q' (say EOQ) and the corresponding reorder point z' on total inventory cost E(z', Q').

Table 1 shows that a predetermined Q' from the EOQ model in Eq.(2) would significantly underestimate the optimal order quantity, Q, which leads to higher total inventory cost.

Table 1: Numerical Example of the (z, Q) Model
(*D* = 5000 *h* = \$1 *b* = \$10 *K* = \$100 P=0.91)

λ	z	Q	E(z, Q)	z'	z %	Q'	Q%	E(z', Q')	E%
				1128	-30.0%	1049		\$3334	18.8%
				1289	-20.0	1049		3167	12.8
				1450	-10.0	1049		3049	8.6
				1600	-0.7	1049	-43.3%	2977	6.1
5	1,611	1,850	\$2,807	1611	0.0	1049		2973	5.9
				1772	10.0	1040		2931	4.4
				1933	20.0	1049		2920	3.9
				2094	30.0	1049		2940	4.7
				435	-30.0%	1049		\$1900	11.3%
				498	-20.0	1049		1849	8.3
				560	-10.0	1049		1812	6.2
10	622	1,507	1,707	622	0.0	1049		1786	4.6
				640	2.9	1049	-30.4%	1781	4.3
				684	10.0	1049		1771	3.7
				746	20.0	1049		1764	3.3
				809	30.0	1049		1766	3.5
				51	-30.0%	1049		\$1100	1.2%
				58	-20.0	1049		1098	1.0
				66	-10.0	1049		1096	0.8
				72	-1.4	1049	-11.0%	1095	0.7
30	73	1,178	1,087	73	0.0	1049		1095	0.7
				80	10.0	1049		1094	0.6
				88	20.0	1049		1094	0.6
				95	30.0	1049		1094	0.6

Legend:

λ = Parameter of the exponential lead time
z = *rD*, the reorder point of the Liberatore's model in Eq.(6)
Q = *qD*, the order quantity of the model in Eq.(6)
Q'=Predetermined order quantity Q from the EOQ model in Eq.(2)
z'=The new reorder point corresponding to the predetermined Q'=EOQ
z%= Percentage difference between z' and z
Q%=Percentage difference between Q' and Q
E(z, Q) = Total relevant cost of the model in Eq.(6)
E(z', Q') = Total relevant cost of the model in Eq.(6) with a predetermined Q' and z'
E%=Percentage difference between E(z', Q') and E(z, Q)

The first column in Table 1 shows the parameter of the exponential lead time, λ, where the mean lead time is μ=1/λ and the standard deviation of the lead time is σ =. For a given λ, columns 2-4 are the resulting the optimal inventory policy in terms of the reorder point, z, the order quantity, Q=1/λ, and the total relevant inventory cost, E(z, Q), via numerical search on Eq.(6). Column 5 is the reorder point z' that Chopra *et al.* try to analyze based upon a predetermined order quantity, Q'. Column 6 is the percentage difference between the optimal z and the z'. Column 7 is the predetermined order quantity, Q', which is based on the EOQ model in Eq. (2). Column 8 is the percentage difference between the optimal Q and the Q'. Column 9 is the total relevant cost, E(z', Q') and column 10 is the percentage difference between the optimal total relevant cost E(z, Q) and the E(z, Q').

For example, when λ=5, demand D=5,000 per unit time, setup cost K=\$100 per order, holding cost h=\$1 per unit, and shortage cost b=\$10 per unit, the fill rate can be derived as P=b/(b+h)=0.91 and Q'=EOQ=1,049. Line 5 of columns 2-4 in Table 1 shows that the optimal order quantity is *Q= 1,850*, the optimal reorder point is *z= 1,612*, and the optimal total relevant cost of the stochastic inventory system with exponential lead time with parameter λ=5 is *E(z, Q)=\$2,807*. This optimal inventory policy is compared with the (z', Q') inventory system, where *z'*= μD+kσD=1,600 and where k is a function of p(k)=Q/(σD)(1-P) [6, p. 272] for a predetermined Q'=1,049. The resulting total relevant inventory cost is *E(z', Q')=\$2,977* (line 4 of column 9), which is 6.1% (line 4 of the last column) higher than the optimal *E(z, Q)=\$2,807* (line 5 of column 4). This is due to the fact that the predetermined Q'=1,049 is 43.3% smaller (line 4 of column 7) than the optimal Q=1,850 (line 5 of column 3) although the reorder point z'=1,600 (line 4 of column 5) using Chopra *et al.* approach is only 0.7% smaller (line 4 of column 6) than the optimal reorder point z=1,611 (line 5 of column 2).

It is worth noting that reducing the z' in Table 1 will increase the corresponding total inventory cost, E(z', Q'); whereas increasing the z' will reduce the total inventory cost initially and then increase after the z' is about 20% more than the initial estimate. For example, when z' is reduced from 1,600 to 1,450, 1,289, and 1,128 (lines 3, 2, and 1 of column 5), the total inventory cost is increased from \$2,977 to \$3,049, \$3167, and 3,334 (lines 3, 2, and 1 of column 9), respectively. However, when z' is increased from 1,600 to 1,772, 1,933, and 2,094 (lines 6, 7, and 8 of column 5), the total inventory cost is first reduced from \$2,977 to \$2,931 and \$2,920 (lines 7 and 8 of column 9), but then increased again from \$2,920 to \$2,940 (line 8 of column 9).

As we can see from Table 1 when λ=10, the same *Q'* = 1049 is used, with a new z' and E(z', Q'), or *z'*= μD+kσD=5000/10+0.28x500=640 and *E(z', Q')=\$1,781*. Thus we conclude that the characteristics described above when λ=5 hold true for λ=10 and λ=30. That is, a predetermined Q' from the EOQ model underestimats the optimal order quantity, Q, when the lead time is stochastic. Consequently, the resulting total relevant inventory cost, E(z', Q') is higher than the corresponding optimal total cost, E(z, Q) even though the z' derived based on Q' is almost identical to the optimal z.

Figure 2 shows that when λ=5, E(z', Q') > E(z, Q) within a reasonable range of z' (z' ≤ 2,227 or no more than 38% above the optimal z). Moreover, when z' decreases, the gap between E(z', Q') and E(z, Q) increases. Similarly, Figure 3 shows that when λ=10, E(z', Q') > E(z, Q) within a reasonable range of z' (for z' ≤ 950 or no more than 53% above the optimal z). By the same token, Figure 4 shows when λ=30, E(z', Q') > E(z, Q) within a reasonable range of z' (z' ≤ 157 or no more than 115% above the optimal z).

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Figure 2. Total Inventory Cost with λ=5 and fill rate P=0.91
For Q=1850, the minimun cost is $2,807 when z=1611
For Q=1049, the minimun cost is $2,920 when z=1925

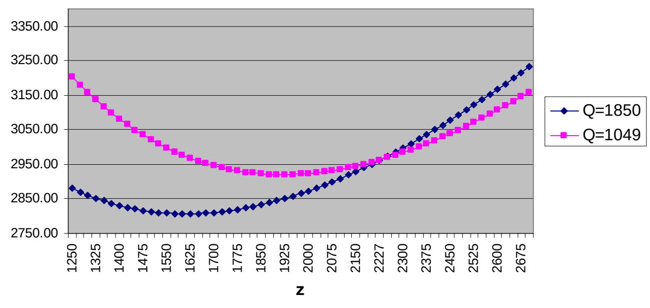


Figure 3. Total Inventory Cost with λ=10 and fill rate P=0.91
For Q=1507, the minimum cost is $1,707 when z=622
For Q=1049, the minumun cost is $1,764 when z=775

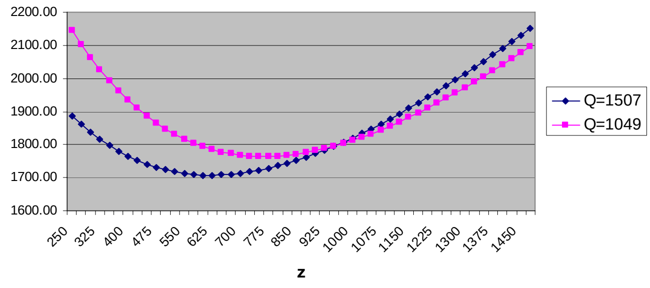
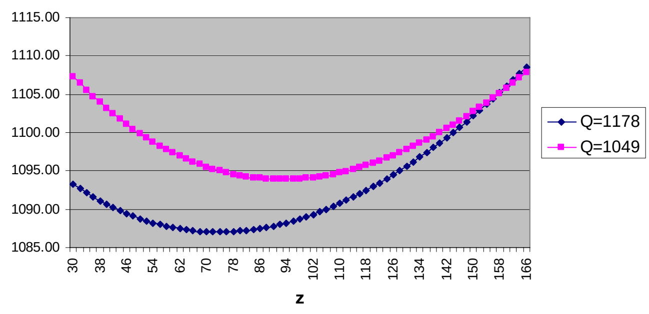


Figure 4. Total Inventory Cost with λ=30 and fill rate P=0.91
For Q=1178, the minimun cost is $1,087 when z=73
For Q=1049, the minimun cost is $1,094 when z=92



IV. IMPACT OF LEAD TIME VARIABILITY

Based on Liberatore's model, He *et al.* develop an inventory system with constant demand and truncated stochastic lead time, $r \le x \le r+q$, with mean μ and variance σ^2. By introducing *the lead-time variability factor, F,* they find that this truncated stochastic lead time inventory system has a closed form solution similar to Eqs. 2-4, with

$$F = \sqrt{1 + \frac{Dh\,\sigma^2}{2K(1-P)}}\,. \quad (7)$$

In other word, this model can be expressed in terms of *F* and the base model (*s,* Q) in Eqs.(2)-(4):

$$Q(F) = F\sqrt{\frac{2KD}{hP}} = QF, \quad (8)$$

$$s(F) = -Q(F)(1-P), \quad (9)$$

$$TRC(z,Q,F) = F\sqrt{2DKhP} = QFhP = TRC(s,Q)F. \quad (10)$$

Consequently, the reorder point is
z(F) = μ*D*+*s(F)*.

Now we analyze the impact of lead-time variability and its effect on the inventory policy, by means of what we call, for the lack of anything better, '*the lead-time variability factor, F*. As we shall see, *the lead-time variability factor, F,* could measure the intensity of lead-time variability.

From Eq.(7), *F* is a function of the variance, σ^2, and the system parameters *D, h, P,* and *K.* The mean, μ, does not show up in Eqs.(8)-(11), meaning that it is σ^2 and not μ that affects the inventory policy and the total relevant cost in a stochastic lead-time inventory system. For $\sigma^2 > 0$, $F > 1$, and *TRC(z, Q, F)* > *TRC(s, Q).* When σ^2 approaches zero, *F* converges to 1. As seen in Eq.(11), *F* is increasing in σ^2, the rate of increase being

$$\frac{\partial F}{\partial \sigma^2} = \frac{Dh}{4K(1-P)\sqrt{1 + \dfrac{Dh\sigma^2}{2K(1-P)}}} > 0 \quad (11)$$

Thus, the higher the lead-time variability, the higher the total inventory cost. Likewise, the higher the lead-time variability, the higher the *F*, and the larger the reorder quantity, *Q(F)* (see Eq. (8)). And as we can see from Eq.(9), the impact of *F* on *s(F)* is negatively proportional to *Q(F)*, with a discount rate of (1-P). Consequently, the impact of lead-time variability on the adjusted reorder point, *z(F)*, depends both on the mean, μ, and the variance, σ^2.

V. SUMMARY

This research is significant since it aims at managerial prescriptions as to how to reduce inventories in terms of safety stocks and ultimately to reduce inventory costs without hurting the level of services provided to customers. Safety stock is a function of the cycle service level, the demand uncertainty, the replenishment lead time, and the lead time variability. Managers have been under increasing pressure to decrease inventories as supply chains attempt to become leaner by reducing lead time variability and average lead time. They believe these are two areas they are able to make a difference since the other two variables, the service level and the demand uncertainty, cannot be reduced due to global competition and shortened product life cycles.

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