Dynamic Model for Portfolio Optimization

Cristinca Fulga *

Abstract —**In this paper we consider a multiperiod model where the investor chooses a portfolio at the beginning of each period facing uncertainty associated with the prices of the assets in portfolio at future dates. The model is considered over a finite horizon, with transaction costs, a risk averse utility function and the uncertainty being modeled using the scenario approach. We propose a solving procedure that utilizes stochastic programming combined with decomposition and approximating techniques. The effectiveness of the proposed approach is proved by the experimental results.**

Keywords: dynamic portfolio optimization; progressive hedging algorithm; approximate dynamic programming; performance measure

1 Introduction

Over the last years, we witness a highly increased interest of private and institutional investors for techniques and tools aimed at a more efficient forecast of the dynamics of securities prices and to a rational management of investment capital. The growing interest for the optimal portfolio problem is demonstrated by the publication of numerous papers on this subject. Following this trend, our paper focuses on a dynamic portfolio model in which the investor has to decide the composition of risky assets at predetermined dates. To deal with the two main features of the multistage decision making problem, uncertainty and dynamics, we rely on stochastic programming techniques. A long period of time, almost all computational work in stochastic programming was based on the L-shaped decomposition method as described in Van Slyke and Wets [19]. For a detailed discussion of the techniques that help to manage uncertainty in solving problems with stochastic programming, see Birge and Louveaux [2]. When a probabilistic description of the unknown elements is not available, a common approach in practice is to rely on scenario analysis. An important contribution in algorithmic research is the method of scenario aggregation and the Progressive Hedging Algorithm introduced by Rockafellar and Wets [16]. Contributions from Mulvey and Vladimirou [11], Helgason and Wallace [6], Berland and Haugen [1] show the potential of the method. Decomposition of a stochastic program across scenarios partitions the problem into manageable subproblems, allowing the use of parallel processors. This becomes essential for large-scale problems. The present work combines the scenario decomposition approach and the techniques of Approximate Dynamic Programming. We also take into consideration the transaction costs for the amount of asset traded; we note that there is an extensive literature treating the portfolio optimization in the presence of transaction costs, see for example Patel and Subrahmanyam [12], Konno and Yamamoto [9], Konno, Akishino and Yamamoto [10], Kellerer, Mansini and Speranza [8], Fulga and Pop [4], [5] and Choi, Jang and Koo [3].

The rest of the paper is organized as follows: in Section 2, we give the mathematical formulation of the portfolio optimization problem. In Section 3 we propose a method for solving the portfolio optimization problem based on the Progressive Hedging Algorithm (PHA) combined with techniques of the Approximate Dynamic Programming (ADP). Computational results can be found in Section 4.

2 The portfolio model formulation

We consider the problem of a decision maker who is concerned with the management of a dynamic portfolio model over a finite horizon. The problem has a dynamic structure that involves portfolio rebalancing decisions in response to new information on market future prices (returns) of the risky assets in his portfolio. Rebalancing decisions are manifested in a sequence of successive revisions of holdings through sales and purchases of assets. We assume that the assets are sufficiently liquid that market impacts can be neglected. We choose the utility function approach to capture the decision maker's risk bearing preferences.

The decision maker starts (at $t = 0$) with an initial portfolio in assets $x_{1,0}, ..., x_{n,0}$ and in cash $x_{n+1,0}$ and has full knowledge of the current asset prices $p_0 = (p_{1,0}, ..., p_{n,0})'$. Thus, at $t = 0$, individual asset holdings, as well as the entire portfolio, can be accurately valued. We denote the initial investment $x_0 = (x_{1,0}, ..., x_{n,0}, x_{n+1,0})' \in R^{n+1}$. The decision maker rebalances each period his portfolio to achieve best return on his initial investment over time. The planning horizon is divided into periods $t \in \{0, ..., T-1\}$ corresponding to the times at which portfolio rebalancing decisions can be made. The represen-

*The research work has been supported by PN II, Program IDEI, Grant No. 1778. The date of the manuscript submission: 19.01.2009. Author affiliation: Academy of Economic Studies, Department of Mathematics, Piata Romana 6, 010174 Bucharest, Roumania, Fax: +4021.319.18.99, Email: fulga@csie.ase.ro

tation of uncertainty in input parameters of the portfolio optimization model is a critical step in the modelling process. The key random inputs in our portfolio management problem are the assets prices $\{p\}$ at future dates within the planning horizon. Plausible evolutions of the random parameter during the planning horizon are specified in terms of a scenario tree. The tree has a depth equal to the number of periods (decision stages). The root node corresponds to the initial state at the present time ($t = 0$). All input data associated with the root node are known with certainty. The tree branches out from the root depict progressive outcomes in the values of the random variable at subsequent periods. For example, the branches emanating from the root reflect the possible outcomes during the first period $t = 1$. Each scenario distinguishes a sequence of realizations of the random variable during the planning horizon. Thus, there is an one-to-one correspondence with a leaf (terminal node) of the tree. We denote by S the discrete set of scenarios generated by the collection of all realizations of the uncertain quantities, with cardinality $|S|$. The probability (weight) of scenario $s \in S$ is π_s, $\pi_s > 0$, $\forall s = \overline{1, |S|}$ and $\sum_{s=1}^{|S|} \pi_s = 1$.

The action taken on asset i at time t is denoted by $u_{i,t}$ and represents the amount of the i^{th} asset, purchased/sold at time $t \in \{0, ..., T-1\}$. At each time period t the investor can either hold the asset i $(u_{i,t} = 0)$, buy more $(u_{i,t} > 0)$, or sell off part of asset i $(u_{i,t} < 0)$. The transition from the state at time t, denoted by $x_t = (x_{1,t}, ..., x_{n,t}, x_{n+1,t})'$, to the state at time $t+1$, x_{t+1}, is governed by the decision $u_t = (u_{1,t}, ..., u_{n,t})'$ therefore, x_{t+1} depends on x_t and u_t. We assume that no short selling and no borrowing are allowed therefore, $x_t = (x_{1,t}, ..., x_{n,t}, x_{n+1,t})' \geq 0$ for all $t \in \{0, ..., T\}$. Buying and selling causes transaction costs which we assume to be proportional to the amount of asset traded. We denote by $100c^p$ the transaction costs expressed as a percentage associated with buying one unit of asset i and with $100c^s$ the transaction costs expressed as a percentage associated with selling one unit of asset $i \in \{1, ..., n\}$. Therefore, buying one unit of asset i requires $1 + c^p$ units of cash and selling one unit of asset i results in $1 - c^s$ units of cash. Through buying and selling, the investor can restructure his portfolio in each time period t. The assets in the portfolio are then kept constant till the next time period. We denote by $\overline{W}_t = (W_{1,t}, ..., W_{n,t}, W_{n+1,t})'$ the wealth vector at time $t \in \{0, ..., T\}$, where $W_{i,t}$, $i \in \{1, ..., n\}$, represents the wealth associated with the asset i calculated using the prices $p_{i,t} \geq 0$ at time t. For the symmetry of the notation, sometimes we will use $W_{n+1,t}$ for designating $x_{n+1,t}$ but we keep in mind that the two notations represent the same amount of cash disposable in the portfolio at time t. We assume that this liquidity component of the portfolio has a risk-free and constant over the time periods return r. We note that in our problem, it is possible to break down the effect of decisions and information on the variables of the problem. For the wealth vector, recall that $W_{i,t}$ represents the wealth associated with the asset i calculated using the number of assets $x_{i,t}$ of type i in the portfolio at that particular moment and the prices $p_{i,t}$ at t, $W_{i,t} = p_{i,t} x_{i,t}$, $i \in \{1, ..., n\}$. We will call $\overline{W}_t$ the predecision wealth vector or simply, the wealth vector. Let $\overline{W}^u_t = \left(W^u_{1,t}, ..., W^u_{n,t}, W^u_{n+1,t}\right)'$ be the post decision wealth vector which captures the effect of the decision $(u_{1,t}, ..., u_{n,t})'$ on the wealth vector $\overline{W}_t$. For the components of the post decision wealth vector at time t we have:

$$W^u_{i,t} = p_{i,t}\left(x_{i,t} + u_{i,t}\right), \ i \in \{1, ..., n\}, \tag{1a}$$

$$W^u_{n+1,t} = x_{n+1,t} + \sum_{i=1}^{n} p_{i,t} u_{i,t} \cdot$$
$$\cdot \left(-\left(1 + c^p\right) \sigma\left(u_{i,t}\right) + \left(1 - c^s\right) \sigma\left(-u_{i,t}\right)\right), \tag{1b}$$

where $\sigma(a) = \begin{cases} 1, & \text{if } a > 0 \\ 0, & \text{otherwise.} \end{cases}$

The components of the predecision wealth vector at time $t+1$ are

$$W_{i,t+1} = p_{i,t+1} x_{i,t+1}, \tag{2}$$
$$\text{where } x_{i,t+1} = x_{i,t} + u_{i,t}, \ i \in \{1, ..., n\},$$
$$W_{n+1,t+1} = (1 + r) W^u_{n+1,t}. \tag{3}$$

Let $W_t = \sum_{i=1}^{n+1} W_{i,t}$ be the total (predecision) wealth at time t and $W^u_t = \sum_{i=1}^{n+1} W^u_{i,t}$ be the total postdecision wealth at time t. One of the assumptions of the model is that there is no exogenous intervention on the amount of money involved in transactions during the time period $[0, T]$ therefore, the total (predecision) wealth must coincide with the total postdecision wealth at each time t. In fact, at each beginning of a time period $[t, t+1]$, a redistribution of the wealth take place, hence we have

$$W_t = W^u_t.$$

Remark 1 *In the formulation of the problem, all the variables whose values depend on the realization of a scenario will be indexed by s.*

Remark 2 *A policy is a function assigning to each scenario s a sequence of decisions $u_s = (u_{s,0}, ..., u_{s,T-1})'$, where $u_{s,t} = (u_{1,s,t}, ..., u_{n,s,t})'$, $t \in \{0, ..., T-1\}$. Several scenarios may reveal identical values for the uncertain quantities up to a certain period. In order to avoid dependence of hindsight, we impose the condition* *of nonanticipativity: scenarios that share common information history up to a specific period must yield the same decisions up to that period. Following the approach in Helgason and Wallace [6], let s^t be the equivalence class of all scenarios having a common information history up to t. A policy is called implementable if for all $t \in \{0,...,T-1\}$ the t^{th} decision is common to all scenarios in the same class s^t i.e. $u_{s,t} = u_{s',t}$, whenever $s^t = s'^t$. This means that, to be implementable, a decision made at time t can only depend on the history of the process known at that time and not on its future. This non-anticipativity constraints represent the links between the scenarios, deriving from the information structure of the problem.*

The investor's objective is to maximize the expected utility of the wealth generated by allocating it between the n risky assets and the riskless one, the liquidity component, over the investment horizon $[0,T]$. The liquidity component yields a riskless return r, the source of the uncertainty being the prices of the risky assets. The deterministic equivalent program which hedges against the outcomes specified by the set S of postulated scenarios is

$$\max_{u_{s,t}, t=\overline{0,T-1}} \sum_{s=1}^{|S|} \pi_s \sum_{t=1}^{T} U\left(W_{s,t}\right) \tag{4}$$

$$x_{i,s,t+1} = x_{i,s,t} + u_{i,s,t},\ i = \overline{1,n}, \tag{5}$$

$$\begin{aligned} x_{n+1,s,t+1} &= (1+r)\,[x_{n+1,s,t} + \\ &+ \sum_{i=1}^{n} p_{i,s,t} u_{i,s,t} \cdot \left(-\left(1+c^p\right)\sigma\left(u_{i,s,t}\right) + \right. \\ &\left. + \left(1-c^s\right)\sigma\left(-u_{i,s,t}\right)\right)] \end{aligned} \tag{6}$$

$$x_{s,0} = x_0, \tag{7}$$

$$x_{s,t} \geq 0,\ t = \overline{0,T}, \tag{8}$$

$$W_{s,t} = W^u_{s,t}, \tag{9}$$

$$u_{s,t} = u_{s',t},\ \text{if } s^t = s'^t,\ s,s' \in S. \tag{10}$$

$$\text{for all } t = \overline{0,T-1},\ s = \overline{1,|S|}.$$

where $U : R \to R$ is the decision maker's risk-averse utility function. Constraint (5) represents the asset inventory constraints and (6) are the cash balance equations. The constraint (9) means that the decisions at time t induce a redistribution of the wealth, without any exogenous intervention on the amount of money involved in transactions. As well, (8) means that no short selling and no borrowing are allowed.

We notice that if the nonanticipativity constraint (10) is removed, problem (4)-(9) decomposes into individual scenario problems but the solutions are not necessarily implementable. Rockafellar and Wets [16] have developed the so-called Progressive Hedging Algorithm (PHA) which allows to obtain an implementable optimal solution.

3 The Progressive Hedging Algorithm

Its fundamental idea is to add the constraints that tie together the different scenarios to the objective function via Lagrangian multipliers. The Lagrangian as well as the constraints are separable with respect to scenarios hence the optimization problem decomposes into smaller problems, with respect to scenarios. Penalties for deviating from implementability are added. Therefore, the problem decomposes into solving (approximately) iteratively for each scenario $s \in S$ the subproblem:

$$(P^\nu_s) \left\{ \begin{array}{c} \displaystyle\max_{u_{s,t}, t=\overline{0,T-1}} \sum_{t=0}^{T} \left[U\left(W_{s,t}\right) - \left(\Lambda^{\nu-1}_{s,t}\right)' u_{s,t} - \right. \\ \left. -\frac{1}{2}\gamma \left\| u_{s,t} - \widehat{u}^{\nu-1}_{s,t} \right\|_2^2 \right] \\ \left(u_{s,0},...,u_{s,T-1}\right) \in \Omega. \end{array} \right.$$

where Ω is the set of feasible but not necessarily implementable solutions, ν is the current iteration index, the vector $\widehat{u}^{\nu-1}_{s,t}$ stands for an estimate of $u_{s,t}$ from which we do not want to stray too far, $\Lambda^{\nu-1}_{s,t} \in R^n$ is a price vector and $\gamma > 0$ a penalty parameter.

By solving the subproblems (P^ν_s) the algorithm produces a well hedged solution to the underlying problem which performs well under all scenarios, relative to some weighting of scenarios. The solving of the time-discrete optimization problem (P^ν_s) could be done using the Dynamic Programming (DP). Suppose that, under the scenario $s \in S$ and at the moment $t \in \{0,...,T-1\}$ the wealth in hand is $W_{s,t}$. Let $u_{s,t} = \left(u_{1,s,t},...,u_{n,s,t}\right)'$ be our decision at time t. The transition function $(2)-(3)$ tell us that if we take the decision $u_{s,t}$ we are going to reach the wealth $W_{s,t+1}$. Suppose also that we had a function $V_{s,t+1}\left(W_{s,t+1}\right)$ that told us the value of being in state $W_{s,t+1}$ by giving us the value of the path from this state onward. Then, we could evaluate each possible decision $u_{s,t}$ and choose the decision that has the largest one-period contribution plus the value of reaching the wealth $W_{s,t+1}$, denoted by $V_{s,t+1}\left(W_{s,t+1}\right)$. Therefore, for each $t \in \{0,...,T-1\}$ we consider the problem:

$$(P^\nu_{s,t}) \left\{ \begin{array}{c} \displaystyle\max_{u_{s,t}} \left[U\left(W_{s,t}\right) - \left(\Lambda^{\nu-1}_{s,t}\right)' u_{s,t} - \right. \\ \left. -\frac{1}{2}\gamma \left\| u_{s,t} - \widehat{u}^{\nu-1}_{s,t} \right\|_2^2 + V_{s,t+1}\left(W_{s,t+1}\right) \right] \\ u_{s,t} \in \Omega_t \end{array} \right.$$

where the set of feasible solution Ω_t is defined by the constraints (5), (6), (9) and $x_{s,t+1} \geq 0$, $x_{n+1,s,t+1} \geq 0$.

Instead of DP, we propose an algorithm based on the Approximate Dynamic Programming techniques. Stepping forward in time, we have not computed the value function so, we have to turn to an approximation in order to make decisions. A simple and effective approximation is to use separable, piecewise linear approximations for the value function V. The accuracy of the approximation depends on the iterative process of slopes update.

The algorithm

Step 1. Initialization

1. Initialize $N_{\max}$, γ, $\Lambda^0_{s,t}$ and $\widehat{u}^0_{s,t}$ for all stages t and all scenarios s (one can take $\Lambda^0_{s,t} = 0$ and $\widehat{u}^0_{s,t} = 0$).
2. Set $\nu = 1$.
3. Set $s = 1$.

Step 2. Solving (P^ν_s)

1. Solve the problem $\left(P^\nu_{s,t}\right)$ for all $t = \overline{0, T-1}$. Let $u^\nu_{s,t}$ be the optimal solution of $\left(P^\nu_{s,t}\right)$, $t = \overline{0, T-1}$.
2. If $s < |S|$, increment s. Go to Step 2.1.

Step 3. Updating

For $t = \overline{0, T}$ and all scenarios $s \in \{1, ..., |S|\}$ compute the implementable solutions

$$\widehat{u}^\nu_{s,t} = \sum_{s \in s^t} \pi'_s u^\nu_{s,t},$$

where $\pi'_s = \frac{\pi_s}{\sum\limits_{s \in s^t} \pi_s}$, and update the implementability multipliers

$$\Lambda^\nu_{s,t} = \Lambda^{\nu-1}_{s,t} + \gamma \left(u^\nu_{s,t} - \widehat{u}^\nu_{s,t}\right).$$

Step 4. Checking stopping conditions

1. If, for all $t = \overline{0, T-1}$,

$$\left[\left\| \widehat{u}^\nu_{s,t} - \widehat{u}^{\nu-1}_{s,t} \right\|^2_2 + \sum_{s \in s^t} \pi'_s \left(u^\nu_{s,t} - \widehat{u}^\nu_{s,t}\right)^2 \right]^{\frac{1}{2}} < \varepsilon,$$

then, STOP.

2. Increment ν; if $\nu \leq N_{\max}$ go to Step 1.3. Otherwise STOP.

4 Computational results

The proposed algorithm was implemented in a custom program. The utility function and the quadratic term were approximated by piecewise-linear functions and for the optimization step the Revised Simplex Algorithm was used. The convergence of the algorithm that we use to solve the problem $\left(P^\nu_{s,t}\right)$ relies on Theorem 1 from Powell, Ruszczynski and Topaloglu [14].

To illustrate the behavior of the proposed method, we considered the optimization of a portfolio of assets transactioned at the Bucharest Stock Exchange. The portfolio is composed of fifteen assets, the most attractive in the last period of time, and the initial investment is divided equally among the assets. We simulated the evolution of the portfolio over 10 periods of time, one period representing one month, and a scenario tree with nine branches was built starting from the historical data (prices of the assets).

As the evolution of the portfolio depends on each investor, on his attitude towards risk, different utility functions were considered in the simulation:

$$\begin{aligned} U_1(W_t) &= \ln\left(\frac{W_t}{W_{t-1}} + 0.01\right), \\ U_2(W_t) &= -0.1\left(\frac{W_t}{W_{t-1}}\right)^{-4}, \\ U_3(W_t) &= -100\left(\frac{W_t}{W_{t-1}}\right)^{-3}. \end{aligned}$$

One iteration of the wealth is made of ten time steps; the value functions are updated at each moment of time - ten times for each iteration. The value functions are updated using different sequences of stepsizes $\{\alpha'_n\}_{n\geq 1}$, $\{\alpha''_n\}_{n\geq 1}$,... which depend on the iteration number. The effect of using different stepsizes rules it is apparent in the final solution and it is reflected in the Sharpe Ratio, see Table 1.

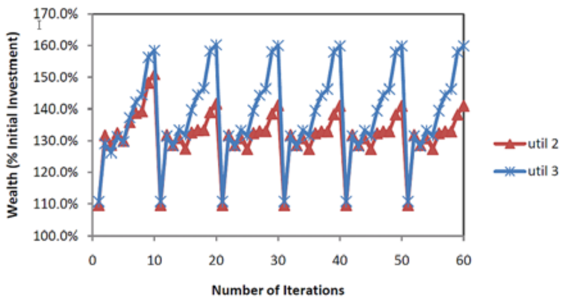

Figure 1: Comparative dynamics of the portfolio wealth

The wealth is represented across iterations as it can be seen in Figure 1. The evolution of the portfolio for two different investors is presented as percentage of the initial investment, in relation with the iterations and time horizon T. One investor is more risk-averse, (the evolution of the wealth in his case is given by the lower curve, comparative with the upper curve which describes the

ISBN: 978-988-17012-7-5

dynamics of the wealth of the second investor), and the corresponding wealth is lower. The shape of the curve for the evolution of the wealth converges rapidly, as it can be seen in the figure, which means that the value functions stabilize rapidly.

In Figure 2 the evolution of the final wealth, at time $t = T$, is shown in relation with the scenarios. There are nine scenarios, and for each scenario the total wealth is shown for the last four iterations. For example, the green bars represent the level reached by the final wealth at the iteration $N-1$, where the N^{th} iteration is the last one, when the algorithm stops and the optimum is attained.

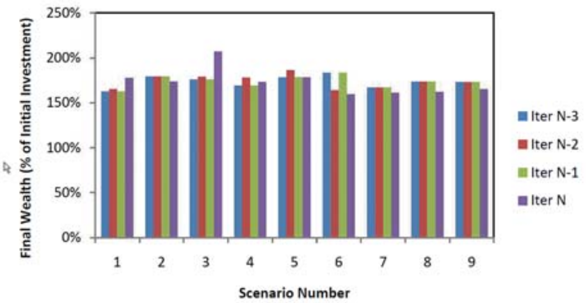

Figure 2: Final wealth variation vs. scenarios

The classical performance measures by Treynor [17], Sharpe [15], Jensen [7], as well as Treynor and Black [18] are central for any kind of performance evaluation. In this paper, we refer to Sharpe ratio which tells us whether a portfolio's returns are due to good decisions or a result of excess risk. The Sharpe Ratio is calculated by the formula $S = \frac{E(R-R_f)}{\sigma}$, see Sharpe [15]. The numerator of the ratio is the expected return that the portfolio is expected to provide above the risk free rate. The risk free rate was taken 10%, the actual rate of government bonds. The denominator is the standard deviation of the portfolio. This measurement is very useful because although one portfolio has higher returns than others, it is a good investment only if those higher returns do not come with too much additional risk.

The Sharpe ratio was calculated for each scenario and the values obtained are represented in Table 1. To outline the dependence of the solution on the stepsizes, we represented the Sharpe ratio for two cases, for $\alpha'_n = \frac{8}{n+20}$ and $\alpha''_n = \frac{1}{n^{2/3}}$.

SR\Scenario	1	2	3	4	5	6	7	8	9
SR for α'_n	1.11	1.16	0.95	1.07	1.22	0.97	0.78	1.16	0.82
SR for α''_n	1.08	0.96	1.14	1.04	1.06	0.76	0.84	0.85	1.02

Table 1: Sharpe Ratio variations across scenarios

5 Conclusions

In this paper we developed a technique for solving the multiperiod portfolio optimization problem. This new technique relies on the Progressive Hedging Algorithm to manage the nonanticipativity constraints and to decompose the problem across scenarios, combined with an algorithm for solving the problems $\left(P^{\nu}_{s,t}\right)$ based on Approximate Dynamic Programming techniques, well suited for large-scale problem solving, which offer an optimal solution by taking advantage of time decomposition and fully exploiting the properties of the portfolio model formulation.

References

[1] Berland, N.J., Haugen, K.K., "Mixing stochastic programming and Scenario Aggregation", *Ann Oper Res,* V64, pp. 1-19, 1996.

[2] Birge, J.R., Louveaux, F., *Introduction to stochastic programming*, Springer, Berlin, 1997.

[3] Choi, U.J., Jang, B.G., Koo, H.K., "An algorithm for optimal portfolio selection problem with transaction costs and random lifetimes", *Appl Math Comput*, V191, pp. 239–252, 2007.

[4] Fulga, C., Pop, B., "Portfolio Selection with Transaction Costs", *Bull Math Soc Sci Math Roumanie*, V50, N4, pp. 317-330, 2007.

[5] Fulga, C., Pop, B., "Single Period Portfolio Optimization with Fuzzy Transaction Costs", *Proc of EURO Conference Continuous Optimization and Knowledge-Based Technologies*, Neringa, Lithuania, pp. 125-131, 2008.

[6] Helgason T., Wallace, W., "Approximate scenario Solutions in the Progressive Hedging Algorithm", *Ann Oper Res,* V31, pp. 425-444, 1991.

[7] Jensen, M.C., "The Performance of Mutual Funds in the Period 1956–1964", *J Financ*, pp. 389–416, 1968.

[8] Kellerer, H., Mansini, R., Speranza, M.G., "Selecting Portfolios with Fixed Costs and Minimum Transaction Lots", *Ann Oper Res*, V99, pp. 287–304, 2000.

[9] Konno, H., Yamamoto, R., "Global Optimization Versus Integer Programming in Portfolio Optimization under Nonconvex Transaction Costs", *J Global Optimiz,* V32, pp. 207–219, 2005.

[10] Konno, H., Akishino, K., Yamamoto, R., "Optimization of a Long-Short Portfolio under Nonconvex Transaction Cost", *Comput Optimiz Appl*, V32, pp. 115–132, 2005.

[11] Mulvey, J.M., Vladimirou, H., "Applying the progressive hedging algorithm to stochastic generalized networks", *Ann Oper Res,* V31, pp. 399-424, 1991.

[12] Patel, N.R., Subrahmanyam, M.G., "A Simple Algorithm for Optimal Portfolio Selection with Fixed Transaction Costs", *Manage Sci*, V28, pp. 303–314, 1982.

[13] Powell, W.B., *Approximate Dynamic Programming: Solving the Curses of Dimensionality*, Wiley-Interscience, 2007.

[14] Powell, W.B., Ruszczynski, A., Topaloglu, H., "Learning Algorithms for Separable Approximations of Discrete Stochastic Optimization Problems", *Math Oper Res,* V29, pp. 814-836, 2004.

[15] Sharpe, W.F., "The Sharpe Ratio", *J Portfolio Manage,* V21, N1, pp. 49-58, 1994.

[16] Rockafellar, R.T., Wets, R.J.B., "Scenario and policy aggregation in optimization under uncertainty", *Math Oper Res,* V16, N1, pp. 1-29, 1991.

[17] Treynor, J.L., "How to Rate Management of Investment Funds", *Harvard Business Review*, January/February, pp. 63–75, 1965.

[18] Treynor, J.L., Black, F., "How to Use Security Analysis to Improve Portfolio Selection", *J Bus*, pp. 66–86, 1973.

[19] Van Slyke, R., Wets, R.J.B., "L-shaped linear programs with applications to optimal control and stochastic programming", *SIAM J Appl Math,* V17, pp. 638-663, 1969.