Multi-Point Special Boundary-Value Problems and Applications to Fluid Flow Through Porous Media

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Abstract—**In this paper we propose a numerical scheme based on finite differences for the numerical solution of nonlinear multi-point boundary value problems over adjacent domains. In each subdomain the solution is governed by different equation. The solutions are required to be smooth across the interfaces. The approach is based on using finite difference approximation of the derivatives at the interface nodes. A modified multidimentional Newton's method is proposed for solving the nonlinear system of equations. The accuracy of the proposed scheme is validated by examples whose exact solutions are known. The method is then applied to solve for the velocity profile of fluid flow through multilayer porous media.**

Keywords:* *Multi-dimensional Newton's method, Porous media, Multilayer flows, interface region

1 Introduction

Many physical phenomenons are modeled by differential equations, ordinary or partial, linear or nonlinear. The solutions, exact or numerical, of such differential equations offer valuable insights into these phenomenons. Exact solutions of these differential equations, especially of the nonlinear ones, are in most of the time not easily obtainable. For this reason, a numerical approach is often adopted to approximate the solution. Although many physical problems are modeled by a single or a system of differential equations over a finite domain, there are also applications where the system is modeled by different differential equations over subdomains of the overall domain, and require the solution to satisfy certain conditions across the subdomains boundaries, in addition to conditions at the overall domain boundaries. This is what constitutes a multi-point boundary value problem.

In the context of this paper, a second order multi-point boundary value problem is defined as the following sequence of boundary value problems.

$$y'' = f_i(x, y, y'),\ c_{i-1} \le x \le c_i,\ 1 \le i \le M, \qquad (1)$$

where $[c_{i-1}, c_i]$, $1 \le i \le M$, is the ith subdomain, M is the number of subdomains, $[c_0, c_M] = [a, b]$ is the overall domain, and f_i, $1 \le i \le M$, are functions defining the differential equations in the ith subdomain. The solution y is to satisfy the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. In addition, the solution has to satisfy smoothness conditions at the *interface* nodes c_i, $1 \le i \le M-1$. Specifically, the solution is assumed to be smooth at each of the c_i, i.e.,

$$y(c_i^-) = y(c_i^+) \text{ and } y'(c_i^-) = y'(c_i^+),\ 1 \le i \le M-1. \qquad (2)$$

The above mentioned multi-point boundary value problem is of interest to us in this paper because it occurs in many areas of engineering applications such as in modeling the flow of fluid such as water, oil and gas through ground layers, where each layer constitutes a subdomain. In fact, our motivation for this work comes from modeling fluid flow through multilayer porous medium, where many works have been conducted in this regard [9]–[13]. The present work generalizes earlier works on two-layer porous medium, [9, 10].

Boundary value problems have been extensively studied theoretically and numerically [1]–[5]. Many of the numerical schemes are based on discretization of the space and finite difference approximations of the derivatives. Other schemes are based on expansion methods [4]. Theorems pertaining to the existence and uniqueness of solutions of boundary value problems are contained in a comprehensive survey in a book by Agarwal [5].

In this paper, we consider the multi-point boundary value problem given by (1)–(2) and propose a numerical approach based on finite differences. The idea is to express the solution at the interface nodes $y(c_i)$ in terms of the value of the solution at neighboring internal points. This is accomplished by equating the left and the right approximations, of various orders, of the derivative $y'(c_i)$, since the solution is assumed to be smooth at each c_i. This results is a system of nonlinear equations for the solution at the discretization points. The nonlinear system is solved iteratively using a third order modified form of the classical multidimensional Newton's method.

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The paper is organized as follows. In section 2, we present the problem formulation and present the derivation which leads to the nonlinear system. In section 3, a modified Newton's method with cubic convergence is outlined. In section 4, we test the method using examples whose exact solutions are known in order to measure the accuracy of the method. In section 5, we present an application of the proposed method to the resolution of the velocity profile of fluid in multilayer porous media.

2 Problem formulation and description of the algorithm

The problem we consider is the following second order multi-point boundary value problem:

$$y'' = f_i(x, y, y'),\ c_{i-1} < x < c_i,\ 1 \le i \le M, \tag{3}$$

$$y(a) = \alpha,\ y(b) = \beta, \tag{4}$$

$$y(c_i^-) = y(c_i^+),\ y'(c_i^-) = y'(c_i^+),\ 1 \le i \le M-1, \tag{5}$$

where $[c_{i-1}, c_i]$, $1 \le i \le M$, is the ith subdomain, M is the number of subdomains, $[c_0, c_M] = [a, b]$ is the overall domain, and f_i, $1 \le i \le M$, are functions defining the differential equations in the ith subdomain. Here, we assume that functions f_i satisfy necessary conditions for the existence of solution. Conditions (5) states that the solution must be smooth at the interface nodes c_i. Such a condition is very important in many areas of engineering such as in modeling fluid flow through multilayer porous media, where the velocity and shear stress are to be continuous across the layers [14].

The finite difference approach we propose starts with a the classical uniform discretization of each subdomain $[c_{i-1}, c_i]$ with a *local* step size h_i, $i = 1, 2, \dots M$, where $h_i = \frac{c_i - c_{i-1}}{N_i}$ and N_i, assumed to be an integer, is the number of subintervals in the ith subdomain. The discretization produces the doubly-indexed mesh points

$$x_k^{(i)} = c_{i-1} + kh_i,\ 1 \le i \le M,\ 0 \le k \le N_i,$$

where the superscript (i) refers to the subdomain and the subscript k refers to the mesh point in that subdomain. Note that $x_0^{(1)} = a$, $x_{N_M}^{(M)} = b$, and $x_{N_i}^{(i)} = x_0^{(i+1)} = c_i$. It is important to note here that we have chosen to discretize each subdomain with a different step size. We could have chosen a uniform step size h for the whole domain $[a, b]$, but this does not guarantee that the interface nodes, c_i, will be mesh points. Moreover, choosing a different step size for each subdomain offers more flexibility.

The differential equations (3) are then discretized at the internal mesh points $x_k^{(i)}$, excluding the interface nodes c_i. The interface nodes c_i are excluded because the second derivative at these points is in general not continuous. In each subdomain i, with local step size h_i, the derivatives are approximated by the central difference formulae: $y'(x_k^{(i)}) = \frac{y(x_{k+1}^{(i)}) - y(x_{k-1}^{(i)})}{2h_i} + O(h_i^2)$, $y''(x_k^{(i)}) = \frac{y(x_{k+1}^{(i)}) - 2y(x_k^{(i)}) + y(x_{k-1}^{(i)})}{h_i^2} + O(h_i^2)$.

In general the discretization produces a rectangular system of algebraic equations in the unknowns $y_k^{(i)} \approx y(x_k^{(i)})$:

$$y_{k+1}^{(i)} - 2y_k^{(i)} + y_{k-1}^{(i)} = h_i^2 f_i(x_k^{(i)}, y_k^{(i)}, \frac{y_{k+1}^{(i)} - y_{k-1}^{(i)}}{2h_i}), \tag{6}$$

where $1 \le i \le M,\ 1 \le k \le N_i - 1$. We note here that the above algebraic system is under determined. It has $(N - M)$ equations and N unknowns $y_k^{(i)}, 1 \le k \le N_i, 1 \le i \le M$, where $N = \sum_{i=1}^{M} N_i$ and $N_i = \frac{c_i - c_{i-1}}{h_i}$.

In theory, under determined systems, if they admit a solution, they admit infinitely many. Thus, we need to somehow transform the system to a square system whose solution, if it exists, is unique. This can be done by eliminating enough unknowns. In our case we need to eliminate M unknowns.

To render system (6) square, we eliminate the M unknowns $y_{N_i}^{(i)} \approx y(c_i)$ by expressing them in terms of neighboring unknowns $y_{N_i - j}^{(i)}$ and $y_j^{(i+1)}$, $j = 1, 2$ or 3, depending on the desired accuracy. This will be accomplished by imposing the smoothness conditions at the interface nodes: $y'(c_i^-) = y'(c_i^+), 1 \le i \le M$.

From the right and left Taylor series expansions of $y(x)$ about c_i, $1 \le i \le M-1$, we have the following backward and forward first-, second- and third-order approximations for $y'(c_i)$.

First order:

$$\text{Backward:}\quad y'(c_i) \approx \frac{y_{N_i}^{(i)} - y_{N_i-1}^{(i)}}{h_i}. \tag{7}$$

$$\text{Forward:}\quad y'(c_i) \approx \frac{y_1^{(i+1)} - y_{N_i}^{(i)}}{h_{i+1}}. \tag{8}$$

Second order:

$$\text{Backward:}\quad y'(c_i) \approx \frac{3y_{N_i}^{(i)} - 4y_{N_i-1}^{(i)} + y_{N_i-2}^{(i)}}{2h_i}. \tag{9}$$

$$\text{Forward:}\quad y'(c_i) \approx \frac{-3y_{N_i}^{(i)} + 4y_1^{(i+1)} - y_2^{(i+1)}}{2h_{i+1}}. \tag{10}$$

Third order:

Backward:

$$y'(c_i) \approx \frac{11y_{N_i}^{(i)} - 18y_{N_i-1}^{(i)} + 9y_{N_i-2}^{(i)} - 2y_{N_i-3}^{(i)}}{6h_i}. \tag{11}$$

Forward:

$$y'(c_i) \approx \frac{-11y_{N_i}^{(i)} + 18y_1^{(i+1)} - 9y_2^{(i+1)} + 2y_3^{(i+1)}}{6h_{i+1}}. \tag{12}$$

Then using (7) – (12), equating the forward approximation to the backward approximation, we find the following approximations for $y^{(i)}_{N_i} \approx y(c_i)$ in terms of $y^{(i)}_{N_i-j}$ and $y^{(i+1)}_j, j = 1, 2$ or 3:

First order:

$$y^{(i)}_{N_i} = \frac{h_{i+1} y^{(i)}_{N_i-1} + h_i y^{(i+1)}_1}{h_i + h_{i+1}}. \tag{13}$$

Second order:

$$y^{(i)}_{N_i} = \frac{-h_{i+1} y^{(i)}_{N_i-2} + 4h_{i+1} y^{(i)}_{N_i-1} + 4h_i y^{(i+1)}_1 - h_i y^{(i+1)}_2}{3(h_i + h_{i+1})}. \tag{14}$$

Third order:

$$\begin{aligned} y^{(i)}_{N_i} &= [2h_{i+1} y^{(i)}_{N_i-3} - 9h_{i+1} y^{(i)}_{N_i-2} \\ &\quad +18h_{i+1} y^{(i)}_{N_i-1} + 18h_i y^{(i+1)}_1 \\ &\quad -9h_i y^{(i+1)}_2 + 2h_i y^{(i+1)}_3]/[11(h_i + h_{i+1})]. \end{aligned} \tag{15}$$

When either (13), (14) or (15) is substituted for each $y^{(i)}_{N_i}$, $i = 1, 2, \ldots, M$, in the system (6), we obtain a square nonlinear system of size $(N - M)$ in the unknowns $y^{(i)}_k$, $1 \le i \le M$, $1 \le k \le N_i - 1$. The resulting nonlinear algebraic system can then be solved by a suitable iterative method. The solution at the interface nodes c_i, i.e., $y(c_i) \approx y^{(i)}_{N_i}$, can be recovered using the appropriate approximation formula (13), (14) or (15). In the next section we describe a modified Newton's method which may be used to solve the resulting nonlinear algebraic system.

3 A Modified Newton's Method with Cubic Convergence

It is known that the multidimensional Newton's method iterative scheme

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - \mathbf{J}_n^{-1}\, \mathbf{F}(\mathbf{x}^{(n)}) \tag{16}$$

converges quadratically to the (simple) root $\mathbf{x}^* = (x^*_1, x^*_2, \ldots, x^*_d)^T$ of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{F} : R^d \longrightarrow R^d$ ($\mathbf{F} = (F_1, F_2, \ldots, F_d)^T$), $\mathbf{x} = (x_1, x_2, \ldots, x_d)^T$, and $\mathbf{J}_n^{-1} \equiv \mathbf{J}^{-1}(\mathbf{x}^{(n)})$ is the inverse of the Jacobian matrix of $\mathbf{F}$, $\mathbf{J}_{ij} = \frac{\partial F_i}{\partial x_j}$, evaluated at $\mathbf{x}^{(n)}$.

Recently, a lot of work has been done to derive modified versions of Newton's method which converge cubically. In the one-dimensional case, cubically convergent modified Newton's schemes were derived in [6, 8] and in references therein. A generalization of the cubically convergent scheme given in [6] to the multivariate case was later given in [7] and is formally stated below.

Theorem 3.1 *Let $\mathbf{F} : R^d \to R^d$ be a sufficiently smooth function in a neighborhood of its root $\mathbf{x}^*$, where the Jacobian of $\mathbf{F}$ at $\mathbf{x}^*$, $\mathbf{J}(\mathbf{x}^*)$, is invertible. Assume that*

$$\max_{1 \le i,j,k \le d} \left| \frac{\partial^3 \mathbf{F}}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}) \right| \le C$$

holds in the neighborhood of $\mathbf{x}^*$ for some constant C . Then the iterative scheme

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - \mathbf{J}^{-1}(\mathbf{x}^{(n)} - \frac{1}{2}\mathbf{J}^{-1}(\mathbf{x}^{(n)})\mathbf{F}(\mathbf{x}^{(n)}))\mathbf{F}(\mathbf{x}^{(n)})$$

converges cubically to $\mathbf{x}^*$.

It is important to note that as in the univariate case the inverse $\mathbf{J}^{-1}$ is never computed in practice. The scheme in Theorem 3.1 proceeds as follows. Given $\mathbf{x}^{(n)}$, let $\mathbf{z}^{(n)} = \mathbf{x}^{(n)} - \frac{1}{2}\mathbf{J}^{-1}(\mathbf{x}^{(n)})\mathbf{F}(\mathbf{x}^{(n)})$. Then $\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - \mathbf{J}^{-1}(\mathbf{z}^{(n)})\mathbf{F}(\mathbf{x}^{(n)})$. Therefore, at every step, given $\mathbf{x}^{(n)}$, $\mathbf{x}^{(n+1)}$ is obtained by solving, in order, the following two systems of linear equations with coefficient matrix $\mathbf{J}$:

$$\mathbf{J}(\mathbf{x}^{(n)})\mathbf{z}^{(n)} = \mathbf{J}(\mathbf{x}^{(n)})\mathbf{x}^{(n)} - \frac{1}{2}\mathbf{F}(\mathbf{x}^{(n)}) \qquad \text{for } \mathbf{z}^{(n)} \text{ then}$$
$$\mathbf{J}(\mathbf{z}^{(n)})\mathbf{x}^{(n+1)} = \mathbf{J}(\mathbf{z}^{(n)})\mathbf{x}^{(n)} - \mathbf{F}(\mathbf{x}^{(n)}) \qquad \text{for } \mathbf{x}^{(n+1)}.$$

In our numerical experiments we use the scheme given in Theorem 3.1 with the stopping criteria $\|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}\| < \varepsilon = 10^{-6}$.

4 Numerical Examples

In this section we apply our proposed algorithm as described in Section 2 to two examples whose exact solutions are known thereby validating its accuracy.

Example 4.1 *As a first example, we consider the same ODE*

$$y'' = yy' - y + \cos(2x), \qquad 0 \le x \le 2\pi, \tag{17}$$

over three neighboring intervals $[0, \pi/2]$, $[\pi/2, 5\pi/4]$, *and* $[5\pi/4, 2\pi]$, *with the boundary conditions at $x = 0$ and at $x = 2\pi$:*

$$y(0) = 1 \quad \textit{and} \quad y(2\pi) = 1, \tag{18}$$

and the continuity and smoothness conditions at $c_1 = \pi/2$ and $c_2 = 5\pi/4$:

$$\begin{aligned} &y(\pi/2^-) = y(\pi/2^+),\ y'(\pi/2^-) = y'(\pi/2^+), \\ &y(5\pi/4^-) = y(5\pi/4^+),\ y'(5\pi/4^-) = y'(5\pi/4^+). \end{aligned} \tag{19}$$

It can be verified that the exact solution of (17) subject to (18) is

$$y_e(x) = \cos(x) - \sin(x)$$

which is smooth, hence verifies (19).

The suggested algorithm as described in section 2 has been applied to the above example and the results are shown in Figure 1 and in Table 1. The following parameter values have been used. The step sizes used are $h_1 = \pi/200 \approx 0.015708$, $h_2 = 3\pi/800 \approx 0.011781$, and $h_3 = 3\pi/400 \approx 0.0235619$. Figure 1 displays the absolute error between the exact and the numerical solution for second- and third-order approximations.

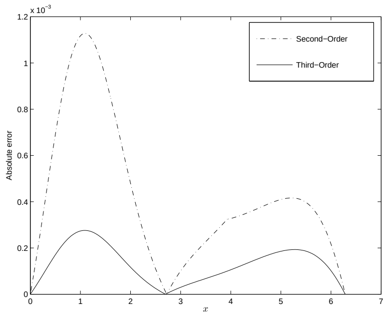

Figure 1: The absolute error between the exact and numerical solution of Example 1, for second- and third-order approximations.

	$(y(\pi/2), y'(\pi/2))$	$(y(5\pi/4), y'(5\pi/4))$
O(1)	(-1.006254,-1.001467)	(-0.004125,1.414787)
O(2)	(-1.000878,-0.999192)	(0.000324,1.414522)
O(3)	(**-1.000216**,**-0.999786**)	(**0.000099**,**1.414294**)
Exact	(**-1.000000**,**-1.000000**)	(**0.000000**,**1.414214**)
Error	(**2.16**, **2.14**)$\times 10^{-4}$	(**0.99**,**0.8**)$\times 10^{-4}$

Table 1: Approximate values of $y(c_i)$ and $y'(c_i)$ for $c_1 = \pi/2$ and $c_2 = 5\pi/4$.

This first example shows that the proposed method is accurate in resolving the smoothness of the solution at the interface nodes, as can be seen by the results displayed in Table 1, where the absolute error is of the order of 10^{-4} when third order approximation of the derivative is used. We note that if higher order approximations are used, certainly the absolute error will be less. This is, of course, on the expense of having a lesser dense matrix.

Example 4.2 *As a second example, consider the 4-subinterval problem:*

$$\begin{aligned}
&y'' = 2y^3, \qquad -1 \le x < 0,\\
&y'' = -(y')^2 - \tfrac{20}{9}y' - \tfrac{100}{81}, \qquad 0 < x < 1/2,\\
&y'' = 2(y - 14x/9 - \ln(3x) + 3)^3 - \tfrac{1}{x}\\
&\qquad +(y' - 14/9)^2, \qquad 1/2 < x < 1,\\
&y'' = -(y - 5x/9 + 1) + \ln(3x)\\
&\qquad -(y' - 5/9)^2, \qquad 1 < x \le 2,
\end{aligned}$$

with the boundary conditions $y(-1) = \frac{1}{2}$, $y(2) = \ln(6) + \frac{1}{9}$, *and the smoothness conditions at the interface nodes* $c_1 = 0$, $c_2 = 1/2$ *and* $c_3 = 1$*:* $y(0^-) = y(0^+)$, $y'(0^-) = y'(0^+)$, $y(1/2^-) = y(1/2^+)$, $y'(1/2^-) = y'(1/2^+)$, $y(1^-) = y(1^+)$, $y'(1^-) = y'(1^+)$.

It is easy to check that the exact solution to the above problem is

$$y(x) = \begin{cases} \frac{1}{x+3}, & -1 \le x < 0,\\ \ln(x+1) - \frac{10}{9}x + \frac{1}{3}, & 0 \le x < 1/2,\\ \ln(3x) + \frac{1}{x} + \frac{14}{9}x - 3, & 1/2 \le x < 1,\\ \ln(3x) + \frac{5}{9}x - 1, & 1 \le x \le 2. \end{cases} \tag{20}$$

The results of the algorithm for this example are shown in Fig. 2 and Table 2. The following parameter values have been used. The step sizes used are $h_1 = h_4 = 0.01$ and $h_2 = h_3 = 0.005$.

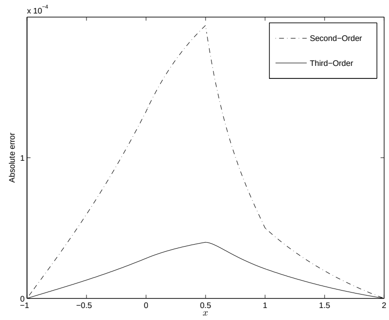

Figure 2: The absolute error between the exact and numerical solution of Example 2, for second- and third-order approximations.

	Order (1,2,3)	Exact	**Order3** *Abs.Error* ($\times\mathbf{10^{-5}}$)
$y(0)$	(0.3378,0.3335,**0.3334**),	**0.3333**	**2.8445**
$y'(0)$	(-0.1059,-0.1110,**-0.1111**)	**-0.1111**	**3.4837**
$y(1/2)$	(0.1903,0.1834,**0.1833**)	**0.1832**	**3.9887**
$y'(1/2)$	(-0.4399,-0.4444,**-0.4444**)	**-0.4444**	**1.4985**
$y(1)$	(0.6559,0.6542,**0.6542**)	**0.6542**	**2.0979**
$y'(1)$	(1.5478,1.5554,**1.5555**)	**1.5556**	**3.1434**

Table 2: $y(c_i)$ and $y'(c_i)$ for $c_1 = 0, c_2 = 1/2, c_3 = 1$.

The results of this second example again show that the proposed algorithm can accurately resolve the continuity and smoothness of the solution at the interface nodes as seen in Table 2.

5 Flow through porous media

In this section we apply the proposed algorithm to solve for the velocity profile of fluid flow through multilayer porous media. The media consists of a many porous layers, where the upper and lower layers are bounded above and below, respectively, by solid walls, see Fig. 3, which depicts a six-layer porous media configuration.

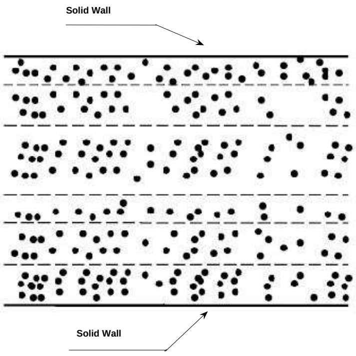

Figure 3: Configuration of 6 layer-porous media.

In each layer, the governing equation, after suitable transformations, in dimensionless form, is the following second-order differential equation.

$$\frac{d^2u}{dy^2} = Re\,C + \frac{u}{k} + \frac{Re\,C_d}{\sqrt{k}}\,u^2, \qquad (21)$$

where $u(y)$, $-1 \leq y \leq 1$, is the velocity of the fluid, where $y = 1$ corresponds to the upper solid wall boundary and $y = -1$ corresponds to the lower solid wall boundary. The various physical parameters are defined as follows. $Re = \rho U_\infty L/\mu$ is the Reynolds number, ρ is the fluid density, U_∞ is the free-stream characteristic velocity, μ is the fluid viscosity, L is the channel characteristic length, k is the permeability of the porous channel, C_d is the form drag coefficient, and $C < 0$ is a dimensionless pressure gradient. For detailed derivations of (21) see [10].

The model given by equation (21) is referred to as the Darcy-Lapwood-Forchheimer-Brinkman (DFB) model, [11]. When the drag coefficient $C_d = 0$, we have the linear model

$$\frac{d^2u}{dy^2} = Re\,C + \frac{u}{k}, \qquad (22)$$

which is known as Darcy-Lapwood-Brinkman (DLB) model.

A lot of work on fluid flow through porous media have been done, see [9]-[14] and references therein. Two-layer and three-layer configurations have been considered in [10] and [13], respectively, where in [13], exact solutions for the velocity of the flow have been obtained for the three-layer configuration with the middle layer is finite and the outer layers are assumed to be of infinite widths.

In our present work, our aim is to test our algorithm on configurations of more than three layers. In fact, the algorithm is designed to handle any number of layers.

Since the upper (lower) layer is bounded above (below) by solid impermeable wall, a no-slip condition at the solid boundaries ($y = \pm 1$) is assumed, i.e., $u(\pm 1) = 0$. At the interface boundaries between different layers, we assume that the velocity and the shear stress are continuous, that is, $u(0^-) = u(0^+)$ and $u'(0^-) = u'(0^+)$. This assumption is realistic and makes it possible to determine the fluid velocity at the interface.

In our simulations, we have considered 2 and 5 layer-configurations, where each layer is either governed by the DFB or the DLB model. In all experiments we fix the following parameter values. The Reynolds number $Re = 10$, $C = -10$ and $C_d = 0.55$.

5.1 A two-layer configuration

This case is considered here for comparison purposes with the results obtained in [10]. The flow in the top layer is modeled the DLB model and in the lower layer by the DFB model. The permeability of the top layer is set to $k_t = 1$ and that of the lower layer is varied. The results are shown in Fig. 4 and Table 3.

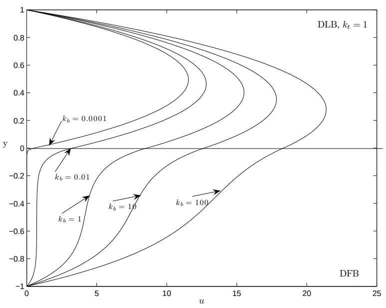

Figure 4: Velocity profile for the DFB/DLB two-layer channel.

k_b	$u(0)$	$u'(0)$)
	Order (1,2,3)	Order (1,2,3)
0.0001	(0.73124, 0.56957,0.52049)	(44.7556, 45.462, 45.5279)
0.01	(3.43867,3.28921,3.26835)	(41.2142, 41.8911, 41.9199)
1	(8.67902, 8.58168 ,8.5728)	(34.3595,34.9422, 34.955)
10	(12.7381, 12.6822,12.6772)	(29.05, 29.5582,29.5657)
100	(18.3294, 18.3301, 18.3278)	(21.7363,22.1426,22.1464)

Table 3: Velocity and shear stress at the interface for various permeability k_b.

We remark here that our results for the two-channel configuration are similar in nature but not in values to those obtained in [10].

5.2 A five-layer configuration

The final simulation was performed on a five-layer configuration. The interface nodes were set at $c_1 = -0.6, c_2 = 0, c_3 = 0.4$ and $c_4 = 0.6$. We assumed that all layers are modeled by the DFB model. The permeabilities were set to $k_1 = 0.1, k_2 = 1, k_2 = 0.001, k_3 = 10, k_4 = 0.1, k_5 = 0.001$, where k_1 corresponds to bottom layer and k_5 corresponds to top layer. The step sizes were chosen to be $h_1 = 0.004, h_2 = 0.006, h_3 = 0.004, h_4 = 0.002$ and $h_5 = 0.004$. The velocity profile is shown in Fig. 5.

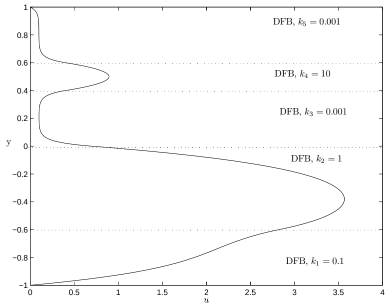

Figure 5: Velocity profile for the DFB five-layer channel.

We particularly note here that eventhough layer 4 has a larger permeability ($k_4 = 10$) than layer 2 ($k_2 = 1$), the velocity of the fluid in layer 2 is larger than in layer 4. This is due to the fact that layer 4 is sandwiched between two low-permeability layers (layers 3 and 5 with $k_3 = k_5 = 0.001$). This suggests that the flow in the low-permeability layers affect the flow in the high-permeability layers.

6 Conclusion

In this paper, we have developed a numerical procedure to solve multipoint special second-order boundary-value problems. The algorithm was based on finite differences. The discretization is done locally to each subdomain to ensure that the interface nodes are not missed and constitute mesh points. The smoothness requirement of the solution at the interface nodes was used to render the obtained algebraic system a square one. A cubic convergent modified Newton's method was introduced and used in the experiments to solve the nonlinear system.

The algorithm proved to be very accurate on two examples with known exact solutions. Also, the algorithm proved to be very accurate and effective in a more realistic problem of fluid flow thorough multi-layer porous media. In this application, it was accurately possible to solve for the flow velocity profiles across any number of layers. The results of the experiments, especially in the five-layer configuration, show that the flow in low-permeability layers affect the flow in high-permeability layers. The implementation of the algorithm was done using the software package *Mathematica*.

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