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A New Finite Difference Approximation for Numerical Solution of Simplified 2-D Quasilinear Unsteady Biharmonic Equation

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Abstract— **Using numerical methods, based on discretization rules has been one of the most popular and successful methods for numerically solving of partial differential equations for several years. In this paper, a new discretization method based on mixing finite difference scheme is applied to a simplified 2-D quasilinear unsteady biharmonic equation. The partial differential equation is defined with subject to initial and boundary conditions, but no fictitious points in discretization are appeared. This method is implicit and its corresponding local truncation error is at least of order six in space and four in time respectively. Stability, convergence, and consistency are considered for the new method and several numerical examples are given to validate our results. This method can also be extended for some partial differential equations with more complex nonlinear terms especially for the 2-D unsteady biharmonic equation.**

Index Terms— **High order finite difference scheme, Local truncation error, Unsteady quasilinear biharmonic equation, Stability.**

I. INTRODUCTION

Family of biharmonic equations has an extra range of forth order partial differential equations. In real world, many physical phenomena concern with biharmonic equation in several forms. For instances we can refer to the problem of determining the deflection of a thin clamped plate under the action of a distributed load f [3] or the problems related to blending surface [9]. In especial cases, nonlinear biharmonic equation arises in the study of traveling waves in suspension bridges or in the study of static deflection of an elastic plate in a fluid [2], [4].

There are some numerical methods for each type of biharmonic equation. Most of these methods based on variational calculus, finite elements methods, integral equations, and finite difference methods [10], [11]. The accuracy of these numerical schemes goes back to the nature and type of the equation family and its related initial-boundary value problems. In complicated geometry, some domain decomposition methods can simplify obtaining numerical results [1].

To introduce our underlying numerical scheme, at first we consider an initial-boundary value problem concerning a simplified 2-D fourth order quasilinear partial differential equation with variable coefficients

$$A(x,y,t)\frac{\partial^4 u}{\partial x^4}+B(x,y,t)\frac{\partial^4 u}{\partial y^4}+\frac{\partial u}{\partial t} = f(x,y,t,u,u_x,u_{xx},u_{xxx},u_y,u_{yy},u_{yyy}),(x,y)\in\Omega,\ t>0 \quad (1)$$

$$u(x,y,0)=\varphi(x,y) \quad (2)$$

$$u(x,y,t)=p(t), \qquad (x,y)\in\partial\Omega, t\geq 0 \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2}=q(t), \qquad (x,y)\in\partial\Omega, t\geq 0 \quad (4)$$

$$\frac{\partial^2 u}{\partial y^2}=r(t), \qquad (x,y)\in\partial\Omega, t\geq 0 \quad (5)$$

in which $\varphi(x,y), p(t), q(t), r(t)$ are known functions.

By defining v as $\frac{\partial^2 u}{\partial x^2}=v$ and w as $\frac{\partial^2 u}{\partial y^2}=w$ the above initial-boundary problem can be written as follows.

$$\frac{\partial^2 u}{\partial x^2}=v \quad (6)$$

$$\frac{\partial^2 u}{\partial y^2}=w \quad (7)$$

$$A(x,y,t)\frac{\partial^2 v}{\partial x^2}+B(x,y,t)\frac{\partial^2 w}{\partial y^2}+\frac{\partial u}{\partial t}= f(x,y,t,u,u_x,v,v_x,u_y,w,w_y),(x,y)\in\Omega,\ t>0 \quad (8)$$

$$u(x,y,0)=\varphi(x,y) \quad (9)$$

$$u(x,y,t)=p(t), \qquad (x,y)\in\partial\Omega, t\geq 0 \quad (10)$$

$$v(x,y)=q(t), \quad (x,y)\in\partial\Omega, t\geq 0 \quad (11)$$

$$w(x,y)=r(t), \quad (x,y)\in\partial\Omega, t\geq 0 \quad (12)$$

With subject to the conditions:

The equation (1) along with its corresponding initial-boundary conditions (2)-(5) is a 2-D unsteady quasilinear biharmonic problem in which the mixed

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derivative terms have been omitted. So, its title comes after the word 'simplified'. The analytic solution of the initial-boundary problem can not be found for any arbitrary components, so numerical methods are applied to approach these problems numerically in their corresponding physical models. at the first time, an implicit finite difference scheme using three spatial grid points for solving 1-D unsteady quasilinear biharmonic problem was proposed by Mitchell [13] with *A*=1 and *f*=0. Based on this approach, we have constructed a numerical method for the simplified 2-D unsteady quasilinear biharmonic equation. In this method, combination of values of function *u* and its second derivatives respect to *x* and *y* is used to extract discrete difference formula. This combination causes fictitious points not to be appeared for boundary values in discrete scheme. Depending on the function *f*, the derived set of equations may be nonlinear. So, appropriate iterative methods can be useful for decreasing the computational cost of solving set of equations. In this paper, our numerical method will be introduced and analyzed due to convergence and stability criteria. Then, some numerical examples will support our numerical method.

This paper is organized as follows. In Section 2, the numerical scheme and its theoretical aspects are introduced. Error and stability considerations are given in Section 3 which is followed by some numerical examples validating our results. Section 4 deals with some concluding remarks and useful suggestions.

Discrete structures formulas, approaching our finite difference scheme, together with pseudo codes of the main system matrices, are presented in Appendices A and B respectively.

II. Description of difference scheme

In this section, we introduce the difference scheme, approaching the numerical solution of equations (1)-(5). For the sake of simplicity, the domain is considered as a unit square. The region $\Omega = [0,1]\times[0,1]$ is partitioned by rectangle cells as it is in all finite difference methods. Each rectangular cell is denoted by $\Omega_{i,j,k}$ which is used for presenting square with grid indices (*ih*, *jl*), ((*i+1)h*, *jl*), (*ih*,(*j+1)l*) and, ((*i+1)h*, *(j+1)l*) at k^{th} level of time. According to these splitting in space and time, we have (*N+1*)*h*=1, (*M+1*)*l*=1 for $0 \le i \le N+1, 0 \le j \le M+1$ respectively.

For introducing our numerical scheme, at first, proper approximations for function and its partial derivatives are presented and then, second order derivatives are approximated by second order finite difference method. Enforcement of initial and boundary conditions will be made at the end of this computational sequence. At the first stage like what can be seen in [13], some appropriate approximations are introduced for function elements of initial-boundary value problem, presented by equations (9)-(13).

Assume that $u_{i,j}^k$, $v_{i,j}^k$ and $w_{i,j}^k$ denote the approximation values $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ at point (*ih*, *jl*, *kn*) of the cell $\Omega_{i,j,k}$ corresponding to exact values $U_{i,j}^k$, $V_{i,j}^k$ and $W_{i,j}^k$ respectively; large variety of discrete set of equations corresponding to their coefficients can be derived in which all of matrices are strip or block tri-diagonal. By using suitable dedicated iterative methods, arisen discrete equations can be solved efficiently.

For example, let us consider a simple case of initial-boundary value problem (1)-(5) with $A(x,y,t) = B(x,y,t) = 1$ and simplified function *f*.

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial u}{\partial t} = f(x,y,t), \quad x \in \Omega,\ t > 0 \tag{13}$$

$$u(x,y,0) = \varphi(x,y) \tag{14}$$

$$u(x,y,t) = p(t), \qquad (x,y) \in \partial\Omega, t \ge 0 \tag{15}$$

$$\frac{\partial^2 u}{\partial x^2} = q(t), \qquad (x,y) \in \partial\Omega, t \ge 0 \tag{16}$$

$$\frac{\partial^2 u}{\partial y^2} = r(t), \qquad (x,y) \in \partial\Omega, t \ge 0 \tag{17}$$

By applying the mentioned approaches to the above system, the following matrix equation is obtained.

$$AX^{j+1} = (-A+B)X^j + b \tag{18}$$

or

$$X^{j+1} = (-I + A^{-1}B)X^j + A^{-1}b \tag{19}$$

In which, *A* and *B* are square matrices of size $3N^2$, *b* is the vector consisting of known values corresponding to initial and boundary equations and function *f*, and *X* is unknown variable vectors. Matrices *A* and *B* are block tri-diagonal sparse matrices.

III. Error estimations and stability considerations

A forth and a sixth order approximation method, for respectively space and time is modified and extended to solve a simplified 2-D unsteady biharmonic equation, numerically. In this section at first, the error estimations of the new discretization scheme obtained from discrete formulas are discussed and then, the stability considerations for some special cases will be presented.

III.1. Error estimations of the discretization approaches

In this part, we discuss how the new discretization formulas are gained, and what is the convergence behavior of the scheme?

According to the approximating discretization formulas, the initial-boundary problem (8)-(12) can be written by the following equations.

$$\delta_x^2 \overline{u}_{i,j}^k = \frac{h^2}{12}\left[\overline{v}_{i+1,j}^k + \overline{v}_{i-1,j}^k + 10\overline{v}_{i,j}^k\right] \tag{20}$$

$$\delta_y^2 \overline{u}^k_{i,j} = \frac{l^2}{12}\left[\overline{w}^k_{i,j+1} + \overline{w}^k_{i,j-1} + 10\overline{w}^k_{i,j}\right] \tag{21}$$

$$\begin{aligned}
&\left[\overline{A}^k_{i,j} - \frac{h^2}{12}\left(2\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\overline{A}^k_{x\,i,j} - \overline{A}^k_{xx\,i,j}\right)\right]\delta_x^2 \overline{v}^k_{i,j} \\
&+\left[\overline{B}^k_{i,j} - \frac{l^2}{12}\left(2\frac{\overline{B}^k_{y\,i,j}}{\overline{B}^k_{i,j}}\overline{B}^k_{y\,i,j} - \overline{B}^k_{yy\,i,j}\right)\right]\delta_y^2 \overline{w}^k_{i,j} \\
&+\frac{h^2}{24}\left[\left(1-h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i+1,j} + \left(1+h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i-1,j} + 10\overline{u}^k_{t\,i,j}\right] \\
&+\frac{l^2}{24}\left[\left(1-l\frac{\overline{B}^k_{y\,i,j}}{\overline{B}^k_{i,j}}\right)\overline{u}^k_{t\,i,j+1} + \left(1+l\frac{\overline{B}^k_{y\,i,j}}{\overline{B}^k_{i,j}}\right)\overline{u}^k_{t\,i,j-1} + 10\overline{u}^k_{t\,i,j}\right] \\
&=\frac{h^2}{24}\left[\left(1-h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{F}^k_{i+1,j} + \left(1+h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{F}^k_{i-1,j} + 10\overline{F}^k_{i,j}\right] \\
&+\frac{l^2}{24}\left[\left(1-l\frac{\overline{B}^k_{y\,i,j}}{\overline{B}^k_{i,j}}\right)\overline{F}^k_{i,j+1} + \left(1+l\frac{\overline{B}^k_{y\,i,j}}{\overline{B}^k_{i,j}}\right)\overline{F}^k_{i,j-1} + 10\overline{F}^k_{i,j}\right]
\end{aligned} \tag{22}$$

Considering convergence behavior of the above discrete approach is being done by calculating an upper bound for the local truncation error. This can easily be done by using 1-dimension and 2-dimension Taylor series expansion formula. Simplifying equation (22), it can be re-written efficiently by the following formula.

$$\begin{aligned}
&a^k_{i,j}\delta_x^2 \overline{v}^k_{i,j} + b^k_{i,j}\delta_y^2 \overline{w}^k_{i,j} + \frac{h^2}{24} \\
&\left[\begin{array}{l}\left(1-h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i+1,j} + \left(1+h\frac{\overline{A}^k_{x\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i-1,j} \\ +10\overline{u}^k_{t\,i,j}\end{array}\right] \\
&+\frac{l^2}{24}\left[\begin{array}{l}\left(1-l\frac{\overline{A}^k_{y\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i,j+1} + \left(1+l\frac{\overline{A}^k_{y\,i,j}}{\overline{A}^k_{i,j}}\right)\overline{u}^k_{t\,i,j-1} \\ +10\overline{u}^k_{t\,i,j}\end{array}\right] \\
&=\frac{h^2}{24}c^k_{i,j} + \frac{l^2}{24}d^k_{i,j}
\end{aligned} \tag{23}$$

By using the approximation formulas presented in Appendices A and B and following the method given in [13], differentiate equations (6)-(8) are differentiated with respect to t at point (ih, jl, kn). Omitting details, we have the following estimations.

$$AV_{txx} + A_tV_{xx} + BW_{tyy} + B_tW_{yy} + U_{tt} \tag{24}$$

$$\begin{aligned}
&= f_t + f_uU_t + f_{u_x}U_{tx} + f_vV_t + f_{v_x}V_{tx} \\
&+ f_{u_y}U_{ty} + f_wW_t + f_{w_y}W_{ty}
\end{aligned} \tag{25}$$

$$U_{txx} = V_t$$

$$U_{tyy} = W_t \tag{26}$$

Following relations are resulted according to Taylor expansion formula.

$$\overline{A}^{k}_{i,j} = A^{k}_{i,j} + \theta n A_t + O(n^2) \tag{27}$$

$$\overline{B}^{k}_{i,j} = B^{k}_{i,j} + \theta n B_t + O(n^2) \tag{28}$$

$$\overline{A}^{k}_{x_{i,j}} = A^{k}_{x_{i,j}} + \theta n A_{tx} + O(n^2) \tag{29}$$

$$\overline{B}^{k}_{y_{i,j}} = B^{k}_{y_{i,j}} + \theta n B_{ty} + O(n^2) \tag{30}$$

$$\overline{A}^{k}_{xx_{i,j}} = A^{k}_{txx_{i,j}} + \theta n A_{txx} + O(n^2) \tag{31}$$

$$\overline{B}^{k}_{yy_{i,j}} = B^{k}_{yy_{i,j}} + \theta n B_{tyy} + O(n^2) \tag{32}$$

$$\overline{u}^{k}_{i,j} = U^{k}_{i,j} + \theta n U_t + O(n^2) \tag{33}$$

$$\overline{v}^{k}_{i,j} = V^{k}_{i,j} + \theta n V_t + O(n^2) \tag{34}$$

$$\overline{w}^{k}_{i,j} = W^{k}_{i,j} + \theta n W_t + O(n^2) \tag{35}$$

$$\overline{u}^{k}_{i\pm1,j} = U^{k}_{i\pm1,j} + \theta n U_t + O(\pm nh + n^2) \tag{36}$$

$$\overline{u}^{k}_{i,j\pm1} = U^{k}_{i,j\pm1} + \theta n U_t + O(\pm nl + n^2) \tag{37}$$

$$\overline{v}^{k}_{i\pm1,j} = V^{k}_{i\pm1,j} + \theta n V_t + O(\pm nh + n^2) \tag{38}$$

$$\overline{v}^{k}_{i,j\pm1} = V^{k}_{i,j\pm1} + \theta n V_t + O(\pm nl + n^2) \tag{39}$$

$$\overline{w}^{k}_{i\pm1,j} = W^{k}_{i\pm1,j} + \theta n W_t + O(\pm nh + n^2) \tag{40}$$

$$\overline{w}^{k}_{i,j\pm1} = W^{k}_{i,j\pm1} + \theta n W_t + O(\pm nl + n^2) \tag{41}$$

$$\overline{u}^{k}_{t_{i,j}} = U^{k}_{t_{i,j}} + \frac{n}{2} U_{tt} + O(n^2) \tag{42}$$

$$\overline{u}^{k}_{t_{i\pm1,j}} = U^{k}_{t_{i\pm1,j}} + \frac{n}{2} U_{tt} + O(\pm nh + n^2) \tag{43}$$

$$\overline{u}^{k}_{t_{i,j\pm1}} = U^{k}_{t_{i,j\pm1}} + \frac{n}{2} U_{tt} + O(\pm nl + n^2) \tag{44}$$

$$\overline{u}^{k}_{x_{i,j}} = U^{k}_{x_{i,j}} + \theta n U_{tx} + \frac{h^2}{6} U_{xxx} + O(n^2 + nh^2 + h^4) \tag{45}$$

$$\overline{u}^{k}_{y_{i,j}} = U^{k}_{y_{i,j}} + \theta n U_{ty} + \frac{l^2}{6} U_{yyy} + O(n^2 + nl^2 + l^4) \tag{46}$$

$$\overline{v}^{k}_{x_{i,j}} = V^{k}_{x_{i,j}} + \theta n V_{tx} + \frac{h^2}{6} V_{xxx} + O(n^2 + nh^2 + h^4) \tag{47}$$

$$\overline{v}^{k}_{y_{i,j}} = V^{k}_{y_{i,j}} + \theta n V_{ty} + \frac{l^2}{6} V_{yyy} + O(n^2 + nl^2 + l^4) \tag{48}$$

$$\overline{w}^{k}_{x_{i,j}} = W^{k}_{x_{i,j}} + \theta n W_{tx} + \frac{h^2}{6} W_{xxx} + O(n^2 + nh^2 + h^4) \tag{49}$$

$$\overline{w}^{k}_{y_{i,j}} = W^{k}_{y_{i,j}} + \theta n W_{ty} + \frac{l^2}{6} W_{yyy} + O(n^2 + nl^2 + l^4) \tag{50}$$

$$\bar{u}^{k}_{x_{i\pm1,j}} = U^{k}_{x_{i\pm1,j}} + \theta n U_{tx} - \frac{h^2}{3} U_{xxx} + O(\pm nh \pm h^3 + n^2 + nh^2 + h^4) \tag{51}$$

$$\bar{u}^{k}_{x_{i,j\pm1}} = U^{k}_{x_{i,j\pm1}} + \theta n U_{tx} - \frac{l^2}{3} U_{xxx} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{52}$$

$$\bar{u}^{k}_{y_{i\pm1,j}} = U^{k}_{y_{i\pm1,j}} + \theta n U_{ty} - \frac{h^2}{3} U_{yyy} + O(\pm nh \pm h^3 + n^2 + nh^2 + h^4) \tag{53}$$

$$\bar{u}^{k}_{y_{i,j\pm1}} = U^{k}_{y_{i,j\pm1}} + \theta n U_{ty} - \frac{l^2}{3} U_{yyy} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{54}$$

$$\bar{v}^{k}_{x_{i\pm1,j}} = V^{k}_{x_{i\pm1,j}} + \theta n V_{tx} - \frac{h^2}{3} V_{xxx} + O(\pm nh \pm h^3 + n^2 + nh^2 + h^4) \tag{55}$$

$$\bar{v}^{k}_{x_{i,j\pm1}} = V^{k}_{x_{i,j\pm1}} + \theta n V_{tx} - \frac{l^2}{3} V_{xxx} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{56}$$

$$\bar{v}^{k}_{y_{i\pm1,j}} = V^{k}_{y_{i\pm1,j}} + \theta n V_{ty} - \frac{h^2}{3} V_{yyy} + O(\pm nh \pm h^3 + n^2 + nh^2 + h^4) \tag{57}$$

$$\bar{w}^{k}_{y_{i,j\pm1}} = W^{k}_{y_{i,j\pm1}} + \theta n W_{ty} - \frac{l^2}{3} W_{yyy} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{58}$$

$$\bar{w}^{k}_{x_{i,j\pm1}} = W^{k}_{x_{i,j\pm1}} + \theta n W_{tx} - \frac{l^2}{3} W_{xxx} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{59}$$

$$\bar{w}^{k}_{y_{i\pm1,j}} = W^{k}_{y_{i\pm1,j}} + \theta n W_{ty} - \frac{h^2}{3} W_{yyy} + O(\pm nh \pm h^3 + n^2 + nh^2 + h^4) \tag{60}$$

$$\bar{w}^{k}_{y_{i,j\pm1}} = W^{k}_{y_{i,j\pm1}} + \theta n W_{ty} - \frac{l^2}{3} W_{yyy} + O(\pm nl \pm l^3 + n^2 + nl^2 + l^4) \tag{61}$$

$$\bar{v}^{k}_{xx_{i,j}} = V^{k}_{xx_{i,j}} + \theta n V_{txx} + \frac{h^2}{12} V_{xxxx} + O(n^2 + nh^2 + h^4) \tag{62}$$

$$\bar{w}^{k}_{yy_{i,j}} = W^{k}_{yy_{i,j}} + \theta n W_{tyy} + \frac{l^2}{12} W_{yyyy} + O(n^2 + nl^2 + l^4) \tag{63}$$

By using relations (27)-(63), the following approximations for central difference operators presented in discretization formulas (20)-(22), are resulted.

$$\delta_x^2 \bar{u}^{k}_{i,j} = \delta_x^2 U^{k}_{i,j} + \theta n h^2 U_{txx} + O(n^2 h^2 + nh^4) \tag{64}$$

$$\delta_y^2 \bar{u}^{k}_{i,j} = \delta_y^2 U^{k}_{i,j} + \theta n l^2 U_{tyy} + O(n^2 l^2 + nl^4) \tag{65}$$

$$\delta_x^2 \bar{v}^{k}_{i,j} = \delta_x^2 V^{k}_{i,j} + \theta n h^2 V_{txx} + O(n^2 h^2 + nh^4) \tag{66}$$

$$\delta_y^2 \bar{w}^{k}_{i,j} = \delta_y^2 W^{k}_{i,j} + \theta n l^2 W_{tyy} + O(n^2 l^2 + nl^4) \tag{67}$$

$$e_1 = O(n^2 h^2 + nh^4 + h^6) \tag{68}$$

$$e_2 = O(n^2 l^2 + nl^4 + l^6) \tag{69}$$

$$\begin{aligned} e_3 &= \alpha_1(h,l,n,\theta,f,p,q,r)U_{tt} + \alpha_2(h,l,n,\theta,f,p,q,r)U_{xxx} + \alpha_3(h,l,n,\theta,f,p,q,r)V_{xx} \\ &+ \alpha_4(h,l,n,\theta,f,p,q,r)V_{xxx} + \alpha_5(h,l,n,\theta,f,p,q,r)U_{yyy} + \alpha_6(h,l,n,\theta,f,p,q,r)W_{yy} \\ &+ \alpha_7(h,l,n,\theta,f,p,q,r)W_{yyy} + O(n^2 h^2 l^2 + nh^4 l^4 + h^6 + l^6) \end{aligned} \tag{70}$$

By finding appropriate values for coefficients p, q, r and θ, the accuracy of suggested method is of order of $O(n^2h^2l^2+nh^4l^4+h^6+l^6)$.

III.2. Stability considerations of the method, applied to some simplified biharmonic-type equations

In this section, we consider the stability of difference formulas (20)-(22) applied to some equations of types (1)-(5). When difference formulas (20)-(22), are applied to linear equations (13)-(17), the following matrix equations are obtained.

$$AX^{j+1} = (-A+B)X^j + b \quad (71)$$

or

$$X^{j+1} = (-I + A^{-1}B)X^j + A^{-1}b \quad (72)$$

in which the matrices A and B are block diagonal matrices.

The essential idea defining stability is that the numerical process, applied exactly should limit the amplification of all components of the initial conditions. For linear initial-value and boundary-value problems, Lax and Richtmyer have related stability to convergence via Lax's equivalence theorem by defining stability, in effect, in terms of the boundedness of the solution of the finite difference equations at a fixed time level T as $n \to 0$. It is proven that the practical consequences of this definition of stability is that a norm of matrix $C = -I + A^{-1}B$, compatible with the norm of the unknown vector must be satisfied in $\|C\| \le 1$ when, the solution of the partial differential equation does not increase as t increases, or $\|C\| \le 1 + O(n)$ when the solution of the partial differential equation increases as t increases. These conditions also ensure the boundedness of all rounding errors because they are subject to the same arithmetic operations as the finite difference solution [14].

The coefficient matrix $C = -I + A^{-1}B$ is a sparse block matrix. So, the error equation corresponding to those equations can be written as $e^{j+1}=Ce^j$ in which e^j is a $3n^2$ column matrix representing the error vector of equation (72) at j^{th} iterate. Eigenvalues are often used as a source of information on stability or convergence, and the question arises as to validity of the information gained from these values [6]. Many authors have studied the problem of sensitivity of eigenvalues with respect to perturbations; see for instance [5], [7]-[8]. These studies are usually related to perturbations caused by rounding errors, and not so much by the relevance of the eigenvalues due to the particular presentation of a given problem for instance, the choice of basis. Recently there are some useful techniques and tools available for computing the eigenvalues of large sparse matrices. Calculating the eigenvalues of the sparse matrices can be efficiently done by calculating the correspondences of its building block sub-matrices. Generally, it should be noted that the coefficient matrix C of our problem is not simply a sparse matrix so; the techniques mentioned above are not applicable in this case. The conditional number of matrix C will be calculated in some numerical examples in the next section.

IV. Numerical experiments

In this section, we present some linear and quasilinear numerical examples to validate our proofs. Three numerical examples will be given to show the robustness of the new scheme. The first example is a quasilinear type and the others are linear. In all cases, region Ω is supposed to have rectangular shape in which exact solution of the correspondence PDE, boundary, and initial conditions are known to us. Discretization parameters in each example are determined according to stability criteria.

IV.1. Quasilinear problem

Table I
Quasilinear case: The error presentation corresponding to exact solution $U=(x+y)^4e^{-t}sin(t)$

h=0.001 , k=.01	Error bound norm-infinity	Error bound norm-2
T=0.1	8×10^{-7}	2×10^{-6}
T=1	2×10^{-5}	8×10^{-5}
T=5	9×10^{-5}	2×10^{-5}

Table II
Quasilinear case: The error presentations corresponding to exact solution $U=(x+y)^4e^{-t}sin(t)$

h=0.001 , k=.001	Error bound norm-infinity	Error bound norm-2
T=0.1	7×10^{-6}	2×10^{-5}
T=1	2×10^{-5}	8×10^{-5}
T=5	4×10^{-5}	10^{-5}

In this example, equations (1)-(5) with exact solution $U=(x+y)^4e^{-t}sin(t)$ and the following coefficients are solved via the above-mentioned method.

$$A(x, y, t)=x$$
$$B(x, y, t)=y$$

Discretization and linearization operations are done according to (20)-(22). The norm- error bounds corresponding to some specified time are calculated and shown in Tables I and II.

Table III
Linear case: The error bound and matrix Cond. number corresponding to exact solution $U=sin(x+y)t$

h=0.001 , k=.001	Error bound norm-infinity	Iteration matrix Cond. number
T=0.1	10^{-4}	1.021719800
T=1	10^{-3}	
T=5	9×10^{-3}	

IV.2. Linear problem

In these examples, the equations (1)-(5) with $A(x,y,t) = B(x,y,t) = 1$ are solved with some linear coefficients whose exact solutions, are known to us.

IV.2.1. Equation (1)-(5) with exact solution U=sin(x+y)t

Like the last example the norm-2 and norm error bounds and also conditional number of the iteration matrix for some short time solutions are calculated and shown in Tables III and IV.

Table IV
Linear case: The error bound and matrix Cond. number corresponding to exact solution $U=sin(x+y)t$

h=0.001 , k=.01	Error bound norm	Iteration matrix Cond. number
T=0.1	3×10^{-4}	1.003821651
T=1	2×10^{-3}	
T=5	10^{-2}	

Calculations also have been done for long time solution with different step sizes. Obviously, it should be noted that for long time numerical solution, one must care more about the altitude of the conditional number of the iteration matrix.

Table V
Linear case: The error bound and matrix Cond. number corresponding to exact solution $U=sin(x+y)t$

h=0.00031 , k=.08	Error bound norm	Iteration matrix Cond. number
T=10	6×10^{-3}	1.077179041
T=50	3×10^{-2}	
T=100	2×10^{-1}	
T=200	Not negligible	
T=500	Not negligible	

IV.2.1. Equation (1)-(5) with exact solution $U=(x+y)^4e^{-t}sin(t)$

Table VI
Linear case: The error bound and matrix Cond. number corresponding to exact solution $U=(x+y)^4e^{-t}sin(t)$

h=0.001 , k=.001	Error bound norm	Iteration matrix Cond. number
T=0.1	7×10^{-6}	1.021719800
T=1	2×10^{-5}	
T=5	5×10^{-7}	

Like the former example, short and long time solutions are calculated for the latter problem, shown in Tables VI and VII respectively.

Table VII
Linear case: The error bound and matrix Cond. number corresponding to exact solution $U=(x+y)^4e^{-t}sin(t)$

h=0.0008 , k=.01	Error bound norm	Iteration matrix Cond. number
T=10	5×10^{-7}	1.005075487
T=50	5×10^{-7}	
T=100	5×10^{-7}	
T=200	5×10^{-7}	
T=500	5×10^{-7}	

V. CONCLUSION

In this paper, one type of discretization methods is introduced and extended for numerically solving a simplified 2-D unsteady quasilinear biharmonic problem. This method is an implicit method which does not use fictitious points. Therefore, there are no any other approximations required for estimating those fictitious points. The numerical results of this new scheme are compared with the analytical solution of corresponding problems for both linear and quasilinear ones. In both cases, achieving to a reasonable accuracy along with the stability and the convergence considerations certified the new method successfully.

Extending the new method for finding numerical solution of 2-D unsteady quasilinear biharmonic problem and, with more complicated nonlinear terms and also similar equations suggested as future works.

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