Proceedings of the International MultiConference of Engineers and Computer Scientists 2009 Vol II,
IMECS 2009, March 18 - 20, 2009, Hong Kong

On Natural Frequencies and Mode Shapes of Microbeams

Amir M. Lajimi*, Eihab Abdel-Rahman†, and Glenn R. Heppler ‡

Abstract–**Microbeams, a significant category of inertial sensors, are used in a variety of applications including pressure sensors and accelerometers. This study aims to develop a simplified computational model of the microbeam for computing the natural frequencies and mode shapes of the microstructure. The finite element equations are formed by employing a variational principle including the end-mass as a point mass in the global mass matrix. The natural frequencies of the cantilever microbeam are calculated and compared with accessible closed-form analytical solution.**

Keywords: microbeam, finite element model, natural frequency, mode shape

1 Introduction

Micro-Electro-Mechanical-Systems (MEMS) are divided into three categories: structures with no moving parts, sensors, and actuators. Micro-machined inertial sensors constitute a group utilized for measuring linear or angular acceleration. Production cost has been a considerable factor in manufacturing MEMS, and has therefore limited broad application of the microsensors. By reducing the cost of micromachining, MEMS are becoming more accessible. MEMS are currently used in a wide variety of applications for the automotive industries, such as suspension systems and seat belt control systems.

In precision measurement sensors such as accelerometers, the strain-sensitive component conventionally made of silicon pizoresistors can be altered into a combination of polysilicon microbeams and flexures with a proof mass. This directly converts the acceleration signal into the frequency [1].

There have been an ongoing effort in the field of beam vibration to understand the dynamic characteristics of beams with end-mass, e.g. see [2], [3], and [4]. Laura et al. [5] derived the frequency equation for a clamped-free beam with an end finite mass. Mermertaş and Gürgöze considered a clamped-free beam with a mass-spring system attached to its free-end [6]. The frequency equation of a cantilever with an end rigid mass was derived by Rama Bhat and Wagner [7].

Esmaeili et al. [2] studied the kinematics and dynamics of a clamped-free Euler-Bernoulli microbeam with tip-mass. To imitate the actual application, the base excitation was included in the model, while the torsional vibration was disregarded. According to Esmaeili et al., an objective in modeling micro-gyroscopes with vibrating beams is to quantify the rate of rotation about the longitudinal axis. Therefore, the rotation about the longitudinal axis was included in the formulation. The authors concluded that the natural frequencies of the system with the rotation about the longitudinal axis were similar to those of the system which does not rotate about the same axis [2].

A schematic diagram of the cantilever microbeam is shown in Figure 1. It is noted that the end-mass vibrates in various directions. Such a presentation is typically used to describe the structure of microaccelerometers and microgyroscopes. The aforementioned sensors are widely used in industrial applications.

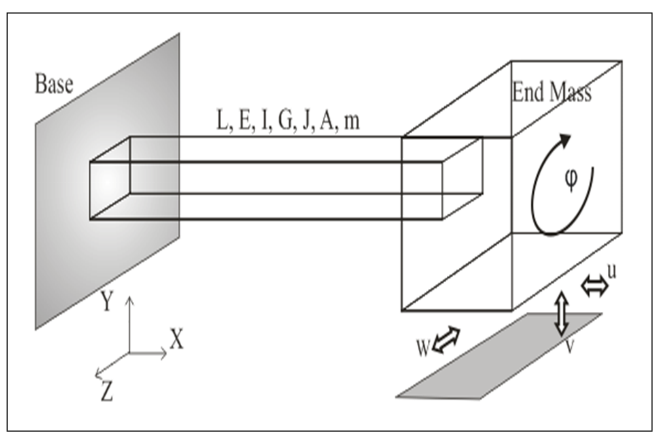

Figure 1: A schematic figure of the microbeam with the end-mass

*S. Amir M. Lajimi (corresponding author), PhD candidate, is with the Department of Systems Design Engineering, University of Waterloo, 200 University Avenue West Waterloo, Ontario, Canada, N2L 3G1, Email: samousav@uwaterloo.ca

†Eihab Abdel-Rahman, associate professor, is with the Department of Systems Design Engineering, University of Waterloo, Canada, N2L 3G1

‡Glenn R. Heppler, professor, is with the Department of Systems Design Engineering, University of Waterloo, Canada, N2L 3G1

ISBN: 978-988-17012-7-5 (revised on 20 September 2011) IMECS 2009
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

2 Finite element modeling

The finite element equations do not include the coupling effects of the axial, torsional, and bending vibrations. The microbeam has a constant squared cross-section with a uniform mass density, Figure 1. The material and geometrical characteristics of the beam are chosen to be the same as those of Esmaeili et al. [2], Table 1. The shear modulus is equal to 80×10^9 Pa.

The finite element analysis is carried out on a one dimensional model of the microbeam, Figure 2. Despite the geometric simplicity of the finite element model, such a model is reasonably expected to imitate the major dynamic behavior of the actual microstructure.

By simultaneously studying the axial, torsional, and flexural vibrations of the microbeam, the scale effects are not ignored, a priori. The element trial function in terms of the basis functions and the undetermined nodal variables is given by

$$\tilde{d}(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \\ w(x,t) \\ \varphi(x,t) \end{pmatrix} = [N(x,t)]\, \tilde{d}_e(t) \tag{1}$$

where $\tilde{d}(x,t)$, $[N(x,t)]$, and $\tilde{d}_e(t)$ represent the element trial function, the matrix of the element basis functions, and the vector of the undetermined nodal variables, respectively. Linear Lagrange polynomials and Hermite cubic basis functions are employed to interpolate the axial and flexural displacements, respectively. The end-mass is considered to be concentrated at the tip of the microbeam with no rotary inertia, Figure . Therefore, the end-mass is assembled to the global mass matrix as a point mass. For numerical integration, Gauss-Legendre quadrature method is applied with two, three, and four integration points.

In micro-scale, the relative magnitude of material properties and geometrical dimensions becomes an influencing parameter. It follows that using actual values of the parameters results in incorrect computations of the natural frequencies and mode shapes. Therefore, a set of nondimensional parameters are used to transform the system of equations to the dimensionless form. The main parameters include

Description	*Numerical value*
Length	0.0004 m
Mass density	$2300\,\mathrm{kg/m^3}$
Modulus of elasticity	160×10^9 Pa
Linear mass density	1.803×10^{-8} kg/m
End-mass	7.2128×10^{-12} kg
Area moment of inertia	$5.1221 \times 10^{-24}\,\mathrm{m^4}$

Table 1: Microbeam specifications [2]

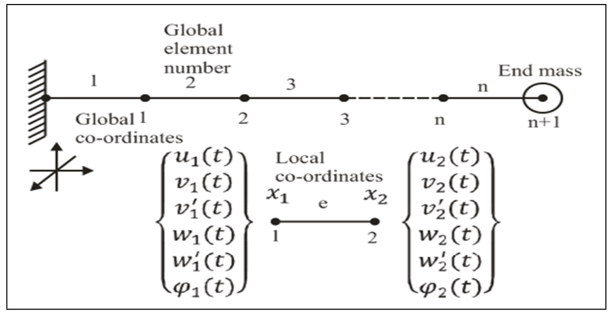

Figure 2: One dimensional finite element model of the microbeam with the end-mass

$$t = \kappa\tau \tag{2}$$
$$x = h\xi \tag{3}$$
$$v = h\vartheta \tag{4}$$

where $\kappa = \sqrt{\frac{\rho A h^4}{EI}}$ is the time constant and h is the width (=height) of the cross-section of the beam. Then the spatial derivatives in energy expressions become

$$\begin{aligned} \frac{\partial v}{\partial x} &= h\frac{\partial \vartheta}{\partial x} \\ &= h\frac{\partial \vartheta}{\partial \xi}\frac{\partial \xi}{\partial x} \\ &= \frac{\partial \vartheta}{\partial \xi} \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial \vartheta}{\partial \xi}) \\ &= \frac{\partial}{\partial \xi}(\frac{\partial \vartheta}{\partial \xi})\frac{\partial \xi}{\partial x} \\ &= \frac{1}{h}\frac{\partial^2 \vartheta}{\partial \xi^2} \end{aligned} \tag{6}$$

Assuming harmonic displacements and substituting $t = \kappa\tau$ in

$$\tilde{d} = \underline{\tilde{d}}\exp^{i\omega t} \tag{7}$$

results in

$$\left(\left[\hat{K}\right] - (\omega\kappa)^2\left[\hat{M}\right]\right)\underline{\tilde{d}} = \underline{0} \tag{8}$$

The kinetic energy of the end-mass is included in the energy expression in the form of

$$T_t = \frac{1}{2}M_t\left(\frac{\partial v(L,t)}{\partial t}\right)^2 \tag{9}$$

The nondimensional end-mass parameter is given by

$$\hat{M}_t = \frac{M_t}{\rho h^3} \tag{10}$$

where ρ denotes the mass density of the beam. Forming the mass and stiffness matrices in dimensionless form and using time scaling improves the stability of the numerical procedure.

3 Results

The analytical relations for computing the natural frequencies of a cantilever beam without tip-mass have been provided in numerous resources, e.g. [8] and [9]. The convergence of the finite element analysis is confirmed by analytical solutions in various cases, Figures 3, 4, and 5.

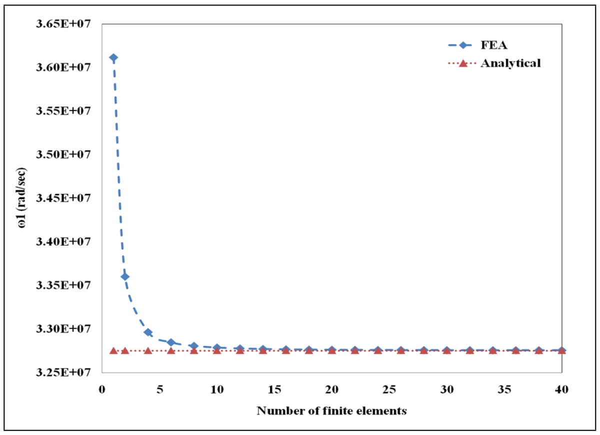

Figure 3: First axial natural frequency

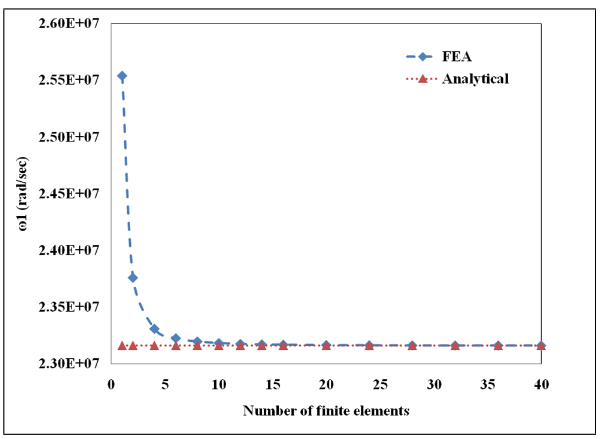

Figure 4: First torsional natural frequency

The effect of adding the end-mass on the first and second axial, torsional, and flexural mode shapes are illustrated in Figures 6, 7, 8, 9, 10, and 11, respectively. The parameter λ is defined as $M_t = 7.2128 \times 10^{-13}\,\lambda$. The impact of the end-mass on the axial and torsional vibration modes is more pronounced than the result of adding end-mass on the bending mode shapes. The effects of adding an end-mass, which is equal to 7.2128×10^{-12} kg,

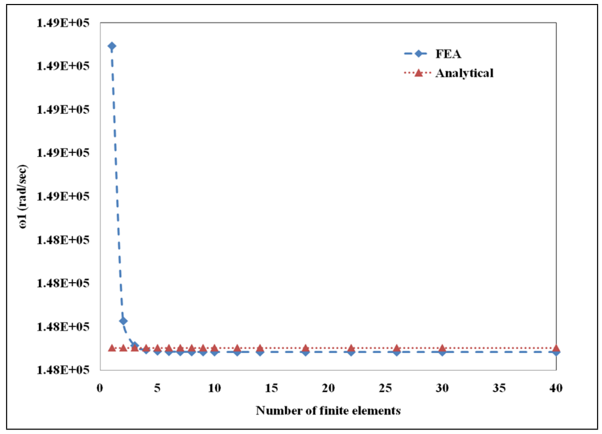

Figure 5: First flexural natural frequency

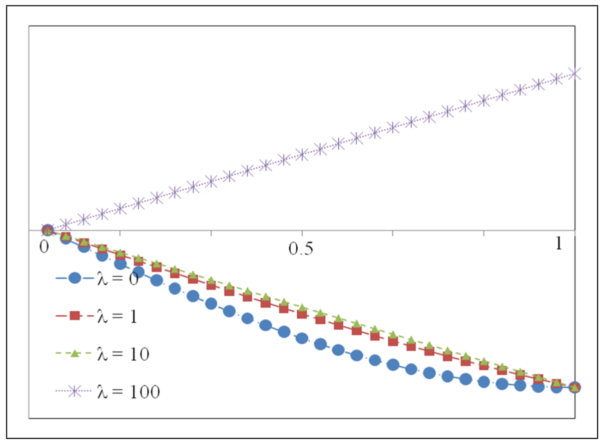

Figure 6: First mode shape, axial vibration

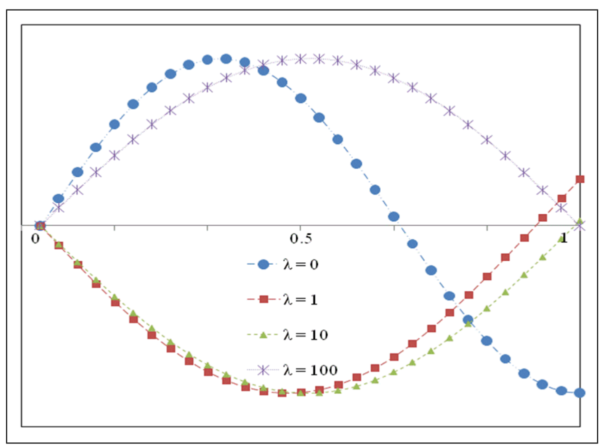

Figure 7: Second mode shape, axial vibration

on the natural axial, torsional, and flexural frequencies are demonstrated and compared in Figure 12. The end-mass reduces the fundamental and second natural frequencies of the microbeam.

ISBN: 978-988-17012-7-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

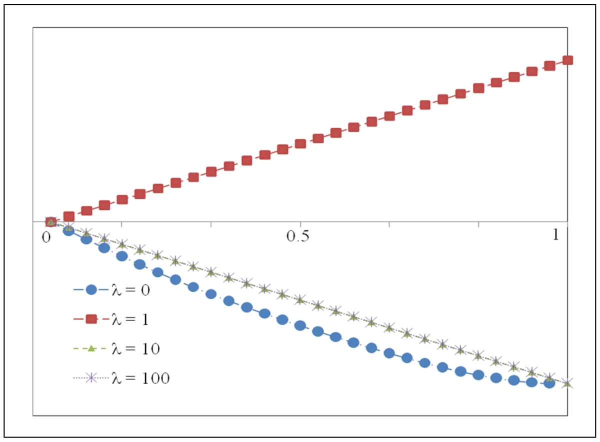

Figure 8: First mode shape, torsional vibration

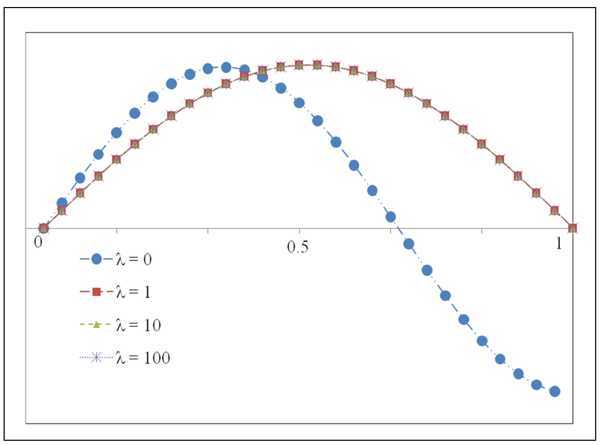

Figure 9: Second mode shape, torsional vibration

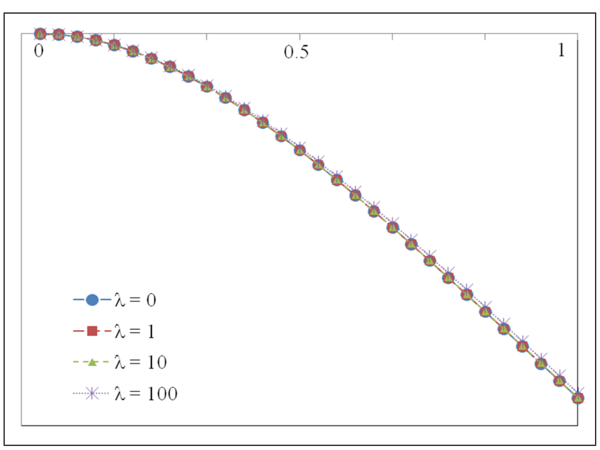

Figure 10: First mode shape, bending vibration

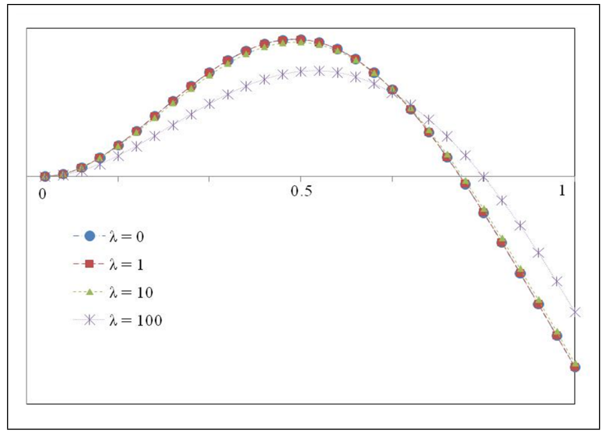

Figure 11: Second mode shape, bending vibration

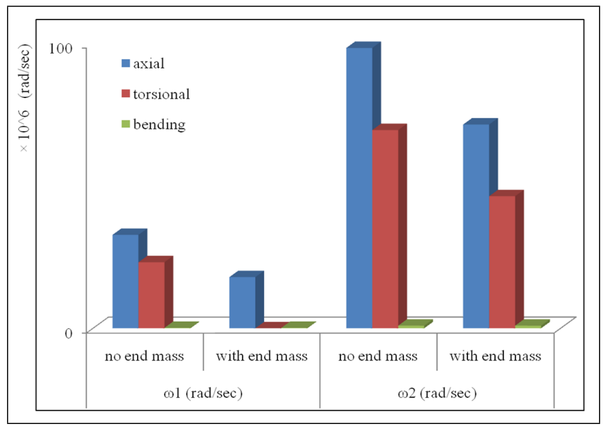

Figure 12: The end-mass effect on the natural frequencies

4 Conclusions and future works

The finite element model has been successfully employed to compute the natural frequencies and modes shapes of the axial, torsional, and flexural vibration of the microbeam. The convergence of the finite element analysis has been verified by available analytical closed-form solutions. When the tip-mass is removed, the results monotonically converge to the exact solutions. As a result of this analysis, the torsional vibration of a clamped-free microbeam cannot be neglected, a priori.

In future studies on the microbeam, the effects of the coupling between axial, torsional, and flexural vibrations could be investigated. A combination of the torsional and bending deformations results in Coriolis forces which may significantly affect the mode shapes. In order to have a realistic model, the effects of the base excitation should be included in the model. Finally, a full three dimensional analysis of the cantilever microbeam could provide an accurate representation of the natural frequencies and mode shapes of the device.

References

[1] Zook, J. D., Burns, D. W., Guckel, H., Sniegowski, J. J., Engelstad, R. L., and Feng, Z., 1992. "Characteristics of polysilicon resonant microbeams". *Sensors and Actuators A: Physical,* ***35***(1), pp. 51–59.

[2] Esmaeili, M., Jalili, N., and Durali, M., 2007. "Dynamic modeling and performance evaluation of a vibrating beam microgyroscope under general support motion". *Journal of Sound and Vibration,* ***301***(1-2), pp. 146–164.

[3] Bhadbhade, V., Jalili, N., and Nima Mahmoodi, S., 2008. "A novel piezoelectrically actuated flexural/torsional vibrating beam gyroscope". *Journal of Sound and Vibration,* ***311***(3-5), pp. 1305–1324.

[4] Vyas, A., Peroulis, D., and Bajaj, A., 2008. "Dynamics of a nonlinear microresonator based on resonantly interacting flexural-torsional mode". *Nonlinear Dynamics,* ***54***, p. 3152.

[5] Laura, P. A. A., Pombo, J. L., and Susemihl, E. A., 1974. "A note on the vibrations of a clamped-free beam with a mass at the free end". *Journal of Sound and Vibration,* ***37***(2), pp. 161–168.

[6] Mermertaş, V., and Gürgöze, M., 1997. "Longitudinal vibrations of rods coupled by a double spring-mass system". *Journal of Sound and Vibration,* ***202***(5), pp. 748–755.

[7] Rama Bhat, B., and Wagner, H., 1976. "Natural frequencies of a uniform cantilever with a tip mass slender in the axial direction". *Journal of Sound and Vibration,* ***45***(2), pp. 304–307.

[8] Lalanne, M., Berthier, P., and Der Hagopian, J., 1983. *Mechanical Vibrations for Engineers.* Wiley, New York, NY.

[9] Thomson, W. T., 1981. *Theory of Vibration with Applications*, 2nd edition ed. Prentice-Hall, Englewood Cliffs, NJ.

Notes: This article has been revised on September 5, 2011. The revision includes correcting typographical errors, paraphrasing and summarizing some of the sentences, changing some of the captions of the figures, and correcting reference style.