LDPC Codes by Circulant Decomposition Based on Difference Family

Zhihua Du and Zhen Ji

Abstract—**We consider in this paper decomposing quasi-cyclic low-density parity check codes (LDPC) based on difference family. The resulting codes can be encoded with low complexity and perform well when iteratively decoded with the sum-product algorithm.**

***Index Terms*—quasi-cyclic codes, difference families and low-density parity-check codes.**

I. Introduction

Low density parity-check (LDPC) codes were first presented by Gallager [1] in 1962 and have created much interest recently when rediscovered and shown to perform remarkably close to the Shannon limit. Decoding with the sum-product algorithm requires only that the parity-check matrix, H , be sparse. However, decoding performance can often be improved if the code is also free of 4-cycles, which occur if two code bits are both checked by the same pair of parity-check equations. Gallager described regular codes, defined by parity-check matrices with constant column and row weights, which were constructed pseudo-randomly to avoid 4-cycles [1]. Randomly realized finite length irregular LDPC codes with block size on the order of 10^4 approach their density evolution threshold closely at rate 1/2. While optimized irregular codes are capable of excellent performance with reasonable decoding complexity, one of the main hurdles in the implementation of LDPC codes is the computational complexity of the encoding algorithm. Encoding is, in general, performed by matrix multiplication and so complexity is quadratic in the code length. One option for efficient encoding is to use algebraic code constructions and exploit the subsequent code structure. In the case of regular codes a number of algebraic constructions have been presented, such as in [2], [3], [4]. Less consideration however has been given to structured irregular codes. The aim of this paper is to give a new construction of irregular quasi-cyclic codes free of 4 cycles by circulant decomposition using difference families to improve their performance with sum-product decoding.

This paper is organized as follows: In section II and section III quasi-cyclic codes and different family are shown respectively, then we give the method of decomposing quasi-cyclic codes in section IV. Finally we present simulation results and conclusion in section V.

II. Quasi-Cyclic Codes

A code is quasi-cyclic if, for any cyclic shift of a codeword by $p \neq 1$ places, the resulting word is also a codeword [5]. A cyclic code is a quasi-cyclic code with $p = 1$. We consider binary quasi-cyclic codes described by a parity-check matrix

$$H = \lfloor B_1, B_2, \ldots B_p \rfloor \qquad (1)$$

where $B_1, B_2, \ldots B_p$ are binary $v \times v$ circulant matrices. Provided that one of the Circulant matrices is invertible (say B_p) the generator matrix for the code can be constructed in systematic form

$$G = \begin{bmatrix} & (B_p^{-1} B_1)^T \\ I_{v(p-1)} & (B_p^{-1} B_2)^T \\ & \\ & (B_p^{-1} B_{p-1})^T \end{bmatrix} \qquad (2)$$

resulting in a quasi-cyclic code of length vp and dimension $v(p-1)$. Encoding can be achieved with linear complexity using $v(p-1)$ -stage shift register in much the same way as for cyclic codes [5].

The algebra of $v \times v$ binary Circulant matrices is isomorphic to the algebra of polynomials modulo $x^v - 1$ over GF (2) [5]. A Circulant matrix B is completely characterized by the polynomial $a(x) = a_0 + a_1 x + \cdots + a_{v-1} x^{v-1}$ with coefficients from its first row, and a code C with parity-check matrix of the form (1) is completely characterized by the polynomials $a_1(x), \ldots, a_p(x)$. Polynomial transpose is defined as

$$a(x)^T = \sum_{i=0}^{n-1} a_i x^{n-i}, x^n = 1.$$

For a binary [n, k] code, length $n = vp$ and dimension $k = v(p-1)$, the $k-bit$ message $\lfloor i_0, i_1, \ldots i_{k-1} \rfloor$ is described by the polynomial $i(x) = i_0 + i_1 x + \ldots + i_{k-1} x^{k-1}$ and the codeword for this message is $c(x) = i(x) \cdot x^k p(x)$ where $p(x)$ is given by

$$p(x) = \sum_{j=1}^{p-1} i_j(x) * (a_p^{-1}(x) * a_j(x))^T \qquad (3)$$

$i_j(x)$ is the polynomial representation of the information bits $i_{v(j-1)}$ to i_{vj-1}

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$$i_j(x) = i_{v(j-1)} + i_{v(j-1)+1}x + \ldots + i_{vj-1}x^{v-1}$$

and polynomial multiplication ($*$) is mod $x^v - 1$.

As an example, consider a rate-1/2 quasi-cyclic code with $v = 5, p = 2$, first circulant described by $a_1(x) = 1 + x$ and second Circulant described by $a_2(x) = 1 + x^2 + x^4$ which is invertible

$$a_2^{-1}(x) = x^2 + x^3 + x^4 .$$

The generator matrix contains a 5×5 matrix described by the polynomial

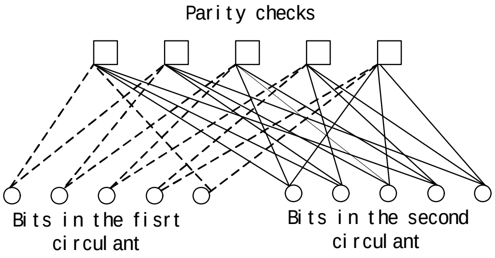
$$(a_2^{-1}(x) * a_1(x))^T = (1 + x^2)^T = 1 + x^3$$

$$H = \left[\begin{array}{ccccc|ccccc} 1 & 1 & & & & 1 & & 1 & & 1 \\ & 1 & 1 & & & 1 & 1 & & 1 & \\ & & 1 & 1 & & & 1 & 1 & & 1 \\ & & & 1 & 1 & 1 & & 1 & 1 & \\ 1 & & & & 1 & & 1 & & 1 & 1 \end{array}\right]$$

a) Parity-check matrix with two circulants

$$G = \left[\begin{array}{ccccc|ccccc} 1 & & & & & 1 & & & 1 & \\ & 1 & & & & & 1 & & & 1 \\ & & 1 & & & 1 & & 1 & & \\ & & & 1 & & & 1 & & 1 & \\ & & & & 1 & & & 1 & & 1 \end{array}\right]$$

b) Generator matrix in systematic form



c) Tanner graph representation

III. DIFFERENCE FAMILY

A difference family is an arrangement of a group of v elements such as Z_v into not necessarily disjoint subsets of equal size, which meet certain difference requirements.

Definition: The t $\gamma -$element subsets of the group $Z_v, D_1, D_2, \ldots D_t$ with $D_i = \{d_{i,1}, d_{i,2}, \ldots d_{i,\gamma}\}$ form a (v, γ, λ) difference family if the differences $d_{i,1} - d_{i,y}$, ($i = 1, \ldots t; x, y = 1, \ldots, \gamma, x \neq y)$ given each nonzero element of Z_v exactly λ times.

For a quasi-cyclic code we define the column weight distribution of a length vp rate $p - 1/p$ code as the vector $W = \lfloor w_1, w_{2,} \ldots w_p \rfloor$ where w_j is the column weight of the columns in the jth circulant.

*Construction*1: To construction a length vp rate $(p-1)/p$ quasi-cyclic code, $H = \lfloor a_1(x), a_2(x), \ldots, a_p(x) \rfloor$, with weight distribution $W = \lfloor w_1, w_2, \ldots w_p \rfloor$, take p sets $D_1, \ldots D_p$ of a $(v, \gamma, 1)$ difference family ,as such that $a_j(x)$ is defined using w_j of the elements of D_j as

$$a_j(x) = x^{d_{j,1}} + x^{d_{j,2}} + \ldots + x^{d_{j,w_j}}$$

To ensure invertibility at least one $a_j(x)$ must divide $x^v - 1$.

*Lemma*1: A pair of elements from Z_v occur together exactly λ times in the set of translates of every set in a (v, γ, λ) difference family.

*Lemma*2: The codes of the Construction 1 have Tanner graphs free of 4-cycles.

Proof: [6].

IV. CIRCULANT DECOMPOSING

Consider a $v \times v$ circulant B_1 over GF (2) with column and row weight w. Because column and row weights of a circulant are the same, for simplicity, we say that B_1 has weight w. For $1 \leq t \leq w$, let $w_1, w_2, \ldots w_t$ be a set of positive integers such that $1 \leq w_{1,} w_2, \ldots w_t \leq w$, and $w_1 + w_2 + \ldots + w_t = w$. Then we can decompose B_1 into t $v \times v$ circulants with weights $w_1, w_2, \ldots w_t$, respectively. Let b_1 be the first column of B_1. We split b_1 into t columns of the same length v, denoted by $b_1^{(1)}, b_1^{(2)}, \ldots b_1^{(t)}$, such that the first w_1 1-components of b_1 are put in $b_1^{(1)}$, the next w_2 1-components of b_1 are put in $b_1^{(2)}, \ldots$, and the last w_t 1-components of b_1 are put in $b_1^{(t)}$. For each new column $b_1^{(i)}$, we form a $v \times v$ circulant $B_{1,i}$ by cyclically shifting $b_1^{(i)}$ downward v times. This results in t $v \times v$ circulants, $B_{11}, B_{12}, \ldots B_{1t}$, with weights $w_1, w_2, \ldots w_t$ respectively. These circulants are called the descendants of B_1. Such a decomposition of B_1 is called column decomposition of B_1. Column decomposition of B_1 results in a $v \times tv$ matrix

$$B_1 = [B_{11} B_{12} \cdots B_{1t}],$$

which is a row of t $v \times v$ circulants. The parameter t is called the column splitting factor. If $t = w$ and $w_1 = w_2 = \ldots = w(t) = 1$ then each descendant circulant B_{1i} of B is a permutation matrix and B is a row of t permutation matrices.

Figure 1 shows a column decomposition of a 5×5 circulant of weight 3 into two descendants with weights 2 and

1, and descendant matrices can be described by $1+x^2, x^4$, respectively.

$$B_1 = \begin{bmatrix} 1 & & 1 & & 1 \\ 1 & 1 & & 1 & \\ & 1 & 1 & & 1 \\ 1 & & 1 & 1 & \\ & 1 & & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & & 1 & & \\ & 1 & & 1 & \\ & & 1 & & 1 \\ 1 & & & 1 & \\ & 1 & & & 1 \end{bmatrix} \begin{bmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$$

Fig 1 A column decomposition of a circulant of weight 3

If no two columns in B_1 have more than one 1-component in common, then no two columns in B_1 have more than one 1-component in common. In this case the null space of B_1 gives a quasi-cyclic LDPC code whose Tanner graph is free of cycles of length 4. If B_1 is a sparse matrix, B_{11} is also a sparse matrix with smaller density than B_1. So after decomposing we extended the H matrix to get the smaller density matrix with a minimum distance of at least 6.

V. CONCLUSION

Using the (101,5,1) difference family from [7],

$$D_1 = \{0,14,42,47,55\}, D_2 = \{0,95,83,52,63\}$$

$$D_3 = \{0,17,51,21,74\}, D_4 = \{0,36,7,92,26\}$$

$$D_5 = \{0,100,98,76,71\}$$

the quasi-cyclic irregular LDPC codes have been constructed:

a rate-2/3,[303,202] code with $a_1 = x^{D_3}$ decomposing to descendant matrices expressed by $b_{11} = 1+x^{17}, b_{12} = x^{21}+x^{51}, b_{13} = x^{74}$

a rate-3/4,[404,303] code with $a_1 = x^{D_3}$ decomposing to descendant matrices expressed by $b_{11} = 1+x^{51}, b_{12} = x^{17}, b_{13} = x^{21}, b_{14} = x^{74}$

a rate-6/7,[707,606] code with $a_1 = x^{D_4}, a_2 = x^{D_2}$ decomposing to descendant matrices expressed by $b_{11} = 1+x^{92}, b_{12} = x^{7}, b_{13} = x^{26}, b_{14} = x^{36}$,

$b_{21} = 1+x^{52}, b_{21} = x^{63}+x^{83}, b_{31} = x^{95}$

where $x^{D_j} = x^{d_{j,1}} + \cdots + x^{d_{j,\gamma}}$

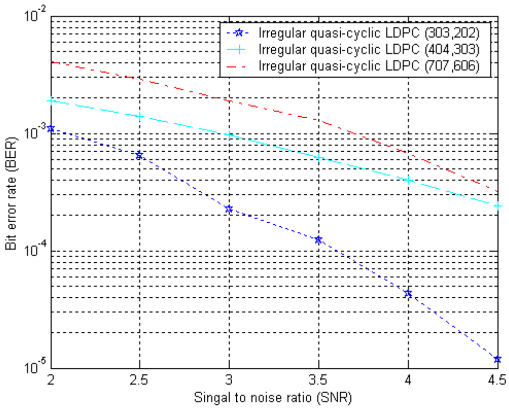


Fig 2. Error correction performance of LDPC codes on an AWGN channel. The rate-2/3 [303,202] irregular quasi-cyclic code with W=[2,2,1], the rate-3/4 [404,303] irregular quasi-cyclic code with W=[2,1,1,1] and the rate-6/7 [707,606] irregular quasi-cyclic code with W=[2,1,1,1,2,2,1]

These new codes are compared to regular quasi-cyclic codes based on difference family. The decoding performance of the quasi-cyclic codes shown in Fig2, Fig 3 presents that it is a modest performance gain to be made over the regular quasi-cyclic codes by using column decomposition method to get irregular quasi-cyclic LDPC codes. Further it has the advantage of a reduced encoding complexity.

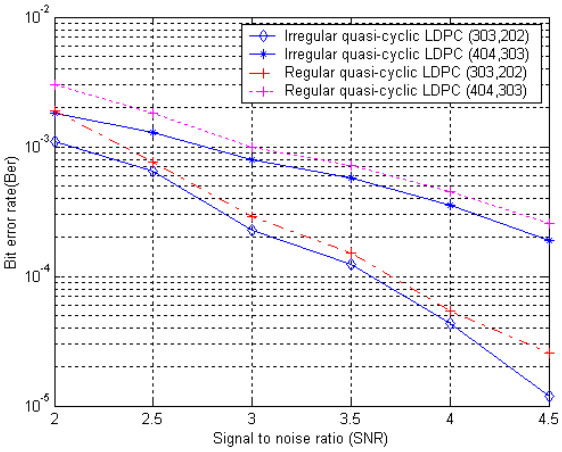


Fig 3 Error correction performance of LDPC codes on an AWGN channel. The rate-2/3 [303,202] irregular quasi-cyclic code with W=[2,2,1] and the rate-3/4 [404,303] irregular quasi-cyclic code with W=[2,1,1,1] compared regular quasi-cyclic LDPC codes.

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