Proceedings of the International MultiConference of Engineers and Computer Scientists 2009 Vol I
IMECS 2009, March 18 - 20, 2009, Hong Kong

Three-player Col played on trees is $\mathcal{NP}$-complete

Alessandro Cincotti *

Abstract— **The game of Col is a two-player map-coloring game invented by Colin Vout where to establish who has a winning strategy on a general graph is a $\mathcal{PSPACE}$-complete problem. However, winning strategies can be found on specific graph instances, e.g., k-ary complete trees. Three-player Col is a three-player version of Col. Because of the possibility to form alliances, cooperation between players is a key-factor to determine the winning coalition and, as a result, three-player Col played on trees is $\mathcal{NP}$-complete.**

Keywords: combinatorial games, $\mathcal{NP}$-complete, three-player Col

1 Introduction

Col is a map-coloring game invented by Colin Vout. Every instance of this game is defined as an undirected graph $G = (V, E)$ where every vertex is uncolored, black or white.

Two players, First and Second, play in turn and First starts the game. In the beginning, all the vertices are uncolored. First has to paint an uncolored vertex using the color black, and Second has to paint an uncolored vertex using the color white. There exists only one restriction: two adjacent vertices cannot be painted with the same color. In normal play, the first player unable to paint an uncolored vertex is the loser.

The value of some Col positions and the description of some general rule for simplifying larger positions is presented in [1], [2]. Recently, the game has been solved on complete k-ary trees [3] and on a specific class of trees where all the internal nodes have at least two children and the depth of all the leaves is either even or odd [4]. Moreover, Col is proved to be a $\mathcal{PSPACE}$-complete problem on a general graph [5].

In this paper we show that in three-player Col, because of the possibility to form alliances, cooperation between players is a key-factor to determine the winning coalition and, as a result, three-player Col played on trees is $\mathcal{NP}$-complete.

*A. Cincotti is with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan (phone: +81-761-51-1293; fax: +81-761-51-1149; e-mail: cincotti@jaist.ac.jp).

2 Three-player Col

Three-player Col is a three-player version of Col. Every instance of this game is defined as an undirected graph $G = (V, E)$ where every vertex is labeled by an integer $j \in \{1, 2, 3\}$. Three players, First, Second and Third, take turns making legal moves in cyclic fashion (First, Second, Third, First, Second, Third, ...) and First starts the game.

In the beginning, all the vertices are unlabeled. First has to label an unlabeled vertex with "1", Second has to label an unlabeled vertex with "2", and Third has to label an unlabeled vertex with "3". There exists only one restriction: two adjacent vertices cannot be labeled with the same number.

In normal play, if one of the players is unable to move, then he/she leaves the game and the remaining players continue in alternation until one of them cannot move. Then that player leaves the game and the remaining player is the winner. In other words, the last player to move wins.

Three-player games [6], [7], are difficult to analyze because of queer games [8], [9], i.e., games where no player has a winning strategy. Moreover, because of the possibility to form alliances, cooperation between players is a key-factor to determine the winning coalition. There exist at least two different ways to establish the winning coalition:

- **Weak coalition convention.** If one of the player is unable to move, then he/she leaves the game but the other players of his/her coalition are still able to play. In other word, the coalition of the player that makes the last move wins.
- **Strong coalition convention.** If one of the player is unable to move, then he/she and all the players of his/her coalition leave the game. In other word, a coalition wins if all its players are able to make a move until the end of the game.

In weak coalition convention, the fact that one of the player is not able to move does not affect the other players of the same coalition because the main goal is to be able to make the last move. Differently, in strong coalition convention, the fact that all the players are able to make a move until the end of the game becomes a crucial point.

In the previous work [10], we studied the complexity of three-player Hackenbush played on strings under weak coalition convention.

In this work, the complexity of three-player Col played on trees under strong coalition convention is investigated.

3 Three-player Col played on trees is $\mathcal{NP}$-complete

We prove that to solve three-player Col played on trees under strong coalition convention is a $\mathcal{NP}$-complete problem.

We briefly recall the definition of Subset Sum Problem.

Definition 1. *Let*

$$\mathcal{U} = \{u_1, \ldots, u_n\}$$

be a set of natural numbers and K a given natural number. The problem is to determine if there exists $\mathcal{U}' \subseteq \mathcal{U}$ such that

$$\sum_{u_i \in \mathcal{U}'} u_i = K$$

This problem is known to be $\mathcal{NP}$-complete [11]. Starting from a general instance of Subset Sum Problem it is possible to create an instance of three-player Col on trees as shown in Fig. 1 where U is the sum of all the elements in $\mathcal{U}$, i.e.,

$$\mathcal{U} = \sum_{u_i \in \mathcal{U}} u_i$$

Who has a winning strategy if Second and Third form a coalition and they play under strong coalition convention?

We observe that in the tree shown in Fig. 1(a), First can make exactly $U + n$ moves. Moreover, in the trees shown in Fig. 1(b) and Fig. 1(c), Second and Third can make respectively K and $U - K$ moves. Therefore, if Second and Third form a coalition, then two different scenarios are possible:

- If Second and Third are able to make respectively $U - K + n$ and $K + n$ moves in the tree shown in Fig. 1(d), then First does not have a winning strategy.
- If Second and Third are not able to satisfy the previous condition, then First has a winning strategy.

The problem to determine if First has a winning strategy or not is strictly connected to Subset Sum Problem as shown in the following theorem.

Theorem 1. *Let G be a general instance of three-player Col played on trees. Then, to establish the outcome of G is a $\mathcal{NP}$-complete problem.*

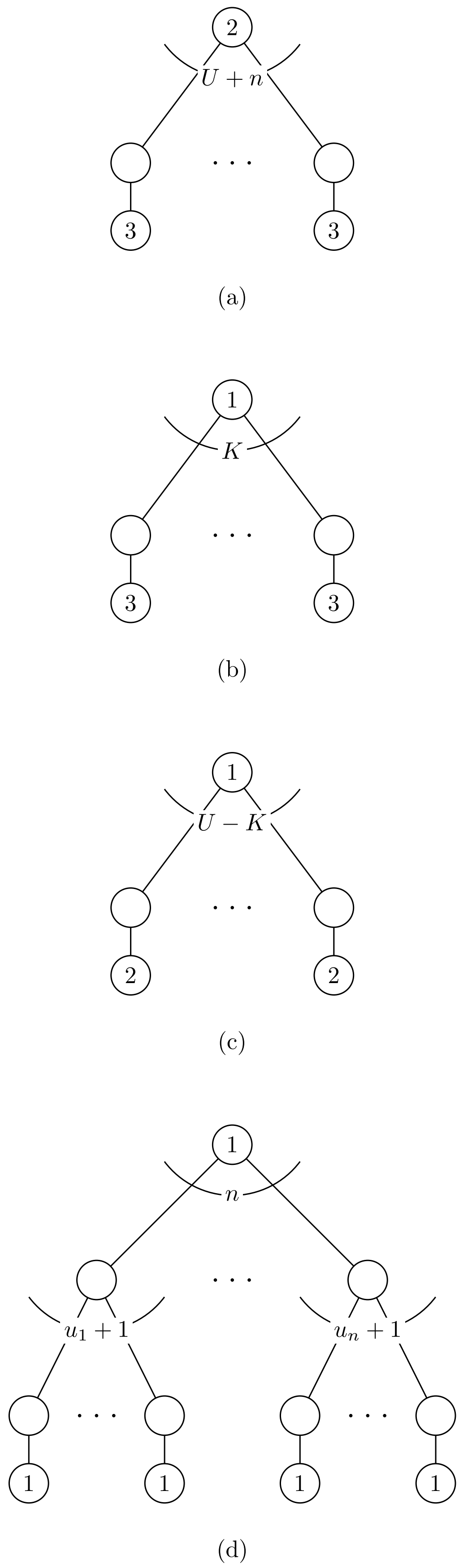

Figure 1: Subset Sum Problem is reducible to three-player Col played on trees.

Proof. The problem is clearly in $\mathcal{NP}$.

We show that it is possible to reduce every instance of Subset Sum Problem to G. Previously we have described how to construct the instance of three-player Col, therefore we just have to prove that Subset Sum Problem is solvable if and only if the coalition formed by Second and Third has a winning strategy, i.e., First does not have a winning strategy.

If $\mathcal{U}'$ is a solution of Subset Sum Problem, then the coalition formed by Second and Third has a trivial way to win the game. All the roots of the sub-trees corresponding to $u_i \in \mathcal{U}'$ will be labeled 2 and all the leaves of such sub-trees will be labeled 3. All the roots of the sub-trees corresponding to $u_i \notin \mathcal{U}'$ will be labeled 3 and all the leaves of such sub-trees will be labeled 2. In this way, Second and Third will make respectively $U - K + n$ and $K + n$ moves in the tree shown in Fig. 1(d).

Conversely, if the coalition formed by Second and Third has a winning strategy, then Second and Third are able to make respectively $U - K + n$ and $K + n$ moves in the tree shown in Fig. 1(d), i.e., all nodes must be labeled.

Because of the rule of the game, two adjacent vertices cannot be labeled with the same number therefore there exist a set of sub-trees corresponding to $\mathcal{U}' \subset \mathcal{U}$ such that

$$\sum_{u_i \in \mathcal{U}'} u_i = K$$

.

Therefore, the problem to establish the outcome of G is $\mathcal{NP}$-hard and $\mathcal{NP}$-complete. □

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