Comparative Mortality Models in Kuwait

Reem A. Al-Jarallah, and Steve Brooks *

Abstract—**The paper compares three different models that have been developed in the literature for modeling and forecasting human mortality rates over the age range. The first model is the quadratic Gompertz model where the quadratic line is fitted for age and sex. The second is the one-dimensional Spline model, which fits linear combinations of the Basis Splines. The last model is the Lee-Carter model which is used to fit each sex to a set of age-specific death rates by fitting Poisson log-bilinear regression model in generalized linear models (GLM). A time-varying index of mortality is then forecasted using a time series linear forecasting model autoregressive integrated moving average (ARIMA). These forecasts are used to generate projected age-specific mortality rates in Kuwait for ages 55 to 90 for the period 2006-2015, based on the mortality data for years 1993-2005 for males and females separately.**

Keywords: mortality Rates, Gompertz model, Spline model, Lee-Carter model, Autoregressive integrated moving average (ARIMA).

1 Introduction

Mortality refers to the decremental process by which living members of a population gradually die out and it is not evenly distributed by age and sex. Therefore, the pattern of mortality depends on the distribution of age and sex of each population and mortality rates are thus measured separately for males and females. Mortality rates are among the most important parameters used in evaluating the population health and social levels, in addition to their importance in determining the level of the population's natural growth, population's growth rate, calculating mortality prospects and creating life tables.

In the last 50 years, human mortality has been seen to follow a continued though frequently irregular declining trend. The prospects of longer life are viewed as a positive change for individuals and a substantial social achievement but have led to concern over their implication regarding public spending on old-age support. The earliest significant measurements of mortality are those of John Graunt who is generally regarded as the father of the demography in England and Wales (Benjamin and Soliman, 1993). Recently, mortality changes are widely studied by demographers, economists and by now there are a considerably growing literature on mortality decline and its individual and social consequences (Tuljapurkar and Boe, 2001).

Basically, there are two mortality patterns: the cause-specific mortality and the total or central mortality rate. The cause-specific mortality is of interest to the health policymakers and public health, medical and pharmaceutical researchers. On the other hand, the total mortality is of interest to the actuarial scientists, insurance companies and economists. In this paper, the second mortality pattern, the central mortality rate is discussed, focusing on the population of age 55 to 90 for each sex from 1993 to 2005. The mortality rates exhibit strong age patterns and varies with respect to many characteristics, such as sex and age. Various researchers have developed methods to capture this structure and use it in forecasting. Therefore, mortality rates will be calculated for each age and sex separately.

In the literature, there were a very few studies that is related to mortality in Kuwait. The Kuwaiti mortality rates remains relatively unexplored. In fact, the demographic components in Kuwait need further investigations. In government planning, it is necessary for a country to forecast future population size and age structure. This is crucial for a country like Kuwait, where the population is increasing rapidly. Therefore to forecast the future population of Kuwait both fertility and mortality rates are needed and the structure of the population can then be forecasted by constructing population projections.

A comparison is conducted between three statistical methods that are applied most frequently in the literature, Gompertz, P-Spline and Lee-Carter model. Starting by an overview of the models, describing the basics; discussing their applications and evaluating their performance.

This paper is divided as follows. In Section 2, we introduce a background on mortality in Kuwait. Section 3 presents the basic three models, Gompertz model, P-Spline model, and Lee-Carter model. The three proposed models are applied to Kuwait mortality data in Section 4. In Section 5 we project the mortality rates. A comparison of the three models is presented in Section 6 followed

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by a general discussion in Section 7.

2 Background on Mortality in Kuwait

We briefly comment on the studies that were conducted in Kuwait. Abaza (1984) presented abridged Kuwaiti life tables by using the Chiang method of constructing Kuwaiti mortality rates for the period 1965-1980. AL-Sabah (1985) introduced the vital statistics in the State of Kuwait among those was the crude mortality rates for the Kuwaiti population by nationality and sex over the year 1970 to 1979. EL-Shalakani (1989) estimated the current level of fertility and mortality among Kuwaiti nationals in Kuwait by applying indirect techniques during 1980 through 1985.

The death rates based on data from Kuwaiti insurance companies were discussed in EL-Mansoury et al. (1991) for the period 1985-1989. AL-Sabah (1992) discussed the level of the natural growth and the directions of population growth in Kuwait focusing on the Infant Mortality rates. The age specific probabilities were estimated by Al-Ramadhan (1995b) for the years 1987-1992 by extrapolating historical life expectancies at birth for Kuwaiti population by sex. The life expectancy at birth for mid-year Kuwaiti population by sex was also estimated from 1992-1999.

More recently, a comparison was conducted by Rowland (2003) between the crude death rate (CDR) of U.K. and Kuwait for the year 2001. The CDR for Kuwait was lower than that for U.K. This was mainly due to the age structure of the population of Kuwait.

Mortality rates discussed here are defined as the ratio of number of people died during that year dividing by the number of people surviving to the start of the following year. Fig. 1 represents the log mortality rates of Kuwait for ages 55-90 during the period of 1993-2005, for male and female respectively. The mortality rates fluctuated over this period and was higher for males than females. The decline in the male mortality rates is more steady than those of the females. The United Nations (ECWA)(1980) described the mortality rate for the population of Kuwait as one of the lowest of all the countries in Asia. These low death rates can be attributed to the high level of health expenditure beginning in 1962 when the Government of Kuwait introduced a comprehensive health care program for the population of Kuwait.

3 Models for Forecasting Mortality

The attempt to find an appropriate mortality curve has a long history in demography and actuarial sciences. Fitting a parametric curve to annual rates was first introduced by Gompertz (1825), Makeham (1860). Different approaches have been developed for forecasting mortality using stochastic models such as Alho (1990, 1992), Alho and Spencer (1985) and Lee and Carter (1992). In this section we introduce the most common methods currently used for forecasting mortality rates in the developed countries. For more discussion on these methods (see Federico and King, 2004).

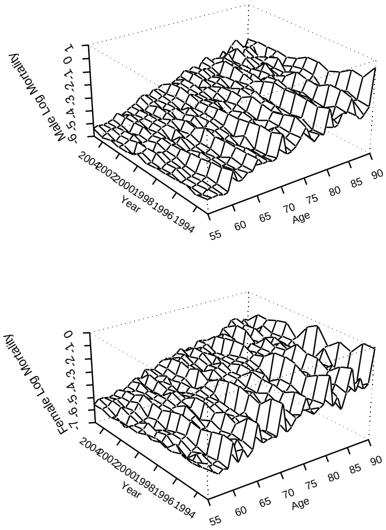

Figure 1: The log mortality rates plot for ages 55-90 in years 1993-2005 for male and female in Kuwait.

3.1 The Gompertz Model

The Gompertz model was firstly introduced by Gompertz (1825) and has played a potential role in developing the theoretical hypotheses on mortality pattern. It describes mortality as increasing exponentially with the age at a constant rate and is used for forecasting mortality changes by considering years to be the covariate for the age-specific mortality rate. The basic model can be adopted to the mortalities of age-groups for single years. The model is constructed as

$$\mu_x = \alpha e^{\beta x},$$

where x denotes the year and μ_x denotes the mortality of a certain age. A log linear extrapolation is given by

$$\log(\mu_x) = \log \alpha + \beta x = a + bx.$$

The parameter α varies with the level of log mortality and β measures the rate of annual decline in age-specific mortality. This model can be extended to a log quadratic known as quadratic Gompertz and is given by

$$\log(\mu_x) = a + bx + cx^2.$$

For many purposes the Gompertz model provides a satisfactory fit to adult morality rates. However, different modifications has been applied to this model as its validity at higher ages was questionable.

3.2 The Spline Model

The Spline model is one of parametric smoothing models, that has been used for mortality graduations for many years. The most commonly applied technique for both smoothing and projecting mortality rates in U.K. is that of penalized-spline regression, known as P-splines. The Spline models, especially the 2-dimensional P-Spline model, was used to construct the recent English Life Tables for the Continuous Mortality Investigation Mortality sub-committee (CMIB)(2004).

Consider the additive model

$$y = g(x) + \epsilon,$$

where y represents the log mortality of one specific age group and x is the predictor variable (Year). We assume that errors ϵ are independent and identically distributed with mean zero, and g is an unknown arbitrary function of the predictor x. The function $g(x)$ can be fitted by using smoothing Splines which is an indirect method of smoothing driven by penalized least squares. At an arbitrary point t, $g(t)$ can be estimated by a locally weighted regression smoother by computing a weighted average of all those values y_j in the sample that have predictors x_j close to t . This can be represented as

$$\hat{g}(t) = \sum_{j=1}^{n} W_\lambda(t, x_j) y_j, \tag{1}$$

where $(x_j, y_j), j = 1, \cdots, n$, is the series of n data points, t is the target point, and W_λ is a weight function parameterized by λ. Smoothing splines exploit a cubic smoothing spline to fit the data in (1) by minimising the penalised residual sum of squares (RSS) given by

$$RSS(g, \lambda) = \sum_{j=1}^{n} (y_i - g(x_i))^2 + \lambda \int (g^{''}(t))^2 dt. \tag{2}$$

The solution is a natural cubic spline with interior and boundary knots at the unique values of x_i, that can be written as

$$g(x) = \sum_{k=1}^{n} \theta_k N_k(x),$$

where $N_k(x)$ are N-dimensional set of basis functions representing this family of natural splines. The fitted smoothing spline is given by

$$\hat{g}(x) = \sum_{k=1}^{n} \hat{\theta}_k N_k(x), \tag{3}$$

where $\hat{\theta} = (N^T N + \lambda \Omega_N)^{-1} N^T y$ with $\{N_{ij}\} = N_j(x_i)$ and $\{\Omega_N\}_{jk} = \int N_j^{''}(t) N_k^{''}(t) dt$. For convenience, we use B-spline rather than the natural spline, thus (3) can be written as

$$g(x) = \sum_{1}^{N+4} \delta_j B_j(x),$$

where δ_j are coefficients and B_j are the cubic B-spline basis functions. Thus the solution can be written as

$$\hat{\delta} = (B^T B + \lambda \Omega_B)^{-1} B^T y. \tag{4}$$

Comparing $\hat{\theta}$ with $\hat{\delta}$, $N \times N$ matrix $N_k(x)$ is replaced by by the $(N+4) \times N$ matrix B, and the $N \times N$ dimensional Ω_N is replaced by $(N+4) \times (N+4)$ penalty matrix Ω_B. For more on the theory of B-splines, see Hastie et al.(2001).

3.3 The Lee-Carter Model

The last model is one of the leading statistical models of mortality in demographic literature, the Lee-Carter model, which was proposed by Lee and Carter (1992) and was used to forecast U.S. mortality to 2065. Since that time, the method has attracted a certain amount of attention. The most recent Census Bureau population forecasts (Hollmann et al. 2000) use this model forecast as a benchmark for their long-run forecast of U. S. life expectancy. Recently, Lee-Carter model become more popular for modeling and forecasting mortality by age and has been adopted widely. The model was applied for many countries and different time periods, for example, U.S. (Lee and Carter, 1992), Canada (Lee and Nault, 1993), Japan (Wilmoth, 1996), Chile (Lee and Rofman, 1994), Belgium (Brouhns et al., 2002), and the seven most economically developed nations (G7) (Tuljapurkar et al., 2000).

The model describes the secular change in mortality as a function of a single time index. The model is based on a log-additive model of age-specific death rates with dominant time component and a fixed relative age components. A matrix decomposition method is used to identify the two components. The time component is projected by using a time series linear forecasting model, autoregressive integrated moving average (ARIMA).

The Lee-Carter model is a simple bilinear model in variables x (age) and t (year) given by

$$\log(\mu_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{5}$$

where $\mu_{x,t}$ is the central death rate at age x and time t. The parameter α_x describes the average age-specific pattern of mortality, β_x is age-specific constants for the relative speed of mortality change in response to changes in κ, and κ_t represents the time-trend index of general mortality level. The term $\epsilon_{x,t}$ is the error term with zero mean and variance σ^2. The error terms indicates age-specific historical influence not captured by the model.

Table 1: The estimated parameters for the log mortality rates of females and males using Gompertz model.

Age	Female		Male	
	β_0	β_1	β_0	β_1
55	191.1630(30.5958)	-0.0982(0.0153)	105.6918(17.3605)	-0.0555(0.0087)
60	215.9261(26.2746)	-0.1101(0.0131)	180.5684(20.0897)	-0.0925(20.0897)
65	227.6358(40.6368)	-0.1157(0.0203)	139.9322(18.6458)	-0.0719(0.0093)
70	210.3217(38.6737)	-0.1066(0.0193)	174.6442(25.4446)	-0.0888(0.0127)
75	179.0954(39.2262)	-0.0909(0.0196)	153.0631(23.5742)	-0.0778(0.0118)
80	229.6621(46.1281)	-0.1158(0.0231)	139.2337(28.8799)	-0.0706(0.0144)
85	147.9094(51.1711)	-0.0884(0.0256)	224.3416(50.5993)	-0.1131(0.0253)
90	217.1079(66.6129)	-0.1362(0.0333)	300.5622(48.4890)	-0.1509(0.0243)

The time component κ_t captures the main time trend on the logarithmic scale in mortality rates at all ages. The age component β_x modifies the main time trend and it is assumed to be invariant overtime. Thus to obtain a unique solution for the system of equations of the model and to ensure identifiability of the model we have

$$\alpha_x = \frac{1}{T}\sum_t log(\mu_{x,t}), \quad \sum_x {\beta_x}^2 = 1, \quad \sum_t \kappa_t = 0.$$

4 Fitting the Three Proposed Models

The three proposed models discussed in the previous section, for modeling and forecasting mortality, are now applied to log mortality data in Kuwait for ages 55 to 90 for the period 1993-2005 for males and females separately.

4.1 Gompertz Model

In Gompertz model we have a single explanatory variable 'year', thus a polynomial regression model can be fitted as

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i \tag{6}$$

where Y is the response variable of log-mortality of one specific age, x is the explanatory variable of years, β is the vector of parameters and ϵ_i is the unknown random error. Fitting Gompertz model, the estimated values of model parameters, as well as the standard errors, for several ages are shown in Table 1, for females and males respectively.

From Table 1 we can notice that the standard errors are relatively large. Similar results were obtained by Zhang (2005) when applying the quadratic Gompertez model to England and Wales mortality experience of age 55-84 for each gender from 1947-1996. The plot of the observed and fitted log mortality are shown in Fig. 4 for age 75 for males and females separately. It is clear that the fitted lines go most of the observed dots which indicates an overall good fit.

As a diagnosis of the model, we examined the residuals to check whether the basic assumptions are satisfied. The histograms and q-q plots of the residuals, reveals that they are normal and independent indicating that the model is adequate. The parameters were found to be significant and the multiple correlation coefficient R^2 have a reasonable values.

4.2 Spline Model

Fitting smooth Spline model to the log mortality data can be seen as fitting an additive model that consists of flexible components. Thus no list of estimated values of model parameters can be obtained.

Fig. 4 shows the smooth spline of log mortality against the year of age 75 category for females and males. It is clear that the curve is smoother than that of Gompertz. Fig. 2 displays the plot of $s(Year)$ against Year as a representation of the additive fit of Year to the log Mortality rate. It is obvious from the plot that x-values are fairly equally scattered with narrow confidence interval.

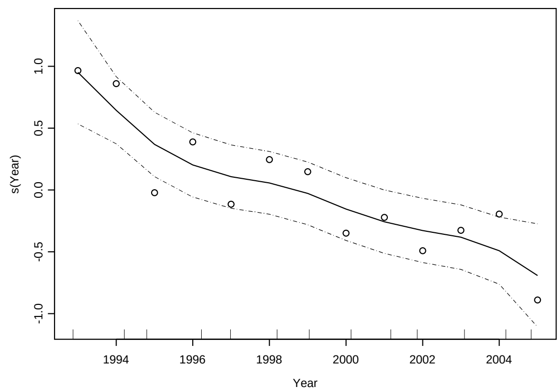

Figure 2: Year smoothing term plot.

We diagnose the Spline model, by examining the residual deviance that are all far smaller than the degrees of freedom which indicates a good fit of data. The F values for nonparametric effect indicate that the nonlinear component $s(Year)$ is significant.

4.3 Lee-Carter Model

Different methods have been proposed to estimate the Lee-Carter model parameters. Lee and Carter (1992) applied a two-stage estimation procedure, Singular Value Decomposition (SVD) in the first stage, and time series method to re-estimate κ_t in the second stage. This was useful in cases where only the total, rather than age-specific, death rates are known in certain years. Wilmoth (1993) developed a weighted least square (WLS) and a maximum likelihood (MLE) technique. Alho (2000) suggested using MLE based on a Poisson number of deaths D_{xt}. The method of maximum likelihood is based on the assumption that the number of deaths is a counting random variable that can be modeled by a Poisson Process.

Let D_{xt} be a random variable representing the death count at age x and time t, d_{xt} is the observed number of deaths, E_{xt} is the exposure-to-risk at age x and time t, and e_{xt} is the observed value. The Poisson specification is

$$\frac{d_{xt}}{e_{xt}} \sim Poisson(e^{\eta_{xt}}, \eta_{xt} = \alpha_x + \beta_x \kappa_t)$$

where η_{xt} is the mean of the Poisson distribution and $\eta_{xt} = m_{xt}E_{xt}$. The difficulty of the MLE method, that maximizes the log-likelihood function, arises because of the bilinear form of the term $\beta_x\kappa_t$. This approach, known as Poisson log-bilinear modeling, is fully described in Brouhns et al. (2002). An iterative method for estimating log-linear models with bilinear terms was first proposed by Goodman (1979). The algorithm is a uni-Dimensional/Elementary Newton Method that uses LEM to solve the likelihood equations. In the iteration step a single set of parameters is updated by fixing other parameters at their current estimate. The detailed algorithm is explained in Appendix A, (see, Zhang, 2005).

The estimated model parameters α_x, β_x and κ_t are shown in Fig. 3 for females and males respectively.

The parameter α_x represents the general age shape of mortality. It is clear that both females and males have upward trend of mortality in general. The parameter β_x describes the tendency of mortality at age x to changes as the general level of mortality κ_t changes. This indicates that when β_x is large, the death rates at age x varies a lot than the general level of mortality change and when β_x is small, then the death rate at that age varies little. The mortality index κ_t, captures the main time trend on the logarithmic scale in the death rates at all ages. In Fig. 3 the females have a significant change of mortality rates in the last year. Moreover, the estimated mortality index κ_t generally exhibits a linear decreasing trend, meaning that the age-specific mortality has yearly declined almost exponentially.

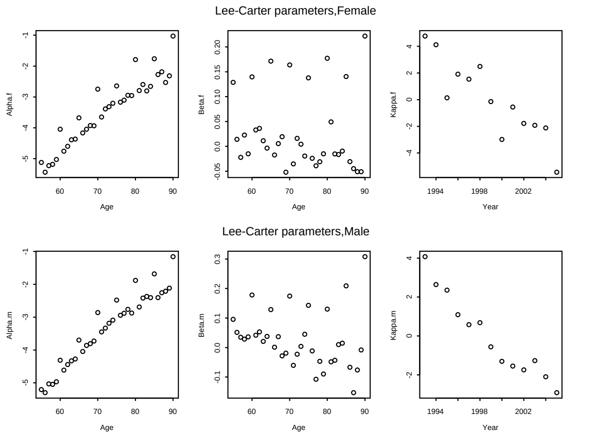

Figure 3: The estimated parameters α_x, β_x and κ_t in Lee-Carter model for females and males.

In the Lee-Carter model, the fitted values of the model are the number of deaths $\hat{D_{xt}}$, the observed log-mortality is $\log(D_{xt}/E_{xt})$ and the fitted log mortality is $\log(\hat{D_{xt}}/E_{xt})$. Fig. 4 represents the observed and fitted log-mortality rates for 75 age category for males and females. It is obvious that the fitted lines and observed values follow a similar pattern.

The model was diagnosed by observing the residual deviance, their values were less than its corresponding degrees of freedom indicating a good model fit, with no over-dispersion. The residuals were also examined where the histogram looks relatively normal and the q-q plot indicates an adequate fit.

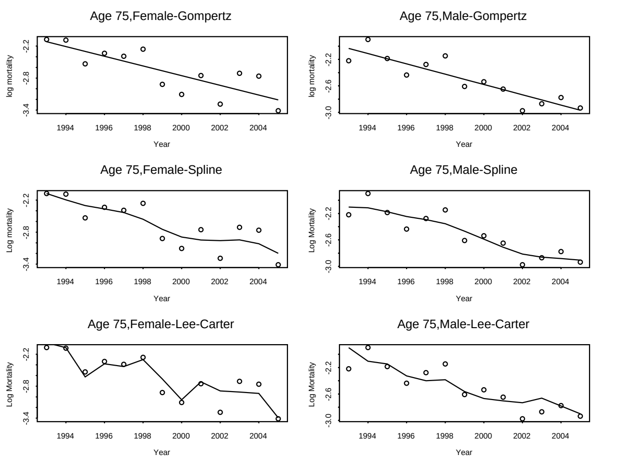

Figure 4: The observed and the fitted log mortality rates for Gompertz, Spline and Lee-Carter model.

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5 Projection of Mortality Rates to 2015

5.1 Gompertz Model

In Gompertz model, the projection of mortality from 2006 to 2015 can be calculated by assuming that the estimated parameters age β_0, β_1 and β_2 are determined by the sample years, varying the explanatory covariate Year to the future years. Fig. 5 shows the projected log mortality of age 75 from 2006 to 2015 with 95% confidence interval. Clearly, the confidence interval in females is much wider than that of the males indicating a greater variability in the data.

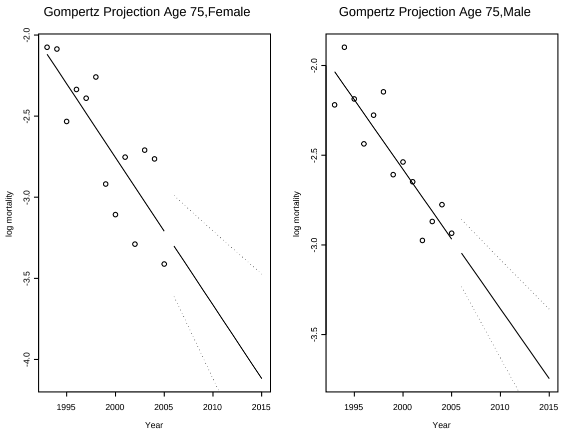

Figure 5: The projected log mortality of age 75 for females and males in Gompertz model.

5.2 Spline Model

For the Spline model, projections is conducted by computing a model matrix using the new data, which is then multiplied by the coefficients extracted from the original object. Fig. 6 represents the projection to year 2015 of age 75 for females and males respectively.

5.3 Lee-Carter Model

One of the advantages of the Lee-Carter approach is that once the data are fitted to the model and the values of the parameters α_x, β_x and κ_t are estimated, only the mortality index κ_t needs to be predicted. Thus to perform projections, the parameters α_x, β_x remains constant over time, while κ_t are forecasted of by using an ARIMA model.

5.3.1 Forecasting the κ in ARIMA Model

Lee and Carter (1992) predicted the mortality index κ_t by a standard univariate time series model ARIMA(0,1,0). They demonstrate that other ARIMA models might be

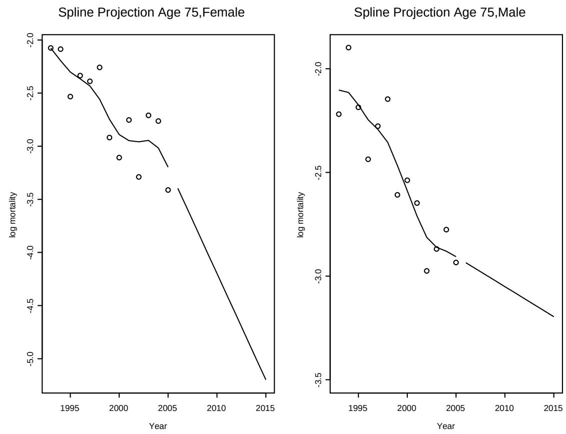

Figure 6: The projected log mortality of the age 75 for females and males in Spline model.

preferable for different data sets. The graphic observation suggests an ARIMA(1,1,1), ARIMA(1,1,0) and ARIMA(0,1,1) models. The Akaike's information criterion (AIC) was applied to select the most appropriate model. The appropriate model for both male and female was ARIMA(1,1,0) with the smallest AIC. The adequacy of the selected ARIMA(1,1,0) model was diagnosed by examining the residual plots which indicated that the residuals are consistent with white noise. Over fitting was also examined, and a second term was not needed.

The appropriate model ARIMA(1,1,0) was used to forecast the parameter κ over the desired time period for both males and females. Fig. 7 illustrates the fitted values of κ from 1993 to 2005 and their forecasts from 2006 to 2015 for both females and males. The estimated values of κ over the base period change in a linear fashion.

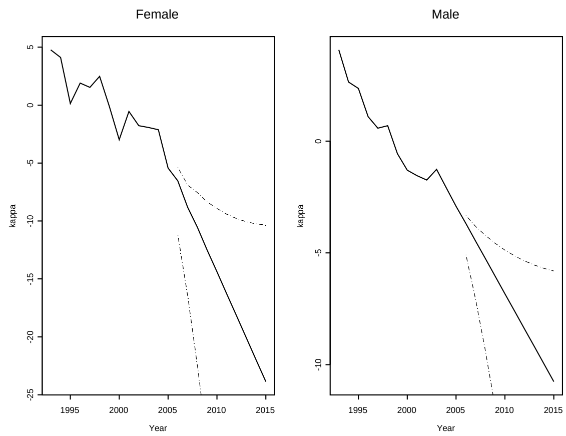

Figure 7: The forecasted Kappa for females and males in Lee-Carter model.

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The adequacy of the ARIMA models was checked by examining different diagnosis plots such as residual plots, function of residual plots. The plot of residual indicates a consistency with white noise, while the plot of standardized residual suggests that the two are stationary series with zero mean and small variance. The ACF of residual plots supports the assumption of independence and P-values for Ljung-Box statistics give an evidence that the model is adequate.

The mortality index κ_t is the only parameter that needs to be predicted, and the next step is to project log mortality to year 2015 using the earlier equation (5) for $\log \mu_{x,t}$, given the previously estimated age specific coefficients α_x and β_x as well as the forecasted κ. Fig. 8 shows the projection of the age 75 category from 2006 to 2015 for the Lee-Carter model.

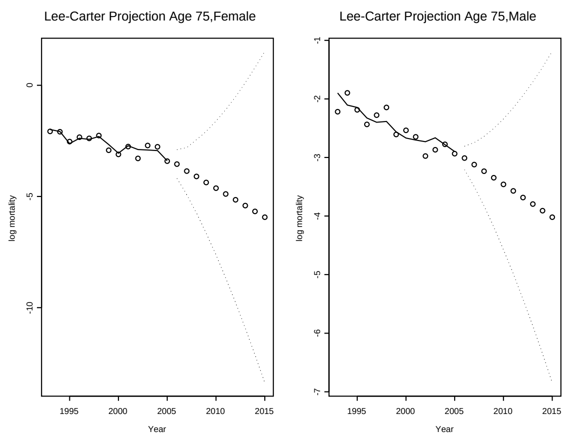

Figure 8: The projected log mortality of the age 75 for females and males in Lee-Carter model.

6 Comparison of the Projection

We evaluate the three proposed models by analyzing the overall errors between the fitted and the observed mortality rates. The method applied here to analyze the errors was discussed by Benjamin and Soliman (1993).

Let e be the difference between the fitted and the observed mortality rate, then the first error type is the average error $e_1 = \sum e/n$, with n is the number of data points. The second error is the average absolute error, $e_2 = \sum |e|/n$, that measures the magnitude of the overall error in projection. The third error type is the root of the square error, $e_3 = \sqrt{\sum e^2/n}$ that measures the standard deviation of the projected values. Table 2 shows the error values for the three fitted models.

For the average error of fitting e_1, it is obvious that the quadratic Gompertz and Smooth spline give produce lower projected mortality rates than the observed rates, while the Lee-Carter method gives higher projected rates overall. For the average absolute error e_2, it is clear that the most accurate method overall is the Spline model with similar values for the Gompertz. The Lee-Carter method produces larger values indicating less accuracy. The standard error e_3 values indicates that the Lee-Carter have greatest confidence interval compared with Gompertz and Spline. The Spline method gives the smallest confidence interval overall with slight difference between the two methods.

Table 2: The error values for Gompertz, Spline and Lee-Carter models for females and males.

	Model	e_1	e_2	e_3
Female	Gompertz	1.8978 $\times 10^{-18}$	0.2213	0.2854
	Smooth-Spline	-2.8348 $\times 10^{-17}$	0.1721	0.2249
	Lee-Carter	26.4788	26.47881	29.4713
Male	Gompertz	1.6605 $\times 10^{-18}$	0.1651	0.2178
	Smooth-Spline	3.0839 $\times 10^{-17}$	0.1207	0.1679
	Lee-Carter	34.9848	34.9848	38.0861

In assessing the overall comment of the fitted mortality rates, the Spline method appears to be the best of the three models and the Gompertz model is the second best. This is true for both males and females. The Smooth spline regression achieves local optimization by which it successfully fits the data.

7 Conclusion

This paper presents a brief review of the main mathematical models that have been developed to describe and explain human mortality patterns over the age range. The projected mortality rates of improvement by age, sex and year were produced by applying the three projection methods which are ultimately chosen for both males and females over the period 2006-2015 based on the data from 1993 to 2005 in Kuwait. First the log mortality was fitted according to Gompertz model using the classical linear regression method. This model was extended by applying Spline model which is equivalent to a polynomial regression with degree two for more flexibility. Then the Lee-Carter method of fitting and projecting the human mortalities was applied.

For the Kuwaiti mortality data, it was found that the smooth Spline regression model gives a better fitting for short-term mortality rates due to its local optimization. The Lee-Carter model is simple but highly structured and two different sources of uncertainty have to be combined, sampling errors in the parameters of Poisson model and forecast errors in the projected ARIMA parameters. However, the Lee-Carter model does not model the observed number of deaths but the logarithms of the force of mortality.

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We summarize the advantages and disadvantages of regression models exemplified by the P-spline model and the Time Series models by Lee-Carter model. We do not single out which is preferable, but here we discuss some of their main features.

An issue that is specific to regression models was that traditional polynomial methods can yield acceptable fits in the region of the data, and yet produce very poor projections out side it. The P-Spline model allows for parameter uncertainty through the variance matrix of the regression coefficients and it can incorporate cohort effect very simple, through the choice of penalty.

The Lee-Carter method has a number of appealing features. The basic model is simple but highly-structured, which introduces a degree of model uncertainty. The model can be made to allow for parameters uncertainty by bootstrapping, however the model does not explicitly allow for cohort effects. The Lee-Carter has been quit well received, but there have also been criticisms. Some have thought that the probability bands are implausibly narrow (e.g. Alho, 1992). Others argued that many age specific rates are so low that they can not realistically be projected to decline much further. Some argue that biomedical information should inform the forecasts, through incorporating expert opinion. Some have called for more within-sample testing of the methods, and others have questioned whether the α_x and β_x should be treated as invariant.

Each of the three models proposed in this paper has its extensions. In the Gompertz model, the difference between model estimates and observed rates reveals systematic underestimation of actual mortality at youngest adult ages and overestimation at oldest ages. The deviation at lower ages is addressed by Makeham (1860) by adding a constant to the Gompertz model that referred to as a background mortality. The deviation at the oldest ages was addressed by different ways, the simples was the logistic model introduced by Thatcher(1999) and Thatcher et al. (1998). More complex logistic models with additional parameters have been proposed, (e. g., Thatcher et al. 1998).

The smooth Spline model is one-dimensional and fitted by penalized least squares. Eiler and Marx (1996) discussed the penalized likelihood in GLM and Currie et al.(2002) extended it to smooth two-dimensional Poisson data where the coefficients of log mortalities live in the age-year plane.

A considerable work has been done to extend the Lee-Carter model. An improved fitting methods has improved by Wilmoth (1993) based on weighted least squares as previously discussed. As the model assumes that age-specific rate of decline remains constant over time, different methods were introduced in the literature to overcome this problem for example, Renshaw and Haberman (2005) extended Lee-Carter model in a wider class of statistical approach. A further area of consideration is to apply these extended models to mortality data in Kuwait and conducting the comparison.

APPENDIX A

Brouhns *et al* (2002) proposed a LEM method to solve the likelihood equations and to get the fitted values of the parameters α_x, β_x and κ_t in the Lee-Carter model.

The iterative method that is used to estimate log-linear models with bilinear terms was first proposed by Goodman (1979). The iteration step is a single set of parameters is updated by fixing other parameters at their current estimate.

$$\hat{\theta}^{(\nu+1)} = \hat{\theta}^{(\nu)} - \frac{\partial L^{(\nu)}/\partial\theta}{\partial^2 L^{(\nu)}/\partial\theta^2}$$

where $L^{(\nu)} = L^{(\nu)}(\hat{\theta}^{(\nu)})$ is the log likelihood function. For the Lee-Carter equation, the updating procedures are as follows:

1. Set the starting value with $\hat{\alpha}_x^0 = 0, \hat{\beta}_x^0 = 1$, and $\hat{\kappa}_t^0 = 0$.

2. Let

$$\hat{\alpha}_x^{(\nu+1)} = \hat{\alpha}_x^{(\nu)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(\nu)})}{-\sum_t \hat{D}_{xt}^{(\nu)}}$$

$$\hat{\beta}_x^{(\nu+1)} = \hat{\beta}_x^{(\nu)}, \hat{\kappa}_t^{(\nu+1)} = \hat{\kappa}_t^{(\nu)}$$

3.

$$\hat{\kappa}_t^{(\nu+2)} = \hat{\kappa}_t^{(\nu+1)} - \frac{\sum_x (D_{xt} - \hat{D}_{xt}^{(\nu+1)})\hat{\beta}_x^{(\nu+1)}}{-\sum_x \hat{D}_{xt}^{(\nu)}(\hat{\beta}_x^{(\nu+1)})^2}$$

$$\hat{\alpha}_x^{(\nu+2)} = \hat{\alpha}_x^{(\nu+1)}, \hat{\beta}_x^{(\nu+2)} = \hat{\beta}_x^{(\nu+1)}$$

4.

$$\hat{\beta}_x^{(\nu+3)} = \hat{\beta}_x^{(\nu+2)} - \frac{\sum_t (D_{xt} - \hat{D}_{xt}^{(\nu+2)})\hat{\kappa}_t^{(\nu+2)}}{-\sum_t \hat{D}_{xt}^{(\nu+2)}(\hat{\kappa}_t^{(\nu+2)})^2}$$

$$\hat{\alpha}_x^{(\nu+3)} = \hat{\alpha}_x^{(\nu+2)}, \hat{\kappa}_t^{(\nu+3)} = \hat{\kappa}_t^{(\nu+2)}$$

The scaling constraint used by LEM is $\hat{\beta} = 1$ is different from Lee-Carter parametrization. In order to obtain the $\sum_x \hat{\beta}_x = 1$, it is necessary to divide the estimates for β_x by $\sum_x \hat{\beta}_x$ and multiply the estimates for κ_t by the same number.

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