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On Multiparametric Analysis in Sum-of-ratios Programming

Sanjeet Singh and Pankaj Gupta [‡]

Abstract—**In this paper, we study multiparametric sensitivity analysis in sum-of-ratios programming problem. We consider two linear ratios for computational ease. We construct critical regions for simultaneous and independent perturbations in the objective function coefficients(both in numerator and denominator) and in the right-hand-side vector. Theoretical results are illustrated with the help of a numerical example.**

Keywords: Multiparametric analysis, sensitivity analysis, sum-of-ratios programming, fractional programming, parametric analysis

1 Introduction

In various applications of nonlinear programming a ratio of two functions is to be maximized or minimized. In other applications the objective function involve more than one ratio of functions. Ratio optimization problems are commonly called fractional programs. One of the earliest fractional programs (though not called so) is an equilibrium model for an expanding economy introduced by Von Neumann [6] in 1937. The model determines the growth rate of an economy as the maximum of the smallest of several output-input ratios. The analysis of fractional programs with only one ratio has largely dominated the literature until about 1980. The first monograph [11] in fractional programming published by Schaible in 1980 extensively covers applications, theoretical results and algorithms for single ratio fractional programs; see also [9, 10]. A series of international conferences was held which demonstrates a shift of interest from the single-ratio to the multi-ratio case [1, 2, 4, 13].

Sum-of-ratios programming problems arise naturally in decision making when several rates are to be optimized simultaneously and a compromise is sought which optimizes a weighted sum of these rates. In light of the applications of single-ratio fractional programming [7, 9, 10] numerators and denominators may be representing output, input, profit, cost, capital, risk or time, for example. A multitude of applications of the sum-of-ratios problem can be envisioned in this way. Included is the case where some of the ratios are not proper quotients. This describes situations where a compromise is sought between absolute and relative terms like profit and return on investment (profit/capital) or return and return/risk, for example [8]. In this paper, we will consider the following sum-of-ratios linear fractional programming:

P(1) Maximize $f(x) = \dfrac{p^T x}{q^T x} + \dfrac{c^T x}{d^T x}$

subject to $Ax = b$,

$x \geq 0$,

where each p, q, c, d is $n \times 1$ column vector, A is $m \times n$ coefficient matrix, b is $m \times 1$ column vector, and x is $n \times 1$ column vector.

In practical applications the data collected may not be precise, we would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. In general, the main focus of sensitivity analysis is on simultaneous and independent perturbations of the parameters. Besides this all the parameters are required to be analyzed at their independent levels of sensitivity. If one parameter is more sensitive than the others, the tolerance region characterized by treating all the parameters at equal levels of sensitivity would be too small for less sensitive parameters. If the decision maker has the prior knowledge that some parameters can be given unlimited variations without affecting the original solution then we consider those parameters as 'nonfocal' and these 'nonfocal' parameters can be deleted from the analysis. Wang and Huang [16, 17] proposed the concept of maximum volume in the tolerance region for the multiparametric sensitivity analysis of a single objective linear programming problem. Their theory allows the more sensitive parameters called as 'focal' to be investigated at their independent levels of sensitivity, simultaneously and independently. This approach is a significant improvement over the earlier approaches primarily because besides reducing the number of parameters in the final analysis, it also handles the perturbation parameters with greater flexibility by allowing them to be investigated at their

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†Sanjeet Singh is with Operations Management Group, Indian Institute of Management Calcutta, DH Road, Joka, Kolkata-700104, India. Tel/Fax:+91-33-24678062/498, Email: sanjeet@iimcal.ac.in

‡Pankaj Gupta is with Department of Operational Research, Faculty of Mathematical Sciences, University of Delhi, Delhi-110007, India.

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independent levels of sensitivity. Singh et. al. [14, 15] extended the results of Wang and Huang [17] to discuss multiparametric sensitivity analysis for different cases of parameter perturbations in linear-plus-linear fractional programming. Also Gupta and Singh [3] studied the multiparametric sensitivity analysis of the constraint matrix in linear-plus-linear fractional programming under general perturbation. More recently, Singh [12] discussed the multiparametric sensitivity analysis of the additive model in data envelopment analysis.

Our objective in this paper is to study multiparametric sensitivity analysis for the linear sum-of-ratios fractional programming. For computational ease, we consider objective function with only two ratios. Simultaneous and independent perturbations of the objective function coefficients (both in numerator and denominators) have been considered to find the critical region where the current optimal basis remains optimal.

2 Notations and optimality Condition

Following notations will be used throughout this study:

$B \subset \{1, 2, ..., n\}$: denotes the index set of basic variables. Without loss of generality, we suppose $B = \{1, 2, ..., m\}$.
$N = \{1, 2, ..., n\} \setminus B$: denotes the index set of nonbasic variables.
A_B: The basis matrix with inverse $\beta = A_B^{-1} = [\beta_{ij}]$.
$\beta_{i\cdot}^T$: i^{th} row of the inverse basis matrix.
$\beta_{\cdot j}^T$: j^{th} column of the inverse basis matrix.
A_N: The submatrix of A corresponding to nonbasic variables.
$\bar{b} = A_B^{-1} b \geq 0$: The vector of the values corresponding to x_B.
$A_B^{-1} A_N$: $[y_{ij}]$.
$y_{\cdot j}^T$: j^{th} column in $A_B^{-1} A_N$.
$c_B = [c_1, c_2, ..., c_m]^T$: the row vector of the cost coefficie nts corresponding to x_B.
$c_N = [c_{m+1}, c_{m+2}, ..., c_n]^T$: the row vector of the cost co efficient corresponding to nonbasic variables.
$A_{\cdot j}$: j^{th} column of the matrix A.
$\bar{\Delta}_j$: The vector of the reduced cost corres-ponding to nonbasic variables.

Under the assumptions $q^T x > 0$ and $d^T x > 0$ over the feasible region, the optimality criteria for the problem P(1) using the simplex type algorithm can be derived as follows:

Let A_B be the some basis and x^* be the corresponding supporting plane

$$x^* = (x_B, 0), \qquad x_B = A_B^{-1} b\,.$$

Then the directions to the neighborhood vertices are determined by the formula [18]

$$s_j = (-A_B^{-1} A_{\cdot j} \,|\, e_j)\,,$$

where e_j is the jth basis vector. We compute $\delta_j = f'(x) s_j$, $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\delta_j &= \frac{dx^*(-c_B A_B^{-1} A_{\cdot j} + c_j) - cx^*(-d_B A_B^{-1} A_{\cdot j} + d_j)}{(dx^*)^2} \\
&\quad + \frac{qx^*(-p_B A_B^{-1} A_{\cdot j} + p_j) - px^*(-q_B A_B^{-1} A_{\cdot j} + q_j)}{(qx^*)^2} \\
&= \frac{dx^*(c_j - z_j^c) - cx^*(d_j - z_j^d)}{(dx^*)^2} + \\
&\quad \frac{qx^*(p_j - z_j^p) - px^*(q_j - z_j^q)}{(qx^*)^2} \\
&= \frac{\bar{\Delta}_j^{cd}}{(dx^*)^2} + \frac{\bar{\Delta}_j^{pq}}{(qx^*)^2}
\end{aligned}$$

where

$$\bar{\Delta}_j^{cd} = dx^*(c_j - z_j^c) - cx^*(d_j - z_j^d), \quad \bar{\Delta}_j^{pq} = qx^*(p_j - z_j^p) - px^*(q_j - z_j^q)\,.$$

If

$$\delta_j = \frac{\bar{\Delta}_j^{cd}}{(dx^*)^2} + \frac{\bar{\Delta}_j^{pq}}{(qx^*)^2} \leq 0\,, \quad j = 1, 2, \ldots, n\,, \tag{1}$$

x^* is the optimal solution.

Remark 1. In general a local optimal solution for the problem P(1) obtained using optimality criteria (1) may not be a global optimal. However, we discuss sensitivity analysis for the local/global optimal solution obtained using optimality criteria (1).

3 Sensitivity results

To address perturbations of the objective function coefficients (both in numerator and denominator) and right-hand-side vector in problem P(1), we consider the following perturbed model:

$$\begin{aligned}
\text{P(2)} \qquad & \text{Maximize } f(x) = \frac{(p + \Delta p)^T x}{(q + \Delta q)^T x} + \frac{(c + \Delta c)^T x}{(d + \Delta d)^T x} \\
& \text{subject to } \quad Ax = b + \Delta b\,, \\
& \qquad\qquad x \geq 0
\end{aligned}$$

where

$$\begin{aligned}
\Delta p &= \left[\sum_{h=1}^{H} p_{jh} \gamma_h\right]_{n \times 1}, \Delta q = \left[\sum_{h=1}^{H} q_{jh} \gamma_h\right]_{n \times 1}, \\
& \Delta c = \left[\sum_{h=1}^{H} c_{jh} \gamma_h\right]_{n \times 1}, \\
\Delta d &= \left[\sum_{h=1}^{H} d_{jh} \gamma_h\right]_{n \times 1}, \Delta b = \left[\sum_{h=1}^{H} b_{jh} \gamma_h\right]_{n \times 1}
\end{aligned}$$

are the multiparametric perturbations defined by the perturbation parameter $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T$. Here, H is the total number of parameters.

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Let,
$u_h = [b_{1h}, b_{2h}, \ldots, b_{mh}]^T$: The vector of the coefficients of parameter γ_h in Δb.
$v_h = [p_{1h}, p_{2h}, \ldots, p_{mh}]^T$: The vector of the partial coefficient of parameter γ_h in Δp.
$w_h = [q_{1h}, q_{2h}, \ldots, q_{mh}]^T$: The vector of the partial coefficient parameter γ_h in Δq.
$r_h = [c_{1h}, c_{2h}, \ldots, c_{mh}]^T$: The vector of the partial coefficient parameter γ_h in Δc.
$s_h = [d_{1h}, d_{2h}, \ldots, d_{mh}]^T$: The vector of the partial coefficient parameter γ_h in Δd.
In the following propositions we construct critical regions for simultaneous and independent perturbations of the objective function coefficients and right-hand-side vector.

Proposition 1. *When p, q, c, d and b are perturbed simultaneously and independently the critical region S of the problem* P(1) *is given by*

$$
\begin{aligned}
S \;=\; & \Bigg\{ \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \Bigg| \bar{b}_i + \sum_{h=1}^{H} (\beta_{i\cdot}^T u_h)\gamma_h \geq 0, \\
& i = 1, 2, \ldots, m; \\
& \left(q_B x_B + \sum_{h=1}^{H} (x_B^T r_h)\gamma_h \right) \left(d_B x_B + \sum_{h=1}^{H} (x_B^T s_h)\gamma_h \right) \neq 0; \\
& \Bigg[\Delta_j^{pq} + \sum_{h=1}^{H} ((p_j - z_j^p)(x_B^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h \\
& + \sum_{h=1}^{H} (q_B x_B (p_{m+j,h} - y_{\cdot j}^T v_h) - (p_B x_B)(q_{m+j,h} - \\
& y_{\cdot j}^T w_h))\gamma_h \\
& + \sum_{h=1}^{H} x_B^T w_h \sum_{h=1}^{H} (p_{m+j,h} - y_{\cdot j}^T v_h)\gamma_h - x_B^T v_h \\
& \sum_{h=1}^{H} (q_{m+j,h} - y_{\cdot j}^T w_h)\gamma_h \gamma_h \Bigg] \\
& \left[d_B x_B + \sum_{h=1}^{H} (x_B^T s_h)\gamma_h \right]^2 + \left[q_B x_B + \sum_{h=1}^{H} (x_B^T r_h)\gamma_h \right]^2 \\
& \Bigg[\Delta_j^{cd} + \sum_{h=1}^{H} ((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d)(x_B^T r_h))\gamma_h \\
& + \sum_{h=1}^{H} (d_B x_B (c_{m+j,h} - y_{\cdot j}^T r_h) - c_B x_B (d_{m+j,h} - \\
& y_{\cdot j}^T s_h))\gamma_h + \sum_{h=1}^{H} \left(x_B^T s_h \sum_{h=1}^{H} (c_{m+j,h} - y_{\cdot j}^T r_h)\gamma_h - \right. \\
& \left. x_B^T r_h \sum_{h=1}^{H} (d_{m+j,h} - y_{\cdot j}^T r_h)\gamma_h \right) \gamma_h \Bigg] \leq 0\,, \\
& j = 1, 2, \ldots, n - m \Bigg\}
\end{aligned}
$$

Proof. Let us assume that $\hat{x}_B$ be the new basic solution, then

$$
\begin{aligned}
\hat{x}_B &= A_B^{-1}(b + \Delta B) = A_B^{-1} b + A_B^{-1} \Delta b \\
&= \bar{b} + A_B^{-1} \left[\sum_{h=1}^{H} b_{1h}\gamma_h, \sum_{h=1}^{H} b_{2h}\gamma_h, \ldots, \sum_{h=1}^{H} b_{mh}\gamma_h \right]^T \\
&= \bar{b} + \left[\sum_{h=1}^{H} (\beta_{1\cdot}^T u_h)\gamma_h, \sum_{h=1}^{H} (\beta_{2\cdot}^T u_h)\gamma_h, \ldots, \sum_{h=1}^{H} (\beta_{m\cdot}^T u_h) \right]^T
\end{aligned}
$$

Now ith component of $\hat{x}_B$ is given by

$$
\hat{x}_{B_i} = \bar{b}_i + \sum_{h=1}^{H} (\beta_{i\cdot}^T u_h)\gamma_h
$$

This new basic solution $\hat{x}_B$ will be feasible if

$$
\bar{b}_i + \sum_{h=1}^{H} (\beta_{i\cdot}^T u_h)\gamma_h \geq 0\,, \quad i = 1, 2, \ldots, m\,.
$$

Now we calculate
$\hat{\Delta}_{pq} = (q_B x_B + \sum_{h=1}^{H} (x_B^T w_h)\gamma_h)(p_j - z_j^p + \sum_{h=1}^{H} (p_{jh} - y_{\cdot j}^T v_h)\gamma_h) - (p_B x_B + \sum_{h=1}^{H} (x_B^T v_h)\gamma_h)(q_j - z_j^q + \sum_{h=1}^{H} (q_{jh} - y_{\cdot j}^T w_h)\gamma_h) = \Delta_{pq} + \sum_{h=1}^{H} ((p_j - z_j^p)(x_B^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h + \sum_{h=1}^{H} (q_B x_B (p_{jh} - y_{\cdot j}^T v_h) - p_B x_B (q_{jh} - y_{\cdot j}^T w_h))\gamma_h + \sum_{h=1}^{H} (x_B^T w_h \sum_{h=1}^{H} (p_{jh} - y_{\cdot j}^T v_h)\gamma_h - x_B^T v_h \times \sum_{h=1}^{H} (q_{jh} - y_{\cdot j}^T w_h)\gamma_h)\,.$

Similarly,
$\hat{\Delta}_{cd} = \Delta_{cd} + \sum_{h=1}^{H} ((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d)(x_B^T r_h))\gamma_h + \sum_{h=1}^{H} (d_B x_B (c_{jh} - y_{\cdot j}^T r_h) - c_B x_B (d_{jh} - y_{\cdot j}^T s_h))\gamma_h + \sum_{h=1}^{H} (x_B^T s_h \sum_{h=1}^{H} (c_{jh} - y_{\cdot j}^T r_h)\gamma_h - x_B^T r_h \sum_{h=1}^{H} (d_{jh} - y_{\cdot j}^T s_h)\gamma_h)$
$\hat{q}_x = \sum_{i=1}^{m} (q_i + \sum_{h=1}^{H} q_{ih}\gamma_h) x_i = q_B x_B + \sum_{h=1}^{H} (x_B u_h)\gamma_h \hat{d}_X = \sum_{i=1}^{m} \left(d_i + \sum_{h=1}^{H} d_{ih}\gamma_h \right) x_i = d_B x_B + \sum_{h=1}^{H} (x_B^T s_h)\gamma_h.$
For the new solution $\hat{x}_B$ to satisfy the optimality condition, the new value $\hat{\delta}_j$'s of δ_j are computed as follows:

$$
\hat{\delta}_j = \frac{\hat{\Delta}_j^{pq}}{(\hat{q}x)^2} + \frac{\hat{\Delta}_j^{cd}}{(\hat{p}x)^2}
$$

Thus, $\hat{\delta}_j$ takes the form

$$
\begin{aligned}
\hat{\delta}_j \;=\; & \Bigg[\delta_j^{pq} + \sum_{h=1}^{H} ((p_j - z_j^p)(x_B^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h \\
& + \sum_{h=1}^{H} (q_B x_B (p_{jh} - y_{\cdot j}^T v_h) - p_B x_B)(q_{jh} - y_{\cdot j}^T w_h))\gamma_h
\end{aligned}
$$

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$$+\sum_{h=1}^{H} x_B^T w_h \sum_{h=1}^{H}(p_{jh}-y_{.j}^T v_h)\gamma_h - x_B^T v_h \sum_{h=1}^{H}(q_{jh} - y_{.j}^T w_h)\gamma_h\gamma_h\Bigg] \Bigg/ \left[q_B x_B + \sum_{h=1}^{H}(x_B^T w_h)\gamma_h\right]^2$$

$$+\Bigg[\Delta_j^{cd} + \sum_{h=1}^{H}((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d)(x_B^T r_h))\gamma_h + \sum_{h=1}^{H}(d_B x_B(c_{jh} - y_{.j}^T r_h) - c_B x_B(d_{jh} - y_{.j}^T s_h))\gamma_h + \sum_{h=1}^{H}\left(x_B^T s_h \sum_{h=1}^{H}(c_{jh} - y_{.j}^T r_h)\gamma_h - x_B^T r_h \times \sum_{h=1}^{H}(d_{jh} - y_{.j}^T s_h)\gamma_h\right)\gamma_h\Bigg] \Bigg/ \left[d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right]^2.$$

Now solution $\hat{x}_B$ will be optimal if

$$\left[d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right]^2 + \Bigg[\Delta_j^{pq} + \sum_{h=1}^{H}(p_j - z_j^p) \times (x_B^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h + \sum_{h=1}^{H}(q_B x_B(p_{m+j,h} - y_{.j}^T v_h) - p_B x_B(q_{m+j,h} - y_{.j}^T w_h))\gamma_h +$$
$$\sum_{h=1}^{H} x_B^T w_h \sum_{h=1}^{H}(p_{m+j,h} - y_{.j}^T v_h)\gamma_h - x_B^T v_h \sum_{h=1}^{H}(q_{m+j,h} - y_{.j}^T w_h)\gamma_h\gamma_h\Bigg]$$
$$+\left[q_B x_B + \sum_{h=1}^{H}(x_B^T r_h)\gamma_h\right]^2 \Bigg[\Delta_j^{cd} + \sum_{h=1}^{H}((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d)(x_B^T r_h))\gamma_h + \sum_{h=1}^{H}(d_B x_B(c_{m+j,h} - y_{.j}^T r_h) - c_B x_B(d_{m+j,h} - y_{.j}^T s_h))\gamma_h$$
$$+\sum_{h=1}^{H} x_B^T s_h \sum_{h=1}^{H}(c_{m+j,h} - y_{.j}^T r_h)\gamma_h - x_B^T r_h \sum_{h=1}^{H}(d_{m+j,h} - y_{.j}^T s_h)\gamma_h\gamma_h\Bigg] \Bigg/ \left[\left(q_B x_B + \sum_{h=1}^{H}(x_B^T r_h)\gamma_h\right)\left(d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right)\right]^2 \le 0, \quad j = 1,2,\ldots,n-m.$$

Thus, the critical region S, is given by $S = \Bigg\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \Bigg| \bar{b}_i + \sum_{h=1}^{H}(\beta_{i.}^T u_h)\gamma_h \ge 0, \quad i = 1,2,\ldots,m;$

$$\left(q_B x_B + \sum_{h=1}^{H}(x_B^T r_h)\gamma_h\right)\left(d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right) \ne 0;$$

$$\Bigg[\Delta_j^{pq} + \sum_{h=1}^{H}((p_j - z_j^p)(x_B^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h + \sum_{h=1}^{H}(q_B x_B(p_{m+j,h} - y_{.j}^T v_h) - (p_B x_B)(q_{m+j,h} - y_{.j}^T w_h))\gamma_h$$
$$+\sum_{h=1}^{H} x_B^T w_h \sum_{h=1}^{H}(p_{m+j,h} - y_{.j}^T v_h)\gamma_h - x_B^T v_h \sum_{h=1}^{H}(q_{m+j,h} - y_{.j}^T w_h)\gamma_h\gamma_h\Bigg]$$
$$\left[d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right]^2 + \left[q_B x_B + \sum_{h=1}^{H}(x_B^T r_h)\gamma_h\right]^2$$
$$\Bigg[\Delta_j^{cd} + \sum_{h=1}^{H}((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d)(x_B^T r_h))\gamma_h + \sum_{h=1}^{H}(d_B x_B(c_{m+j,h} - y_{.j}^T r_h) - c_B x_B(d_{m+j,h} - y_{.j}^T s_h))\gamma_h$$
$$+\sum_{h=1}^{H}\left(x_B^T s_h \sum_{h=1}^{H}(c_{m+j,h} - y_{.j}^T r_h)\gamma_h - x_B^T r_h \sum_{h=1}^{H}(d_{m+j,h} - y_{.j}^T r_h)\gamma_h\right)\gamma_h\Bigg] \le 0,$$
$$j = 1,2,\ldots,n-m\Bigg\}.$$

Hence the proof. ■

To perform multiparametric sensitivity analysis, we decompose the critical region S as follows:

$$S_{dq}^{+} = \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \Big| \left(d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right)\left(q_B x_B + \sum_{h=1}^{H}(x_B^T w_h)\gamma_h\right) > 0\}$$

$$S_{dq}^{-} = \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \Big| \left(d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right)\left(q_B x_B + \sum_{h=1}^{H}(x_B^T w_h)\gamma_h\right) < 0\}$$

$$S_{\Delta} = \Bigg\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \mid \bar{b}_i + \sum_{h=1}^{H}(\beta_{i.}^T u_h)\gamma_h \ge 0; i = 1,2,\ldots,m;$$
$$\left[d_B x_B + \sum_{h=1}^{H}(x_B^T s_h)\gamma_h\right]^2 \Bigg[\Delta_j^{pq} + \sum_{h=1}^{H}((p_j - z_j^p)(x_b^T w_h) - (q_j - z_j^q)(x_B^T v_h))\gamma_h + \sum_{h=1}^{H}(q_B x_B(p_{m+j,h} - y_{.j}^T v_h) - p_B x_B(q_{m+j,h} - y_{.j}^T w_h))\gamma_h$$
$$+\sum_{h=1}^{H} x_B^T w_h \sum_{h=1}^{H}(p_{m+j,h} - y_{.j}^T v_h)\gamma_h - x_B^T v_h \sum_{h=1}^{H}(q_{m+j,h} - y_{.j}^T w_h)\gamma_h\gamma_h\Bigg]$$
$$+\left[q_B x_B + \sum_{h=1}^{H}(x_B^T r_h)\gamma_h\right]^2 \Bigg[\Delta_j^{cd} + \sum_{h=1}^{H}((c_j - z_j^c)(x_B^T s_h) - (d_j - z_j^d))\gamma_h$$
$$\sum_{h=1}^{H} x_B^T s_h \sum_{h=1}^{H}(c_{m_j,h} - y_{.j}^T r_h)\gamma_h - x_B^T r_h \sum_{h=1}^{H}(d_{m+j,h} - y_{.j}^T s_h)\gamma_h\gamma_h\Bigg] \le 0,$$
$$j = 1,2,\ldots,n-m\Bigg\}$$

Then, $S = \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_H)^T \mid \gamma \in \{S_{dq}^{+} \cap S_{\Delta}\}$or $\gamma \in \{S_{dq}^{-} \cap S_{\Delta}\}\}$ can be decomposed into two disjoint regions: $S_1 = \{S_{dq}^{+} \cap S_{\Delta}\}$ and $S_2 = \{S_{dq}^{-} \cap S_{\Delta}\}$.

Recently, Wang and Huang [16,17] have proposed the concept of maximal volume region (MVR) within a tolerance region to investigate the different parameters at their independent levels of sensitivity. The MVR is symmetrically rectangular parallelepiped with the largest volume in a critical region and is characterized by a maximization problem.

Since the critical region is a polyhedral set, there exists $L = [\ell_{i\,j}] \in R^{I\times H}$, $g = \{g_i\} \in R^I$, $I, H \in N$, where I and H are the number of constraints and variables of S, respectively, such that $S = \{\gamma = (\gamma_1, \gamma_2, ..., \gamma_H)^T \mid L\gamma \le g\}$. Also, in practice all the parameters need not be of the same sensitivity level, therefore we classify them as 'focal' and 'non-focal' parameters. Non-focal parameters are less sensitive and hence can be deleted from the analysis. Only the more sensitive parameters called as focal are considered in the final analysis. For focal parameters, it is assumed that $\ell_{.j} \ne 0$ for $j = 1, 2, \dots, H$.

Remark 2. It follows from Proposition 1 that $\gamma = 0$ belongs to S, and thus we have $g \ge 0$.

The (MVR) B_S of a polyhedral set $S = \{\gamma = (\gamma_1, \gamma_2, \dots, \gamma_H)^T \mid L\gamma \le g\} = \{\gamma = (\gamma_1, \gamma_2, \dots, \gamma_H)^T \mid \sum_{j=1}^{H} l_{ij}\gamma_j \le g_i, i = 1, 2, \dots, I\}$, where $g_i \ge 0$ for $i = 1, 2, \dots, I$ and $\sum_{i=1}^{I} |l_{ij}| > 0$ for $j = 1, 2, \dots, H$, is $B_S = \{\gamma = (\gamma_1, \gamma_2, \dots, \gamma_H)^T \mid |\gamma_j| \le k_j^*, j = 1, 2, \dots, H\}$. Here $k^* = (k_1^*, k_2^*, \dots, k_H^*)^T$ is uniquely determined with the following two cases :

(i) If $g_i > 0$ for $i = 1, 2, \dots, I$, then k^* is the unique optimal solution of the problem P(3), where $|L|$ is obtained by changing the negative elements of matrix L to be positive

$$\begin{aligned} P(3) \qquad & \text{Max} \prod k_j \\ & \text{subject to } |L|k \le g \\ & \qquad\qquad k \ge 0\,. \end{aligned}$$

The volume of B_S is $\text{Vol}(B_S) = 2^H \prod k_j^*$.

(ii) If $g_i = 0$ for some i, let $I^\circ = \{i | g_i = 0, i = 1, 2, \dots, I\} \ne \phi$ and $I^+ = \{i | g_i > 0, i = 1, 2, \dots, I\}$ then we have

(a) If $I^+ = \phi$ then $k^* = 0$ is the unique optimal solution

(b) If $I^+ \ne \phi$ then let $\Omega = \bigcup_{i \in I^\circ} \{j \mid l_{ij} \ne 0, j = 1, 2, \dots, H\}$ be the index set of focal parameters that appear in some constraints with right-hand-side $g_i = 0$. Then $k_j^* = 0$ for all j belonging to Ω. The others, k_j^*, $j \notin \Omega$, can be uniquely determined as follows: After deleting all variables γ_j, $j \in \Omega$ and constraints with right-hand-side $g_i = 0$ from the system of constraints S, let the remaining subsystem be in the form of (2) with $g_i' > 0$ for all index i as below:

$$S' = \{\gamma' = [\gamma_j]^T, j \notin \Omega | L'\gamma' \le g'\} \qquad (2)$$

then $k^{*\prime}$ (i.e., $k_j^*, j \notin \Omega$) can be uniquely determined by solving the following problem P(4)

$$\begin{aligned} P(4) \qquad & \text{Max} \prod_{j\notin\Omega} k_j \\ & \text{subject to } |L'|k' \le g' \\ & \qquad\qquad k' \ge 0\,. \end{aligned}$$

The volume of B_S is $\text{Vol}(B_S) = 2^H \prod_{j \notin \Omega} k_j^*$.

Multiparametric sensitivity analysis of the problem P(1) can now be performed as follows :

Obtain the critical region as given in Proposition 1 by considering simultaneous and independent perturbations with respect to the objective function coefficients and right-hand-side vector. Delete all the non-focal parameters from the analysis. The MVR of the critical regions is obtained by solving the problem P(3)/P(4). The problem P(3)/P(4) can be solved by existing techniques such as Dynamic Programming. The detailed algorithm can be found in Wang and Huang [17]. Software GINO [5] can also be used to solve the nonlinear programming problem P(3)/P(4).

4. Numerical Example

$$\mathbf{P(5)} \quad \text{Maximize } f(x) = \frac{x_1 - 2x_2}{3x_1 - 2x_2 + 1} + \frac{x_1 - x_2}{2x_1 + x_2 + 2}$$

$$\begin{aligned} \text{subject to} \quad & 2x_1 - x_2 \le 10 \\ & x_1 + x_2 \quad\ \le 8 \\ & x_j \ge 0, \quad j = 1, 2\,. \end{aligned}$$

The optimal solution is

$$x^* = [x_1^*, x_2^*]^T = [6, 2]^T.$$

Multiparametric Perturbations:
$\Delta p = [\gamma_1 + \gamma_2, 2\gamma_1 - \gamma_2]$, $\Delta c = [3\gamma_1 - 2\gamma_2, 0]$, $\Delta q = [\gamma_2, \gamma_1 + 2\gamma_2]$,
$\Delta b = [4\gamma_1 + 2\gamma_2, \gamma_1 + 3\gamma_3]$
$u_1 = [4, 1]^T, u_2 = [2, 3]^T, v_1 = [1, 2]^T, v_2 = [1, -1]^T, w_1 = [0, 1]^T, w_2 = [1, 2]^T, r_1 = [3, 0]^T, r_2 = [-2, 0]^T$
$S_{dq^+} = \{\gamma = (\gamma_1, \gamma_2)^T | 16 + 2\gamma_1 + 10\gamma_2 > 0\}$.
$S_\Delta = \{\gamma = (\gamma_1, \gamma_2)^T | 6 + 0.6\gamma_1 + 1.67\gamma_2 \ge 0, 2 - 0.67\gamma_1 + 1.33\gamma_2 \ge 0, 26.92 + 5.63\gamma_1 + 3.43\gamma_2 + 2.1\gamma_1^2 + 7.08\gamma_2^2 \le 0, 19.07 - 0.89\gamma_1 + 4.51\gamma_2 + 6\gamma_1^2 + 2.81\gamma_2^2 \le 0\}$.
Therefore critical region S is given by
$S = \{\gamma = (\gamma_1, \gamma_2)^T | \gamma \in S_{dq^+} \cap S_\Delta\}$
$= \{\gamma = (\gamma_1, \gamma_2)^T | 16 + 2\gamma_1 + 10\gamma_2 > 0, 6 + 0.6\gamma_1 + 1.67\gamma_2 \ge 0, 2 - 0.67\gamma_1 + 1.33\gamma_2 \ge 0, 26.92 + 5.63\gamma_1 + 3.43\gamma_2 + 2.1\gamma_1^2 + 7.08\gamma_2^2 \le 0, 19.07 - 0.89\gamma_1 + 4.51\gamma_2 + 6\gamma_1^2 + 2.81\gamma_2^2 \le 0\}$. Because of the involvement of nonlinear inequalities in S, the critical region S is not a rectangular parallelepiped. Moreover, S may not be bounded.

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Therefore, in this case Wang and Huang's approach of finding the Maximal volume region is not applicable. In this situation we can only say that optimal basis will remain unchanged for all those perturbations in parameters for which inequalities in S are satisfied.

4 Conclusions and Future Work

In this paper, we have studied multiparametric sensitivity analysis for the sum-of-ratios programming problem. We have considered objective function with only two ratios in the objective function but our approach can be extended to any number of linear ratios in the objective function. We have derived critical regions for the simultaneous and independent perturbations of the objective function coefficients (both in numerator and denominator) and right-hand-side vector. However, because of the presence of nonlinear inequalities in the critical region, critical region is not rectangular parallelepiped. As a direction for future research further research can be carried out to explore the methods to approximate nonlinear inequalities in the critical region with linear inequalities so that critical region can be approximated as rectangular parallelepiped.

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