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A Fuzzy Approach for Hierarchical Multiobjective Linear Programming Problems

Hitoshi Yano *

Abstract— **In this paper, we focus on hierarchical multiobjective linear programming problems where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints, and propose a fuzzy approach to obtain a satisfactory solution which reflects not only the hierarchical relationships between multiple decision makers but also their own preferences for their membership functions. In the proposed method, instead of Pareto optimal concept, a generalized Λ-extreme point concept is introduced. In order to obtain a satisfactory solution from among a generalized Λ-extreme point set, an interactive algorithm based on linear programming is proposed, and an interactive processes are demonstrated by means of an illustrative numerical example.**

Keywords: hierarchical multiobjective linear programming, decision powers, Λ-extreme points, hyperplane method, fuzzy approach

1 Introduction

In the real-world decision making situations, it is often required that the goal of the overall system is achieved in the hierarchical structure, where many decision makers who belong to its sections or divisions are in action to seek their own goals independently and are affected each other. The Stackelberg games [1, 9] can be regarded as multilevel programming problems with multiple decision makers. Although many kinds of techniques to obtain a Stackelberg solution have been proposed, almost all of such techniques are unfortunately not efficient in computational aspects.

In order to circumvent the computation inefficiency to obtain such a Stackelberg solution and the paradox that the lower level decision power often dominates the upper level decision power, Lai [3], Shih et al.[8] and Lee et al.[4] introduced concepts of memberships of optimalities and degrees of decision powers and proposed fuzzy approaches to multilevel linear programming problems. In their approaches, each decision maker elicits his/her own membership functions for not only the objective functions but also the decision variables. Following the fuzzy decision [5] together with membership functions, the mathematical programming problem of finding the maximum decision is formulated and solved to obtain a candidate of the satisfactory solution. However, in such fuzzy approaches for multilevel linear programming problems, the decision makers are required to elicit each of membership functions for not only the objective functions but also the decision variables, and to update them in each of the iterations. It seems to be very difficult to elicit membership functions for the decision variables.

From a different point of view, Shi [7] especially focused on multiple criteria linear programming problems with multiple decision makers. In his approaches, it is assumed that each decision maker has different resource availability levels for the constraints. He formulated such multiple criteria multiple constraint linear programming problems called MC^2 linear problems and introduced the corresponding solution concept called potential solutions.

In this paper, we especially focus on hierarchical fuzzy multiobjective linear programming problems where multiple decision makers in a hierarchical organization have fuzzy goals for their own multiple objective linear functions together with common linear constraints. In section 2, hierarchical fuzzy multiobjective linear programming problems are formulated and the corresponding solution concept called a generalized Λ-extreme point is introduced. In section 3, an interactive algorithm is proposed to obtain the satisfactory solution from among a generalized Λ-extreme point set, where the corresponding hyperplane problem [6, 10] is solved. In section 4, interactive processes of the proposed method are demonstrated by means of an illustrative numerical example.

2 Hierarchical Fuzzy Multiobjective Linear Programming Problems

We consider the following hierarchical multiobjective linear programming problems (HMOLP), where each decision maker (DM_r) has his/her own multiple objective linear functions together with common linear constraints.

[HMOLP]

first level decision maker : DM_1

*Nagoya City University, School of Humanities and Social Sciences, 1, Yamanohata, Mizuho-cho, Mizuho-ku, Nagoya 467-8501, Japan, Tel: +81-52-872-5182, Fax: +81-52-872-1531, Email: yano@hum.nagoya-cu.ac.jp

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$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_1\boldsymbol{x} = (\boldsymbol{c}_{11}\boldsymbol{x}, \boldsymbol{c}_{12}\boldsymbol{x}, \cdots, \boldsymbol{c}_{1k_1}\boldsymbol{x})^T \quad (1)$$

second level decision maker : $\mathbf{DM}_2$

$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_2\boldsymbol{x} = (\boldsymbol{c}_{21}\boldsymbol{x}, \boldsymbol{c}_{22}\boldsymbol{x}, \cdots, \boldsymbol{c}_{2k_2}\boldsymbol{x})^T \quad (2)$$

....

p-th level decision maker : $\mathbf{DM}_p$

$$\min_{\boldsymbol{x}\in X} \boldsymbol{C}_p\boldsymbol{x} = (\boldsymbol{c}_{p1}\boldsymbol{x}, \boldsymbol{c}_{p2}\boldsymbol{x}, \cdots, \boldsymbol{c}_{pk_p}\boldsymbol{x})^T \quad (3)$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is n-dimensional decision vector, $X \in \mathbf{E}^n$ is a linear constraint set of $\boldsymbol{x}$, and $\boldsymbol{c}_{ri} = (c_{ri1}, c_{ri2}, \cdots, c_{rin})$, $i = 1, \cdots, k_r, r = 1, \cdots, p$ are n-dimensional row vectors, $\boldsymbol{C}_r = (\boldsymbol{c}_{r1}, \boldsymbol{c}_{r2}, \cdots, \boldsymbol{c}_{rk_r})^T$, $r = 1, \cdots, p$ are $(k_r \times n)$-dimensional matrices.

By considering the vague nature of human's subjective judgements, it is quite natural to assume that the decision makers may have fuzzy goals [5] for the objective functions. Through the interaction with the decision maker (DM_r), these fuzzy goals can be quantified by eliciting the corresponding membership functions, which are denoted by $\mu_{ri}(\boldsymbol{c}_{ri}\boldsymbol{x}), i = 1, \cdots, k_r$. Then, HMOLP can be formally transformed to the following hierarchical fuzzy multiobjective linear programming problem (HFMOLP).

[HFMOLP]

first level decision maker : $\mathbf{DM}_1$

$$\max_{\boldsymbol{x}\in X} \boldsymbol{\mu_1}(\boldsymbol{C}_1\boldsymbol{x}) = (\mu_{11}(\boldsymbol{c}_{11}\boldsymbol{x}), \cdots, \mu_{1k_1}(\boldsymbol{c}_{1k_1}\boldsymbol{x}))^T \quad (4)$$

second level decision maker : $\mathbf{DM}_2$

$$\max_{\boldsymbol{x}\in X} \boldsymbol{\mu_2}(\boldsymbol{C}_2\boldsymbol{x}) = (\mu_{21}(\boldsymbol{c}_{11}\boldsymbol{x}), \cdots, \mu_{2k_2}(\boldsymbol{c}_{2k_2}\boldsymbol{x}))^T \quad (5)$$

....

p-th level decision maker : $\mathbf{DM}_p$

$$\max_{\boldsymbol{x}\in X} \boldsymbol{\mu_p}(\boldsymbol{C}_p\boldsymbol{x}) = (\mu_{p1}(\boldsymbol{c}_{p1}(\boldsymbol{x})), \cdots, \mu_{pk_p}(\boldsymbol{c}_{pk_p}\boldsymbol{x}))^T \quad (6)$$

In this paper, we assume that each decision maker (DM_r) in HFMOLP finds his/her satisfactory solution from among Λ_r-extreme point set which can be regarded as a generalized version of Pareto optimal solution set. Λ_r-extreme point [11] is defined by a cone Λ_r in membership space of DM_r as follows.

Definition 1. $\boldsymbol{y}_r^* \in \boldsymbol{\mu}_r(\boldsymbol{C}_r X)$ is said to be a Λ_r-extreme point of $\boldsymbol{\mu}_r(\boldsymbol{C}_r X)$ to FMOLP_r, if there is no $\boldsymbol{y}_r \in \boldsymbol{\mu}_r(\boldsymbol{C}_r X)$ such that $\boldsymbol{y}_r^* \in \boldsymbol{y}_r - \Lambda_r, \boldsymbol{y}_r^* \neq \boldsymbol{y}_r$, where $\boldsymbol{\mu}_r(\boldsymbol{C}_r X) = \{\boldsymbol{\mu}_r(\boldsymbol{C}_r\boldsymbol{x}) \in E^{k_r} \mid \boldsymbol{x} \in X\}$, $\Lambda_r \subset E^{k_r}$ is a cone, and FMOLP_r is DM_r's fuzzy multiobjective linear programming problem formulated as follows:

[$\mathbf{FMOLP}_r$]

$$\max_{\boldsymbol{x}\in X} \boldsymbol{\mu}_r(\boldsymbol{C}_r\boldsymbol{x}) = (\mu_{r1}(\boldsymbol{c}_{r1}\boldsymbol{x}), \cdots, \mu_{rk_r}(\boldsymbol{c}_{rk_r}\boldsymbol{x}))^T \quad (7)$$

In the following, let us assume that each membership function for the objective function is a linear function defined as :

$$\mu_{ri}(\boldsymbol{c}_{ri}\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{c}_{ri}\boldsymbol{x} \leq z_{ri}^1 \\ \dfrac{\boldsymbol{c}_{ri}\boldsymbol{x} - z_{ri}^0}{z_{ri}^1 - z_{ri}^0} & z_{ri}^1 \leq \boldsymbol{c}_{ri}\boldsymbol{x} \leq z_{ri}^0 \\ 0 & \boldsymbol{c}_{ri}\boldsymbol{x} \geq z_{ri}^0 \end{cases} \quad (8)$$

where z_{ri}^0 or z_{ri}^1 denotes the value of the objective function $\boldsymbol{c}_{ri}\boldsymbol{x}$ such that the degree of membership function is 0 or 1 respectively.

According to the notation of Yu [11], let us denote a set of Λ_r-extreme points as $\mathrm{Ext}[\boldsymbol{\mu}_r(\boldsymbol{C}_r X) \mid \Lambda_r]$. Unfortunately, although $\mathrm{Ext}[\boldsymbol{\mu}_r(\boldsymbol{C}_r X) \mid \Lambda_r]$ can be applied to FMOLP_r, $\mathrm{Ext}[\boldsymbol{\mu}_r(\boldsymbol{C}_r X) \mid \Lambda_r]$ can not to be directly applied to HFMOLP, because multiple decision makers $\mathrm{DM}_r, r = 1, \cdots, p$ in the hierarchical structure have to seek their common satisfactory solution to HFMOLP. Therefore, in order to deal with HFMOLP, we introduce the following extended concept called a generalized Λ-extreme point where cones Λ_r, $r = 1, \cdots, p$ are integrated in membership space of $\mathrm{DM}_r, r = 1, \cdots, p$.

Definition 2. $\boldsymbol{y}* \in \boldsymbol{\mu}(\boldsymbol{C}X)$ is said to be a generalized Λ-extreme point to HFMOLP, if there is no $\boldsymbol{y} \in \boldsymbol{\mu}(\boldsymbol{C}X)$ such that $\boldsymbol{y}* \in \boldsymbol{y} - \Lambda, \boldsymbol{y}* \neq \boldsymbol{y}$, where $\boldsymbol{\mu}(\boldsymbol{C}X) = \{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) = (\boldsymbol{\mu}_1(\boldsymbol{C}_1\boldsymbol{x}), \cdots, \boldsymbol{\mu}_p(\boldsymbol{C}_p\boldsymbol{x})) \in \mathbf{E}^{\sum_{r=1}^p k_r} \mid \boldsymbol{x} \in X\}$, and a cone Λ is defined as follows.

$$\Lambda = \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_p \quad (9)$$

where $\otimes$ means Cartesian product.

Similar to $\mathrm{Ext}[\boldsymbol{\mu}_r(\boldsymbol{C}_r X) \mid \Lambda_r]$, let us denote a set of generalized Λ-extreme points in membership space of all decision makers as $\mathrm{Ext}[\boldsymbol{\mu}(\boldsymbol{C}X) \mid \Lambda]$, and the corresponding set of Λ-extreme points in decision space as $\mathrm{Ext}[X \mid \Lambda]$, respectively.

Since it is very difficult to deal with a cone Λ directly, in the following, let us assume that $\Lambda_r, r = 1, \cdots, p$ are polyhedral cones defined as follows:

$$\Lambda_r = \{\sum_{i=1}^{k_r} \alpha_{ri}\boldsymbol{v}_{ri},\ \alpha_{ri} \geq 0, i = 1, \cdots, k_r\} \quad (10)$$

where $\boldsymbol{v}_{ri}, i = 1, \cdots, k_r$ are generators of a cone Λ_r, *i.e.*,

$$\boldsymbol{v}_{ri} = (v_{ri1}, v_{ri2}, \cdots, v_{rik_r})^T \in \mathbf{E}^{k_r}, \quad (11)$$

and $\boldsymbol{v}_{ri}$ is assumed to satisfy the following condition.

$$\| \boldsymbol{v}_{ri} \| = \sqrt{\sum_{j=1}^{k_r} v_{rij}^2} = 1. \tag{12}$$

Using generators $\boldsymbol{v}_{ri}, i = 1, \cdots, k_r$, $(k_r \times k_r)$-dimensional generator matrix $\boldsymbol{V}_r$ of a cone Λ_r can be formulated.

$$\boldsymbol{V}_r = \begin{pmatrix} v_{r11} & v_{r21} & \cdots & v_{rk_r1} \\ v_{r12} & v_{r22} & \cdots & v_{rk_r2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r1k_r} & v_{r2k_r} & \cdots & v_{rk_rk_r} \end{pmatrix} \tag{13}$$

Moreover, on the basis of matrices $\boldsymbol{V}_r, r = 1, \cdots, p$, $(\sum_{r=1}^{p} k_r \times \sum_{r=1}^{p} k_r)$-dimensional generator matrix $\boldsymbol{V}$ of a cone Λ can be formulated as follows.

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{V}_p \end{pmatrix} \tag{14}$$

Then, an integrated cone Λ defined by (9) can be expressed as follows.

$$\Lambda = \boldsymbol{V} \cdot \boldsymbol{\alpha}^T \tag{15}$$

where $\boldsymbol{\alpha_r} = (\alpha_{r1}, \alpha_{r2}, \cdots, \alpha_{rk_r}) \geq \boldsymbol{0}, r = 1, \cdots, p$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_p) \in \mathbf{E}^{\sum_{r=1}^{p} k_r}$.

Since inverse matrices $\boldsymbol{V}_r^{-1}$ for $\boldsymbol{V}_r, r = 1, \cdots, p$ exist, an inverse matrix $\boldsymbol{V}^{-1}$ for $\boldsymbol{V}$ becomes as follows.

$$\boldsymbol{V}^{-1} = \begin{pmatrix} \boldsymbol{V}_1^{-1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}_2^{-1} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{V}_p^{-1} \end{pmatrix} \tag{16}$$

For generating a candidate of the satisfactory solution from among a generalized Λ-extreme point set $\mathrm{Ext}[\boldsymbol{\mu}(\boldsymbol{C}X) \mid \Lambda]$, each decision maker ($\mathrm{DM}_r$) is asked to specify his/her reference membership values $\bar{\boldsymbol{\mu}}_r = (\bar{\mu}_{r1}, \bar{\mu}_{r2}, \cdots, \bar{\mu}_{rk_r})$ [5] which are reference levels of achievement of membership functions. Once the reference membership values are specified, the corresponding generalized Λ-extreme point, which is, in a sense, close to their requirement, is obtained by solving the following hyperplane problem [6, 10].

[HP1($\bar{\boldsymbol{\mu}}$)]

$$\min_{\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1} x_{n+1} \tag{17}$$

subject to

$$\boldsymbol{V}^{-1} \cdot \{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \tag{18}$$

where $\boldsymbol{x}_{n+1} = (x_{n+1}, x_{n+1}, \cdots, x_{n+1})^T \in \mathbf{E}^{\sum_{r=1}^{p} k_r}$.

The relationships between the optimal solution to HP1($\bar{\boldsymbol{\mu}}$) and the corresponding generalized Λ-extreme point set $\mathrm{Ext}[X \mid \Lambda]$ can be characterized by the following theorems.

Theorem 1. If $(\boldsymbol{x}^*, x_{n+1}^*)$ is a unique optimal solution to HP1($\bar{\boldsymbol{\mu}}$), then $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$.

(Proof) Assume $\boldsymbol{x}^* \notin \mathrm{Ext}[X \mid \Lambda]$, then there exist $\boldsymbol{x} \in X$ and $\boldsymbol{\lambda} \in \Lambda$ (or equivalently $\boldsymbol{\alpha} \geq \boldsymbol{0}$) such that

$$\begin{aligned} \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) &= \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{\lambda} \\ &= \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{V} \cdot \boldsymbol{\alpha}^T. \end{aligned}$$

If $(\boldsymbol{x}^*, x_{n+1}^*)$ is an optimal solution to HP1($\bar{\boldsymbol{\mu}}$),

$$\begin{aligned} & \boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{x}_{n+1}^*\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) + \boldsymbol{V} \cdot \boldsymbol{\alpha}^T - \boldsymbol{x}_{n+1}^*\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{x}_{n+1}^*\} \leq -\boldsymbol{\alpha}^T \leq \boldsymbol{0}. \end{aligned}$$

This implies that $\boldsymbol{x}^*$ is not a unique optimal solution to HP1($\bar{\boldsymbol{\mu}}$).

Theorem 2. If $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$, then $(\boldsymbol{x}^*, x_{n+1}^*)$ is an optimal solution to HP1($\bar{\boldsymbol{\mu}}$) for some reference membership values $\bar{\boldsymbol{\mu}}$, where $(\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{x}_{n+1}^*) = \boldsymbol{0}$.

(Proof) Assume that $(\boldsymbol{x}^*, x_{n+1}^*)$ is not an optimal solution to HP1($\bar{\boldsymbol{\mu}}$). Then, there exist $\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1$ such that

$$\boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0},\ x_{n+1} < x_{n+1}^*.$$

Moreover, because of $(\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{x}_{n+1}^*) = \boldsymbol{0}$, the following inequality relations must be satisfied.

$$\begin{aligned} & \boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}\{\bar{\boldsymbol{\mu}}^T - \boldsymbol{x}_{n+1}^* - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) + \boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}\{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) + \boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{0} < \boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1} \in \Lambda$, there exists $\boldsymbol{\alpha} \geq \boldsymbol{0}$ such that $\boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1} = \boldsymbol{V} \cdot \boldsymbol{\alpha}^T$. Therefore, it holds that

$$\begin{aligned} & \boldsymbol{V}^{-1}\{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) + \boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1}\} \leq \boldsymbol{0}, \\ \Leftrightarrow\ & \boldsymbol{V}^{-1}\{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x})\} \leq -\boldsymbol{V}^{-1} \cdot (\boldsymbol{x}_{n+1}^* - \boldsymbol{x}_{n+1}) \\ \Leftrightarrow\ & \boldsymbol{V}^{-1} \cdot \{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x})\} \leq -\boldsymbol{\alpha}^T \leq \boldsymbol{0}. \end{aligned}$$

There exists $\boldsymbol{\beta}$ such that $\boldsymbol{V}^{-1} \cdot \{\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x})\} = -\boldsymbol{\beta}^T \leq \boldsymbol{0}$. This implies that $\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{V} \cdot \boldsymbol{\beta}^T = \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*)$, *i.e.*, $\boldsymbol{x}^* \notin \mathrm{Ext}[X \mid \Lambda]$.

It should be noted here that, in general, the generalized extreme point obtained by solving HP1($\bar{\boldsymbol{\mu}}$) does not reflect the hierarchical structure between p decision makers where the upper level decision maker can take priority for his/her membership functions over the lower level decision makers. In order to cope with such a hierarchical preference structure between p decision makers, we introduce the decision powers $\boldsymbol{w} = (w_1, w_2, \cdots, w_p)^T \in \mathbf{E}^p$

[3] in HP1($\bar{\boldsymbol{\mu}}$), where the r-th level decision maker (DM$_r$) can specify the decision power w_{r+1} in his/her subjective manner and the last decision maker (DM$_p$) has no decision power. In order to reflect the hierarchical preference structure between multiple decision makers, the decision powers $\boldsymbol{w} = (w_1, w_2, \cdots, w_p)^T$ have to satisfy the following inequality condition.

$$w_1 = 1 \geq w_2 \geq \cdots\cdots \geq w_{p-1} \geq w_p > 0 \tag{19}$$

Then, the corresponding modified HP1($\bar{\boldsymbol{\mu}}$) is reformulated as follows:

[HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$)]

$$\min_{\boldsymbol{x} \in X, x_{n+1} \in \mathrm{E}^1} \quad x_{n+1} \tag{20}$$

subject to

$$\boldsymbol{V}^{-1} \cdot \begin{pmatrix} \bar{\boldsymbol{\mu}}_{\boldsymbol{1}} - \boldsymbol{\mu}_1(\boldsymbol{C}_1\boldsymbol{x}) - \boldsymbol{x}_{n+1}/w_1 \\ \bar{\boldsymbol{\mu}}_{\boldsymbol{2}} - \boldsymbol{\mu}_2(\boldsymbol{C}_2\boldsymbol{x}) - \boldsymbol{x}_{n+1}/w_2 \\ \vdots \\ \bar{\boldsymbol{\mu}}_{\boldsymbol{p}} - \boldsymbol{\mu}_p(\boldsymbol{C}_p\boldsymbol{x}) - \boldsymbol{x}_{n+1}/w_p \end{pmatrix} \leq \boldsymbol{0} \tag{21}$$

In the following, let us denote (i,j)-element of $\boldsymbol{V}_r^{-1}$ as q_{rij}. Then, the constraints (21) are equivalently expressed as follows.

$$\sum_{j=1}^{k_r} q_{rij}(\bar{\mu}_{rj} - \mu_{rj}(\boldsymbol{c}_{rj}\boldsymbol{x}) - x_{n+1}/w_r) \leq 0,$$
$$i = 1, \cdots, k_r, r = 1, \cdots, p. \tag{22}$$

The relationships between the optimal solution of HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$) and generalized Λ-extreme points can be characterized by the following theorem.

Theorem 3. If $(\boldsymbol{x}^*, x_{n+1}^*)$ is a unique optimal solution to HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$), then $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$.

It must be observed here that for generating a generalized Λ-extreme point using the above theorem, uniqueness of solution must be verified. In order to test whether a current optimal solution $\boldsymbol{x}^*$ of HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$) is a generalized Λ-extreme point or not, we formulate and solve the following linear programming problem.

[Test problem for $\boldsymbol{x}^*$]

$$\max_{\boldsymbol{x} \in X} \sum_{r=1}^{p} \sum_{i=1}^{k_r} \epsilon_{ri} \tag{23}$$

subject to

$$\boldsymbol{V}^{-1} \cdot (\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*)) = \boldsymbol{\epsilon}^T \tag{24}$$

$$\boldsymbol{\epsilon} = (\epsilon_{11}, \cdots, \epsilon_{1k_1}, \cdots, \epsilon_{p1}, \cdots, \epsilon_{pk_p}) \geq \boldsymbol{0} \tag{25}$$

The following theorem guarantees that the optimal solution $\bar{\boldsymbol{x}}$ of the above test problem is a generalized Λ-extreme point.

Theorem 4. Let $\boldsymbol{x}^* \in X$ be an optimal solution to HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$), and $\bar{\boldsymbol{x}} \in X$ and $\bar{\boldsymbol{\epsilon}} \geq \boldsymbol{0}$ be an optimal solution to test problem for $\boldsymbol{x}^*$. Then, if all $\bar{\epsilon}_{ri} = 0, r = 1, \cdots, p, i = 1, \cdots, k_r$, then $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$. If at least one $\bar{\epsilon}_{ri} > 0$, then $\bar{\boldsymbol{x}} \in \mathrm{Ext}[X \mid \Lambda]$.

(Proof) Let all $\bar{\epsilon}_{ri} = 0, r = 1, \cdots, p, i = 1, \cdots, k_r$. Then, there is no $\boldsymbol{x} \in X$ and $\bar{\boldsymbol{\epsilon}} \geq \boldsymbol{0} (\bar{\boldsymbol{\epsilon}} \neq \boldsymbol{0})$ such that $\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) = \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T$. This means that $\boldsymbol{x}^* \in \mathrm{Ext}[X \mid \Lambda]$. Let some $\bar{\epsilon}_{ri} > 0$. Then, it holds that $\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) = \boldsymbol{\mu}(\boldsymbol{C}\bar{\boldsymbol{x}}) - \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T$. Assume $\bar{\boldsymbol{x}} \notin \mathrm{Ext}[X \mid \Lambda]$. Then, there are some $\boldsymbol{x} \in X$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$ such that $\boldsymbol{\mu}(\boldsymbol{C}\bar{\boldsymbol{x}}) = \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{V} \cdot \boldsymbol{\alpha}^T$. This means that

$$\begin{aligned} & \boldsymbol{\mu}(\boldsymbol{C}\bar{\boldsymbol{x}}) = \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{V} \cdot \boldsymbol{\alpha}^T = \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) + \boldsymbol{V} \cdot \bar{\boldsymbol{\epsilon}}^T, \\ \Leftrightarrow \quad & -\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*) + \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) = \boldsymbol{V} \cdot (\bar{\boldsymbol{\epsilon}}^T + \boldsymbol{\alpha}^T), \\ \Leftrightarrow \quad & \boldsymbol{V}^{-1}(\boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}) - \boldsymbol{\mu}(\boldsymbol{C}\boldsymbol{x}^*)) = (\bar{\boldsymbol{\epsilon}}^T + \boldsymbol{\alpha}^T). \end{aligned}$$

This contradicts that $\bar{\boldsymbol{\epsilon}}$ is an optimal solution of test problem for $\boldsymbol{x}^*$.

3 An Interactive Algorithm

After obtaining a generalized Λ-extreme point $\boldsymbol{x}^*$ by solving HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$), each decision maker must either be satisfied with the current values of membership functions, or update his/her decision power w_r and/or his/her reference membership values $\bar{\boldsymbol{\mu}}_{\boldsymbol{r}} = (\bar{\mu}_{r1}, \cdots, \bar{\mu}_{rk_r})$.

In order to help each decision maker update his/her reference membership values and/or the decision powers, trade-off information [2] is very useful. Such trade-off information is obtainable since it is related to the simplex multipliers of HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$).

Theorem 5. Let $(\boldsymbol{x}^*, x_{n+1}^*)$ be a unique and nondegenerate optimal solution of HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$), and let the constraints with the reference membership values $\bar{\mu}_{rj}, j = 1, \cdots, k_r$ be active. Then, the following relation holds.

$$-\left.\frac{\partial(\mu_{rj_1}(\boldsymbol{c}_{rj_1}\boldsymbol{x}))}{\partial(\mu_{rj_2}(\boldsymbol{c}_{rj_2}\boldsymbol{x}))}\right|_{\boldsymbol{x}=\boldsymbol{x}^*} = \frac{\sum_{i=1}^{k_r} \pi_{ri}^* q_{rij_2}}{\sum_{i=1}^{k_r} \pi_{ri}^* q_{rij_1}} \tag{26}$$

where $\pi_{ri}^* > 0$ is the corresponding simplex multipliers for the constraint (22) of HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$).

Theorem 6. Let $(\boldsymbol{x}^*, x_{n+1}^*)$ be a unique and nondegenerate optimal solution of HP2($\boldsymbol{w}^*, \bar{\boldsymbol{\mu}}$), and let the constraint with the reference membership values $\bar{\mu}_{rj}, j = 1, \cdots, k_r$ be active. Then, the following relation holds.

$$\left.\frac{\partial(\mu_{rj}(\boldsymbol{c}_{rj}\boldsymbol{x}))}{\partial w_r}\right|_{\boldsymbol{x}=\boldsymbol{x}^*} = \frac{x_{n+1}^*}{w_r^{*2}} - \frac{x_{n+1}^*}{w_r^{*3}} \left\{ \sum_{i=1}^{k_r} \pi_{ri}^* \sum_{j=1}^{k_r} q_{rij} \right\} \tag{27}$$

where $\pi^*_{ri} > 0$ is a simplex multiplier for the constraints (22) in HP2($\boldsymbol{w}^*, \bar{\boldsymbol{\mu}}$).

Now, we can construct the interactive algorithm to derive the satisfactory solution of multiple decision makers in a hierarchical organization from among the generalized Λ-extreme point set.

Step 1: Elicit linear membership functions $\mu_{ri}(\boldsymbol{c}_{ri}\boldsymbol{x})$ for the objective functions $\boldsymbol{c}_{ri}(\boldsymbol{x}), i = 1, \cdots, k_r$ from each decision maker (DM$_r$), $r = 1, \cdots, p$.

Step 2: Set the initial decision powers $w_r = 1$ and the initial reference membership values $\bar{\mu}_{ri} = 1, i = 1, \cdots, k_r, r = 1, \cdots, p$.

Step 3: For the specified decision powers and the specified reference membership values, solve HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$), and obtain the corresponding generalized Λ-extreme point $(\boldsymbol{x}^*, x^*_{n+1})$ and trade-off information. If $x^*_{n+1} \geq 0$, then go to Step 4. If $x^*_{n+1} < 0$, then update reference membership values as $\hat{\mu}_{ri} \leftarrow \bar{\mu}_{ri} - x^*_{n+1}/w_r$, $i = 1, \cdots, k_r, r = 1, \cdots, p$. Solve HP2($\boldsymbol{w}, \hat{\boldsymbol{\mu}}$) again, and go to Step 4.

Step 4: If each decision maker is satisfied with the current values of his/her membership functions, then stop. Otherwise, let the s-th level decision maker (DM$_s$) be the uppermost of the decision makers who are not satisfied with the current values. Considering the current values of his/her membership functions and two kinds of trade-off rates, DM$_s$ updates his/her decision power w_{s+1} and/or his/her reference membership values $\bar{\mu}_{si}, i = 1, \cdots, k_s$ according to the following two rules, and return to Step 3.

(1) the rule of updating w_{s+1}: In order to satisfy the condition (19), w_{s+1} must be set as $w_{s+1} \leq w_s$. After updating w_{s+1}, if $w_{s+1} < w_t, s+1 < t \leq p$, w_t is replaced by w_{s+1} ($w_t \leftarrow w_{s+1}$). Here, it should be noted for DM$_s$ that the less value of the decision power w_{s+1} gives better values of membership functions of DM$_r (1 \leq r \leq s)$ at the expense of the ones of DM$_r (s+1 \leq r \leq p)$ for some fixed reference membership values.

(2) the rule of updating $\bar{\mu}_{si}$: After setting $\bar{\mu}_{ri} \leftarrow \boldsymbol{\mu}_{ri}(\boldsymbol{c}_{ri}\boldsymbol{x}*), i = 1, \cdots, k_r, r = 1, \cdots, p, r \neq s$, DM$_s$ updates his/her reference membership values $\bar{\mu}_{si}, i = 1, \cdots, k_s$. Here, it should be stressed for DM$_s$ that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions for some fixed decision powers.

4 A Numerical Example

In order to demonstrate the proposed method and the interactive process, we consider the following hierarchical two-objective linear programming problem.

[HMOLP]

first level decision maker : DM$_1$

$$\max \boldsymbol{C}_1\boldsymbol{x} = \begin{pmatrix} \boldsymbol{c}_{11}\boldsymbol{x} \\ \boldsymbol{c}_{12}\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 10x_1 + 2x_2 + x_3 + x_4 \\ x_1 + 13x_2 + 2x_3 + x_4 \end{pmatrix}$$

second level decision maker : DM$_2$

$$\max \boldsymbol{C}_2\boldsymbol{x} = \begin{pmatrix} \boldsymbol{c}_{21}\boldsymbol{x} \\ \boldsymbol{c}_{22}\boldsymbol{x} \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 11x_3 + x_4 \\ 2x_1 + x_2 + x_3 + 14x_4 \end{pmatrix}$$

subject to

$$\boldsymbol{X} = \{\boldsymbol{x} = (x_1, x_2, x_2, x_4)^T \geq \boldsymbol{0} \mid x_1 + x_2 + x_3 + x_4 \leq 30\}$$

By considering the maximum values and the minimum values of the objective functions $\boldsymbol{c}_{ri}\boldsymbol{x}, i, r = 1, 2$, the hypothetical decision makers (DM$_1$ and DM$_2$) set their membership functions as $\mu_{11}(\boldsymbol{c}_{11}\boldsymbol{x}) = \boldsymbol{c}_{11}\boldsymbol{x}/300$, $\mu_{12}(\boldsymbol{c}_{12}\boldsymbol{x}) = \boldsymbol{c}_{12}\boldsymbol{x}/390$, $\mu_{21}(\boldsymbol{c}_{21}\boldsymbol{x}) = \boldsymbol{c}_{21}\boldsymbol{x}/330$, $\mu_{221}(\boldsymbol{c}_{22}\boldsymbol{x}) = \boldsymbol{c}_{22}\boldsymbol{x}/420$ respectively.

[HFMOLP]

first level decision maker : DM$_1$

$$\max \boldsymbol{\mu}_1(\boldsymbol{C}_1\boldsymbol{x}) = \begin{pmatrix} (10x_1 + 2x_2 + x_3 + x_4)/300 \\ (x_1 + 13x_2 + 2x_3 + x_4)/390 \end{pmatrix}$$

second level decision maker : DM$_2$

$$\max \boldsymbol{\mu}_2(\boldsymbol{C}_2\boldsymbol{x}) = \begin{pmatrix} (x_1 + 2x_2 + 11x_3 + x_4)/330 \\ (2x_1 + x_2 + x_3 + 14x_4)/420 \end{pmatrix}$$

In HFMOLP, let us assume that DM$_1$ and DM$_2$ find their satisfactory solution from Ext[$\boldsymbol{C}_1 X \mid \Lambda_1$] and Ext[$\boldsymbol{C}_2 X \mid \Lambda_2$], where the generators of the polyhedral cones Λ_1 and Λ_2 in membership space are defined as follows:

$$\boldsymbol{V}_1 = (\boldsymbol{v}_{11}, \boldsymbol{v}_{12}) = \begin{pmatrix} 5/\sqrt{26} & -1/\sqrt{65} \\ -1/\sqrt{26} & 8/\sqrt{65} \end{pmatrix}$$

$$\boldsymbol{V}_2 = (\boldsymbol{v}_{21}, \boldsymbol{v}_{22}) = \begin{pmatrix} 7/\sqrt{50} & -1/\sqrt{17} \\ -1/\sqrt{50} & 4/\sqrt{17} \end{pmatrix}$$

According to Step 2, the initial values are set as $\boldsymbol{w} = (w_1, w_2)^T = (1, 1)^T$, and $\bar{\boldsymbol{\mu}} = (\bar{\boldsymbol{\mu}}_1, \bar{\boldsymbol{\mu}}_2)^T = (1, 1, 1, 1)^T$. Then, at Step 3, HP2($\boldsymbol{w}, \bar{\boldsymbol{\mu}}$) is formulated to obtain the corresponding generalized Λ-extreme point.

$$\min_{\boldsymbol{x} \in X, x_5 \in E^1} \quad x_5$$

subject to

$$\begin{pmatrix} 1.045953 & 0.1307441 & 0 & 0 \\ 0.2067246 & 1.033623 & 0 & 0 \\ 0 & 0 & 1.047566 & 0.2618914 \\ 0 & 0 & 0.1527076 & 1.068953 \end{pmatrix}$$

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$$\cdot\begin{pmatrix} \bar{\boldsymbol{\mu}}_{11} - \boldsymbol{c}_{11}\boldsymbol{x}/300 - x_5/w_1 \\ \bar{\boldsymbol{\mu}}_{12} - \boldsymbol{c}_{12}\boldsymbol{x}/390 - x_5/w_1 \\ \bar{\boldsymbol{\mu}}_{21} - \boldsymbol{c}_{21}\boldsymbol{x}/330 - x_5/w_2 \\ \bar{\boldsymbol{\mu}}_{22} - \boldsymbol{c}_{22}\boldsymbol{x}/420 - x_5/w_2 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The optimal solution of HP2$(\boldsymbol{w}, \bar{\boldsymbol{\mu}})$, which is the generalized Λ-extreme point, is obtained as $(x_1, x_2, x_3, x_4, x_5) = (6.967248, 7.779573, 7.275372, 7.977807, 0.665051)$, $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (0.334949, 0.334949, 0.334949, 0.334949)$, $(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) = (0.208719, 0.184302, 0.215918, 0.198964)$. Is should be noted here that $x_5 \geq 0$. According to Theorems 5 and 6, the trade-off rates between membership functions and the decision power w_2 become as follows:

$$\begin{aligned}
-\frac{\partial \boldsymbol{\mu}_{11}(\boldsymbol{c}_{11}\boldsymbol{x})}{\partial \boldsymbol{\mu}_{12}(\boldsymbol{c}_{12}\boldsymbol{x})} &= \frac{\pi_{11}q_{112} + \pi_{12}q_{122}}{\pi_{11}q_{111} + \pi_{12}q_{121}} = 0.8493697 \\
-\frac{\partial \boldsymbol{\mu}_{21}(\boldsymbol{c}_{21}\boldsymbol{x})}{\partial \boldsymbol{\mu}_{22}(\boldsymbol{c}_{22}\boldsymbol{x})} &= \frac{\pi_{21}q_{212} + \pi_{22}q_{222}}{\pi_{21}q_{211} + \pi_{22}q_{221}} = 1.049339 \\
\frac{\partial \boldsymbol{\mu}_{2j}(\boldsymbol{c}_{2j}\boldsymbol{x})}{\partial w_2} &= \frac{x_5}{w_2^2} - \frac{x_5}{w_2^3}\Big\{ \pi_{21}(q_{211} + q_{212}) \\
&+ \pi_{22}(q_{221} + q_{222}) \Big\} = 0.3153653
\end{aligned}$$

At Step 4, let us assume that, DM_1 updates his/her decision power as $w_2 = 0.9$ in order to improve his/her membership functions, and go to Step 3. Then, the corresponding problem HP2$(\boldsymbol{w}, \bar{\boldsymbol{\mu}})$ is solved and the corresponding generalized Λ-extreme point is obtained as $(x_1, x_2, x_3, x_4, x_5) = (8.047015, 9.074458, 6.053371, 6.825155, 0.628341)$, $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (0.371659, 0.371659, 0.301843, 0.301843)$, $(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) = (0.197199, 0.174129, 0.204000, 0.187982$, where DM_1's membership functions are improved at the expense of DM_2's ones. At Step 4, let us assume that DM_1 is satisfied with the current values, and DM_2 updates his/her reference membership values as $(\bar{\mu}_{21}, \bar{\mu}_{22})^T = (0.33, 0.29)^T$ in order to improve $\mu_{21}(\boldsymbol{c}_{21}\boldsymbol{x})$ at the expense of $\mu_{22}(\boldsymbol{c}_{22}\boldsymbol{x})$. According to the rule (2) of Step 4, DM_1's reference membership values are fixed as the optimal values of membership functions, *i.e.*, $(\bar{\mu}_{11}, \bar{\mu}_{12})^T = (0.371659, 0.371659)^T$. Then, at Step 3, the corresponding problem HP2$(\boldsymbol{w}, \bar{\boldsymbol{\mu}})$ is solved and the corresponding generalized Λ-extreme point is obtained as $(x_1, x_2, x_3, x_4, x_5) = (7.941169, 8.883159, 6.861872, 6.313800, 0.003813)$, $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}) = (0.367846, 0.367846, 0.325763, 0.285763)$. At this point, both DM_1 and DM_2 are satisfied, and the interactive processes are terminated.

5 Conclusions

In this paper, hierarchical fuzzy multiobjective linear programming problems (HFMOLP) have been formulated, where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints. In order to deal with HFMOLP, concepts of a generalized Λ-extreme point and decision powers have been introduced and a linear programming based interactive algorithm has been proposed to obtain the satisfactory solution. In the proposed method, not only the hierarchical relationships between multiple decision makers but also their own preferences for their membership functions can be reflected for the satisfactory solution. Applications of the proposed method will require further investigation.

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