A Real Options Game Involving Multiple Projects

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***Abstract*—The equilibrium is derived in a real options game on the basis of a multidimensional state variable. In the game, firms optimize both investment time and project choice in projects that have not been chosen by the leading competitors. We demonstrate how the equilibrium changes with the number of firms, the number of projects, and the correlation between project values. Consistent with previous findings, an increase in the number of firms and a decrease in the number of projects reduce the option value in equilibrium. A new finding suggests that the option value decreases when the numbers of both firms and projects increase by the same amount. Most interestingly, a high correlation between project values plays a positive role in mitigating preemptive competition, unlike in a monopoly. The results complement the literature of both real options games and max-options, and entails new empirical implications.**

***Index Terms*—financial engineering, real options game, options on multiple assets, optimal stopping game, max-option**

I. Introduction

THIS paper investigates the nature of a real options game based on multiple assets. The real options approach, in which option pricing theory is applied to capital budgeting decisions, better enables us to find an optimal investment strategy and project valuation involving uncertainty and flexibility, than the conventional Net Present Value (NPV) method could (see [1]). Although the early literature on real options focuses on a monopolist's investment, many papers have recently investigated real options games, in which game theory, combined with option pricing theory, is applied to strategic interactions among firms competing in the same market.

Studies such as [2], [3], and [4] derive the equilibrium in a duopoly under the preemption game (non-zero-sum optimal stopping game[1]) framework, while [6], [7], and [8] derive the equilibrium in a oligopoly under the Cournot–Nash framework. The competitive equilibrium has been investigated in [1] and [9].[2] The main result of these studies is that competition among firms reduces option value and accelerates the exercise of real options. This prediction has been supported by empirical tests in [12] and [13].

The previous studies on real options games assume one-dimensional Geometric Brownian Motion (GBM) to be the stochastic process (the state variable) that represents the future cash flow from a project. This is because explicit results are more appealing due to the difficulty of model calibration in many real options models; although such simplification could be justified for a problem concerning a single investment project, a problem involving several projects should be modeled by a multidimensional state variable. In fact, several papers have investigated a monopolist's investment decision involving two projects using a model with a bidimensional state variable. For example, [14] investigates land development timing with an alternative land use choice and [15] investigates timing in switching methods of nuclear waste disposal. The former studies a sort of American max-option, while the latter deals with an American spread option.[3]

However, there have been few studies investigating a real options game based on a multidimensional state variable.[4] The contribution of this paper is to derive the equilibrium in a duopoly and oligopoly, taking into account multiple projects of which value follows a multidimensional state variable. We consider the game where firms optimize both investment time and project choice among projects that have not been chosen by leading competitors. In the game, we reveal how the investment strategy and the option value in equilibrium are affected by the number of firms, the number of projects, and the correlation between project values.

In equilibrium, consistent with the main result of real options games, the option value decreases and investment takes place earlier as the number of firms increases. In addition, the option value increases with the number of projects. This result can be considered an extension of previous results regarding max-options. Thus, this paper links the studies on real options games and max-options. Furthermore, this paper reveals how the equilibrium changes when the numbers of both firms and projects change; we show that the option value decreases and investment is hastened when the numbers of both firms and projects increase by the same amount. Although our model exogenously provides the number of firms and the number of projects, in the real world, the number of firms tends to increase with the number of alternatives in the market. Our result enforces the robustness of the main result of real options games.

Another new finding is that a high correlation between the values of alternatives plays a positive role in moderating competition among firms. This is in sharp contrast with the previous findings in a monopoly where, as pointed out in the max-option literature, the high correlation reduces the value of project choice and accelerates investment. In a duopoly and an oligopoly, the high correlation leads to the opposite effects of moderating the competition (positive effect) and reducing the value of project choice (negative effect). The tradeoff determines the sensitivity of the correlation with

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[1]Most of the literature of real options games models competition among rival firms into a non-zero-sum game, while the game options literature, provoked by [5], tends to focus on a zero-sum game for a buyer and a seller. This is a main difference between real options games and game options.

[2]In contrast, [10] and [11] investigate the agency problem in a single firm under the mechanism design framework.

[3]Refer to [16] and [17] for details of American options on multiple assets.

[4]Although in several papers a problem with a bidimensional state variable is reduced to a one-dimensional case by homogeneity, such cases are very restrictive. A noted exception is [18] who investigate the Cournot-Nash equilibrium in the R&D competition with both demand and cost shocks. Due to the model complexity, their results are mainly derived from the numerical examples.

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respect to the option value in equilibrium. In particular, when there is an equal number of projects and firms, the high correlation increases the option value. This paper complements the literature of real options games by revealing the effects of the correlation and complementing the max-option literature in terms of the strategic interactions.

Although the new prediction has yet to be empirically investigated, it has the potential to account for the non-monotonicity pointed out by [13]. Their empirical work finds that, investment in medium-concentration industries takes place earlier than in not only high-concentration industries but also in low-concentration industries. Our results highlight the significance of the correlation between project values in addition to industry concentration.

Finally, we address real-world cases to which the model applies. The model could potentially account for competition in mergers and acquisitions. For instance, in the pharmaceutical industry, large corporations strategically acquire venture businesses that develop new drugs. In a large-scale case, a firm must choose between several targets due to budget constraint. Because many mergers and acquisitions take place by private negotiation rather than through a public bidding process, preemptive competition occurs among the acquiring firms. When a firm is preempted by its rival, it will choose an alternative venture business (plan B). The model is also closely related to strategic interactions among real estate developers. As documented in [14], a developer has several options of land uses. The value of each land use is greatly affected by land development that is done by other developers in the same area. Some developers that are preempted by its rivals are obliged to develop land for an alternative use (plan B).

II. Preliminaries

Consider a firm that has an option to invest in a project. Consider two kinds of projects denoted by $i = 1, 2$. When a firm conducts project i at time t, it receives temporary project value $X_i(t)$.[5] The investment in project i requires an irreversible capital expenditure of $I_i(> 0)$. Assume that project value $X_i(t)$ follows a nonnegative diffusion process under the risk-neutral measure:

$$\mathrm{d}X_i(t) = \mu_i(X_i(t), t)\mathrm{d}t + \sigma_i(X_i(t), t)\mathrm{d}B_i(t), \qquad (1)$$

where $(B_1(t), B_2(t))$ is a bidimensional Brownian Motion (BM) with correlation coefficient ρ. Mathematically, the model is built on the filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ generated by $(B_1(t), B_2(t))$. The set $\mathcal{F}_t$ means the available information set to time t, and a firm optimizes its investment strategy under this information. Let $r(> 0)$ and $T(> 0)$ denote the constant risk-free rate and maturity of the option, respectively.

A. Valuation in a monopoly with a single project

As a benchmark, we consider a firm that has a monopolistic option to invest in a single project i. This option can be regarded as an American call option. At time $t(< T)$ with the state variable $X_i(t) = x_i$, the option value is equal to the value function of the optimal stopping problem:[6]

$$V_i^1(x_i, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^{x_i}[\mathrm{e}^{-r(\tau - t)}(X_i(\tau) - I_i)1_{\{\tau \le T\}}], \qquad (2)$$

where $\mathcal{T}_t$ denotes the set of all stopping times τ satisfying $\tau \ge t$ and $E_t^{x_i}[\cdot]$ is the expectation conditional on $X_i(t) = x_i$. Throughout the paper, the superscript and the subscript on V_i^1 represent the number of firms and available project(s), respectively; that is, V_i^1 in (2) means the value function in a monopoly with a single project i.

We restrict our attention to a diffusion process $X(t)$ satisfying the following assumptions:

Assumption (i) The value function $V_i^1(x_i, t)$ is continuous and strictly increasing with respect to x_i and $\lim_{x_i \downarrow 0} V_i^1(x_i, t) = 0$.

Assumption (ii) There exists a finite threshold $x_i^1(t)$ such that the optimal stopping time $\tau_i^1(t)$ for problem (2) is written as

$$\tau_i^1(t) = \inf\{s \ge t \mid X_i(s) \in [x_i^1(s), \infty)\}. \qquad (3)$$

Define $S_1^1(s) := [x_1^1(s), \infty) \times \mathbb{R}_+$ and $S_2^1(s) := \mathbb{R}_+ \times [x_2^1(s), \infty)$. Then, the optimal investment time $\tau_i(t)$ is written as $\inf\{s \ge t \mid X(s) \in S_i^1(s)\}$. The assumptions are not restrictive. Indeed, we can take a wide range of diffusion processes including a GBM, i.e., $\mu_i(X_i(t), t) = \mu_i X_i(t)$ and $\sigma_i(X_i(t), t) = \sigma_i X_i(t)$ where $\mu_i(< r)$ and $\sigma_i(> 0)$ are constant, and a process with a mean-reverting growth rate, i.e., $\mu_i(X_i(t), t) = \eta(m - X_i(t))$ and $\sigma_i(X_i(t), t) = \sigma_i X_i(t)$ where η, m and σ_i are positive constants.

When $X(t)$ follows a GBM and the maturity is infinite, $V_i^1(x_i, t)$ is explicitly derived independently from time t. In fact, the option value $V_i^1(x_i)$ is expressed as

$$V_i^1(x_i) = \begin{cases} \left(\dfrac{x_i}{x_i^1}\right)^{\beta_i}(x_i^1 - I_i) & (0 \le x_i < x_i^1) \\ x_i - I_i & (x_i \ge x_i^1). \end{cases} \qquad (4)$$

The constant threshold x_i^1 is defined by

$$x_i^1 = \frac{\beta_i}{\beta_i - 1} I_i, \qquad (5)$$

where $\beta_i := 1/2 - \mu_i/\sigma_i^2 + \sqrt{(\mu_i/\sigma_i^2 - 1/2)^2 + 2r/\sigma_i^2}(> 1)$. Similarly, when $X(t)$ follows a process with a mean-reverting process and the maturity is infinite, the option value is explicit and independent of time t. For details, refer to [1].

B. Valuation in a duopoly with a single project

This subsection considers two identical firms that compete for a single project i. Throughout the paper, we assume a winner-take-all game as follows:

Assumption (iii) A firm cannot invest in the project in which the other firm has already invested.

Suppose time t with $X_i(t) = x_i \le I_i$ for $i = 1, 2$. The duopoly game is solved backward. We begin by supposing that one of the firms (the leader) has first invested at time $s \in [t, T]$, and we find the optimal decision of the other (the follower). Because the follower's opportunity to invest is removed, the follower's value is zero. On the other hand,

[5] This is regarded as the discounted cash flow during the lifetime of project i.

[6] When the maturity is infinite, we have only to replace $1_{\{\tau \le T\}}$ with $1_{\{\tau < \infty\}}$.

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the leader's value is $X_i(s) - I_i$ at the time of investment. In the situation where neither firm has invested, firms attempt to preempt each other in order to obtain the leader's project value if $X_i(s) - I_i > 0$. Define $S_1^2(s) := [I_1, \infty) \times \mathbb{R}_+$ and $S_2^2(s) := \mathbb{R}_+ \times [I_2, \infty)$. In equilibrium, both firms attempt to invest at

$$\tau_i^2(t) := \inf\{s \geq t \mid X(s) \in S_i^2(s)\} \tag{6}$$

and hence the option value becomes

$$V_i^2(x_i, t) := 0, \tag{7}$$

where the superscript 2 and the subscript i represent a duopoly with a single project i. In other words, the preemptive competition completely removes the value of option to invest in project i.

Strictly speaking, both firms' investment strategy at (6) proves to be a Nash equilibrium in the optimal stopping game under the assumption that if two firms choose the same timing, one of the firms is chosen as the leader with probability $1/2$. Most studies, including [2] and [3], are built on this assumption. Then, the equilibrium means that one of the firms invests in project i at time (6), while the other cannot undertake the project. The value of the leader, who is selected randomly, is zero because of investing too early. This is the well-known preemptive equilibrium in a real options game. For details of real options games, refer to [19].

C. Valuation in a monopoly with two projects

This subsection considers a firm that has a monopolistic option to invest in a single project between projects $1, 2$. The model applies not only to a case in which two projects are mutually exclusive (e.g., alternative land use) but also to a case where a firm must choose between projects due to budget constraint (e.g., large merger and acquisition transaction). This type of option is classified as American max-options. European max-options have been studied in [20] and [21], while American max-options have been studied in [14], [16], and [17]. Although a max-option commonly has a multidimensional state variable, [22] studies a max-option that is written on a one-dimensional state variable, i.e., $\rho = 1, x_1 \neq x_2$, and $I_1 \neq I_2$, in order to investigate investment timing with an alternative scale choice.

At time $t (< T)$ with $X(t) = x$, the option value is equal to the value function of the optimal stopping problem as follows:

$$V_{1,2}^1(x, t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x [\mathrm{e}^{-r(\tau - t)} \underbrace{\max_{i=1,2} (X_i(\tau) - I_i)}_{\text{project choice}} 1_{\{\tau \leq T\}}]. \tag{8}$$

Recall that $V_{1,2}^1$ in (8) means the value function in a monopoly with projects $1, 2$. The optimal stopping time $\tau_{1,2}^1$ for problem (8) is written as

$$\tau_{1,2}^1(t) = \inf\{s \geq t \mid X(s) \in S_{1,2}^1(s)\}, \tag{9}$$

where the stopping region $S_{1,2}^1(s)$ is defined by

$$S_{1,2}^1(s) := \{x \in \mathbb{R}_+^2 \mid V_{1,2}^1(x, s) = \max_{i=1,2} (x_i - I_i)\}. \tag{10}$$

The stopping region $S_{1,2}^1(s)$ proves to be the union of two disjoint convex sets corresponding to the immediate exercise region of each project, when $X(t)$ follows a GBM. For details, refer to [14], [17].

Let us now focus on two symmetric projects, i.e., $x_1 = x_2$, $\mu_1(\cdot,\cdot) = \mu_2(\cdot,\cdot), \sigma_1(\cdot,\cdot) = \sigma_2(\cdot,\cdot)$, and $I_1 = I_2$. In this case, the larger the correlation coefficient ρ, the more likely it is that the project values $X_1(t)$ and $X_2(t)$ take close values. The option value $V_{1,2}^1$ decreases and the stopping region $S_{1,2}^1$ enlarges with ρ, because the higher ρ reduces the value of project choice. In particular, in the case of the perfect correlation, i.e., $\rho = 1$, the option value $V_{1,2}^1$ and the investment time $\tau_{1,2}^1$, agree with those in a monopoly with a single project, i.e., V_i^1 and τ_i^1, respectively. The effects of the correlation will be compared in detail with that of a duopoly with two projects in Section 3.

The following section is the main contribution of the paper. Although the results can be readily extended to the case of a oligopoly with multiple projects, we present the details of a duopoly with two projects in order to avoid unnecessary confusion.

III. Main results

This section investigates two identical firms that compete for two projects $1, 2$.[7] Recall Assumption (iii). When one of the firms (the leader) undertakes a project, the other (the follower) is deprived of the opportunity to invest in that project. Firms attempt to preempt each other in order to gain the first-mover's advantage in project choice. Assume that the first mover cannot invest in the remaining project. Otherwise, as in Section 2.B, both firms compete for the remaining project and gain no value from the project. Then, it follows from backward reasoning that the equilibrium value becomes zero in the situation where neither firm has invested. As mentioned in Section 1, the model can be applied to strategic interactions in acquisitions and land development.

Consider time $t (< T)$ with $X_i(t) = x_i \leq I_i$ for $i = 1, 2$. As in Section 2.B, the problem is solved backward. Supposed that one of the firms (the leader) has first invested in the better project $i(s)$ at time $s \in [t, T]$, where the function $i(s)$[8] is defined by

$$i(s) := \arg\max_{i=1,2} (X_i(s) - I_i), \tag{11}$$

we find the optimal response of the other firm (follower). Because the follower has the monopolistic option to invest in a single project $i \neq i(s)$, the option value and the optimal investment time coincide with V_i^1 and τ_i^1 (cf. (2) and (3)). On the other hand, the leader's project value is equal to $\max_{i=1,2}(X_i(s) - I_i)$.

Let us return to the situation where neither firm has invested. Intuitively, in equilibrium the leader's advantage in project choice is offset by too early and inefficient investment timing. Define the region $S_{1,2}^{2F}(s)$ where the leader's value dominates that of the follower as follows:

$$\begin{aligned} S_{1,2}^{2F}(s) \quad := \quad & \{x \in \mathbb{R}_+^2 \mid x_1 - I_1 \geq V_2^1(x_2, s)\} \\ & \cup \{x \in \mathbb{R}_+^2 \mid x_2 - I_2 \geq V_1^1(x_1, t)\}. \end{aligned}$$

[7] For simplicity, this paper concentrates on the identical firms. Although similar (but messy) results follow from the same logic in the asymmetric case, interesting insights can be better observed in the symmetric case.

[8] We do not have to be concerned about the value of $i(s)$ when $X_1(s) - I_1 = X_2(s) - I_2$.

Each firm attempts to preempt the competitor when $X(s) \in S_{1,2}^{2F}(s)$. In addition, one of the firms is forced to invest for $X(s) \in S_1^1(s) \cup S_2^1(s)$, if it knows that the other waits until

$$\tau_{1,2}^{2F}(t) := \inf\{s \geq t \mid X(s) \in S_{1,2}^{2F}(s)\}. \tag{12}$$

This is because for $X(s) \in S_1^1(s) \cup S_2^1(s)$ the immediate exercise yields a higher value than the option value to wait until $\tau_{1,2}^{2F}$. Note that, in this equilibrium, the follower's value is higher than that of the leader. For details, refer to the proof of Proposition 1. Therefore, the preemptive investment region $S_{1,2}^2(s)$ becomes

$$S_{1,2}^2(s) := S_{1,2}^{2F}(s) \cup S_1^1(s) \cup S_2^1(s). \tag{13}$$

The preemptive investment takes place at time

$$\tau_{1,2}^2(t) := \inf\{s \geq t \mid X(s) \in S_{1,2}^2(s)\}. \tag{14}$$

It is easily checked that the boundary of $S_{1,2}^2(s)$ can be expressed as

$$\begin{aligned}
&\partial S_{1,2}^2(s) \\
= \; &\underbrace{\{x \in \mathbb{R}_+^2 \mid x_i \leq x_{i'}^1(s) - I_{i'} + I_i, x_i - I_i = V_{i'}^1(x_{i'}, s)\}}_{(a)} \\
&\cup \underbrace{\{x \in \mathbb{R}_+^2 \mid x_{i'} \leq x_{i'}^1(s), x_{i'} - I_{i'} = V_i^1(x_i, s)\}}_{(b)} \\
&\cup \underbrace{\{x \in \mathbb{R}_+^2 \mid x_{i'} = x_{i'}^1(s),}_{(c)} \\
&\underbrace{(V_i^1)^{-1}(x_{i'}^1(s) - I_{i'}) \leq x_i \leq x_{i'}^1(s) - I_{i'} + I_i\}}_{(c)},
\end{aligned} \tag{15}$$

where i and $i'(\neq i)$ (which may depend on s) satisfy

$$x_i^1(s) - I_i \geq x_{i'}^1(s) - I_{i'}. \tag{16}$$

Throughout the paper, we denote by i' for $i' \neq i$. In (16), $(V_i^1)^{-1}(\cdot)$ (which may depend on s) denotes the inverse function for $V_i^1(\cdot, s)$. Note that this function is well defined by Assumption (i).

Figure 1 illustrates the preemptive investment boundary $\partial S_{1,2}^2(s)$. The part (a) is the region where the leader's investment in project i generates the same value as the follower's option value to invest in project i'. Similarly, the part (b) is the region where the leader's investment in project i' generates the same value as the follower's option value to invest in project i. In the part (c), both firms prefer to be the follower with project i to being the leader with project i' due to $X(s) \notin S_{1,2}^{2F}(s)$. In equilibrium, as will be proved in Proposition 1, one of the firms invests when $X(s)$ hits the part (c). We see from (15) that, unlike $S_{1,2}^1$ in a monopoly, the preemptive investment region $S_{1,2}^2$ is independent of the correlation coefficient ρ.

At time $t(< T)$ with $X(t) = x$, the option value of the leader is written as

$$\begin{aligned}
V_{1,2}^2(x,t) &:= \mathbb{E}_t^x[\mathrm{e}^{-r(\tau_{1,2}^2(t)-t)} \max_{i=1,2}(X_i(\tau_{1,2}^2(t)) - I_i) \\
&\times 1_{\{\tau_{1,2}^2(t) \leq T\}}].
\end{aligned} \tag{17}$$

This value is lower than that of the follower if and only if the process $X(t)$ hits the part (c).

So far, we intuitively explain the equilibrium. More precisely, we need to formulate the following optimal stopping

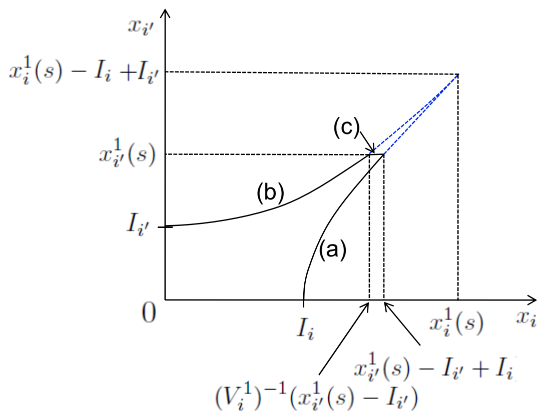

Fig. 1. The preemptive investment boundary $\partial S_{1,2}^2(s)$

game for two identical firms $j = 1, 2$. The set of actions is defined by

$$\begin{aligned}
A(t) \;\; := \;\; &\{(\tau, i) \mid \tau \in \mathcal{T}_t, i : \mathcal{F}_\tau \text{measurable random} \\
&\text{variable taking values in } \{0,1\}\}.
\end{aligned}$$

For firm 1's action $(\tau_1, i_1) \in A(t)$ and firm 2's action $(\tau_2, i_2) \in A(t)$, the payoff of firm 1 is defined by

$$\begin{aligned}
&\pi_1(\tau_1, i_1, \tau_2, i_2) \\
:= \; &\mathbb{E}_t^x[\underbrace{\mathrm{e}^{-r(\tau_1-t)}(X_{i_1}(\tau_1) - I_{i_1})}_{\text{leader's value}} 1_{\{\tau_1<\tau_2\}\cap\{\tau_1 \leq T\}} \\
&+ \underbrace{\mathrm{e}^{-r(\tau_2-t)} V_{i_2'}^1(X_{i_2'}(\tau_2), \tau_2)}_{\text{follower's value}} 1_{\{\tau_1>\tau_2\}\cap\{\tau_2 \leq T\}} \\
&+ \underbrace{\frac{\mathrm{e}^{-r(\tau_1-t)}}{2}(X_{i_1}(\tau_1) - I_{i_1} + V_{i_2'}^1(X_{i_2'}(\tau_2), \tau_2))}_{\text{average of leader's and follower's value}} \\
&\times 1_{\{\tau_1=\tau_2\}\cap\{\tau_1 \leq T\}}].
\end{aligned}$$

The last term corresponds to the assumption that if two firms choose the same timing, one of the firms is chosen as the leader with probability $1/2$. The payoff of firm 2 (denoted by $\pi_2(\tau_1, i_1, \tau_2, i_2)$) is defined symmetrically. A Nash equilibrium $(\tilde{\tau}_1, \tilde{i}_1, \tilde{\tau}_2, \tilde{i}_2) \in A(t) \times A(t)$ of the stopping game satisfies both

$$\pi_1(\tilde{\tau}_1, \tilde{i}_1, \tilde{\tau}_2, \tilde{i}_2) = \max_{(\tau_1, i_1) \in A(t)} \pi_1(\tau_1, i_1, \tilde{\tau}_2, \tilde{i}_2), \tag{18}$$

and

$$\pi_2(\tilde{\tau}_1, \tilde{i}_1, \tilde{\tau}_2, \tilde{i}_2) = \max_{(\tau_2, i_2) \in A(t)} \pi_2(\tilde{\tau}_1, \tilde{i}_1, \tau_2, i_2). \tag{19}$$

We assume that for (17) the diffusion process $X(t)$ satisfies[9]

Assumption (iv)

$$\max_{i=1,2}(x_i - I_i) \leq V_{1,2}^2(x,t) \quad (x \notin S_{1,2}^2(t)).$$

[9]We have not established any proof, but the assumption is satisfied in many cases as far as we can judge from a wide range of computations.

The following proposition shows that the pair of actions $(\tau_{12}^2(t), i(\tau_{12}^2(t)), \tau_{12}^{2F}(t), i(\tau_{12}^{2F}(t))) \in A(t)\times A(t)$ is a Nash equilibrium of the stopping game, where the stopping times $\tau_{12}^2(t), \tau_{12}^{2F}(t)$ are defined by (14),(12), and the functions $i(\tau_{12}^2(t)), i(\tau_{12}^{2F}(t))$ are defined by (11), respectively.

Proposition 1 $(\tau_{12}^2(t), i(\tau_{12}^2(t)), \tau_{12}^{2F}(t), i(\tau_{12}^{2F}(t)))$ is a Nash equilibrium of the stopping game.

Proposition 1 includes the equilibrium in a duopoly with a single project. Indeed, for $x_i > x_{i'} = 0$, the equilibrium in Proposition 1 agrees with that of Section 2.B. Accordingly, Proposition 1 extends the previous results to a more general case in which there are two opportunities to invest in. For most of the diffusion process $X_i(t)$, a higher volatility σ_i leads to a higher option value V_i^1 and a later investment time τ_i^1. If this is the case, by (15) the preemptive investment region $S_{1,2}^2$ decreases, which leads to a higher option value $V_{1,2}^2$ and a later investment time $\tau_{1,2}^2$ in equilibrium. Then, the effects of volatility σ_i in a duopoly remain unchanged from a monopoly.

If $X(t)$ follows a GBM and $T=\infty$, we have an explicit form of the time homogeneous investment boundary $\partial S_{1,2}^2$ by (4), (5) and (15) .

Corollary 1 Assume that $T=\infty$, $\mu_i(X_i(t),t) = \mu_i X_i(t)$, and $\sigma_i(X_i(t),t) = \sigma_i X_i(t)$, where $\mu_i(< r)$ and $\sigma_i(>0)$ are constant for $i = 1,2$. The preemptive investment boundary is equal to

$$\begin{aligned}
&\partial S_{1,2}^2 \\
= \;& \left\{ x_i \le x_{i'}^1 - I_i' + I_i, x_i - I_i = \left(\frac{x_{i'}}{x_{i'}^1}\right)^{\beta_{i'}} (x_{i'}^1 - I_{i'}) \right\} \\
& \cup \left\{ x_{i'} \le x_{i'}^1, x_{i'} - I_{i'} = \left(\frac{x_i}{x_i^1}\right)^{\beta_i} (x_i^1 - I_i) \right\} \\
& \cup \left\{ x_{i'} = x_{i'}^1, (V_i^1)^{-1}(x_{i'}^1 - I_{i'}) \le x_i \le x_{i'}^1 - I_{i'} + I_i \right\},
\end{aligned}$$

where i (which does not depend on s) satisfies (16).

The explicit form of the investment boundary $\partial S_{1,2}^2$ would be useful for applications of the model. The option value $V_{1,2}^2$ (cf. (17)) can be expressed as the solution of the corresponding partial differential equation with the boundary $\partial S_{1,2}^2$. Then, we can compute $S_{1,2}^2$ and $V_{1,2}^2$ without difficulty.

For a general diffusion process $X(t)$ we can show the following properties of the preemptive investment region $S_{1,2}^2(s)$, the timing $\tau_{1,2}^2(t)$, and the option value $V_{1,2}^2(x,t)$.

Proposition 2 The following relationships hold for all $i = 1,2$:

Investment region

$$S_{1,2}^1(s) \subset S_1^1(s) \cup S_2^1(s) \subset S_{1,2}^2(s) \subset S_1^2(s) \cup S_2^2(s), \quad (20)$$

Investment timing

$$\min(\tau_1^2(t), \tau_2^2(t)) \le \tau_{1,2}^2(t) \le \min(\tau_1^1(t), \tau_2^1(t)) \le \tau_{1,2}^1(t), \quad (21)$$

Option value

$$0 = V_i^2(x_i,t) \le V_{1,2}^2(x,t) \le V_i^1(x_i,t) \le V_{1,2}^1(x,t). \quad (22)$$

Proposition 2 reveals that the option value decreases and investment takes place earlier as the number of firms increases. This is in line with both theoretical and empirical results in real options games (e.g., [2], [6], [12], and [13]). The inequality $V_i^2(x_i,t) \le V_{1,2}^2(x,t)$ means that the option value increases with the number of projects in a duopoly. This result extends the previous result for American max-options in a monopoly (e.g., [14], [16], and [17]) into that of a duopoly. Thus, we bridge the gap between the studies on real options games and those on American max-options.

In addition, Proposition 2 reveals how the equilibrium changes when the numbers of both firms and projects change. Indeed, the inequalities, $V_{1,2}^2(x,t) \le V_i^1(x,t), \tau_{1,2}^2(t) \le \min(\tau_1^1(t), \tau_2^1(t))$, demonstrate that the option value decreases and investment is hastened when the numbers of both firms and projects increase by the same amount. While we exogenously provide the numbers of both firms and opportunities, in reality, the number of firms tends to increase with the number of opportunities. Taking this into consideration, our new result can be positioned as an extension of the previous works into a more practical setting.

We now consider two symmetric projects, i.e., $x_1 = x_2$, $\mu_1(\cdot,\cdot) = \mu_2(\cdot,\cdot), \sigma_1(\cdot,\cdot) = \sigma_2(\cdot,\cdot)$, and $I_1 = I_2$. In the sensitivity analysis, we focus on the correlation coefficient ρ because the previous strategic models with a one-dimensional state variable cannot reveal the comparative statics with respect to ρ. For instance, [23] investigates a duopoly with two projects, but they cannot capture the effects of the correlation between project values due to the one-dimensional model. By Proposition 2, we can easily show the following corollary.

Corollary 2 Consider the symmetric projects $i = 1,2$. The following equalities hold for the correlation coefficient ρ:

$$\max_{\rho\in[-1,1]} V_{1,2}^2(x,t) = V_i^1(x_i,t) = \min_{\rho\in[-1,1]} V_{1,2}^1(x,t), \quad (23)$$

where $\rho = 1$ maximizes $V_{1,2}^2(x,t)$ and minimizes $V_{1,2}^1(x,t)$.

Corollary 2 highlights a difference between max-options in a monopoly and in a duopoly. In a monopoly, as is noted in the max-option literature, the high correlation reduces the value of project choice. Conversely, the high correlation in a duopoly plays a positive role in mitigating preemptive competition and increasing the option value. Note that the high correlation reduces the first-mover's advantage in project choice. This finding complements the max-option literature by demonstrating the positive effect of the high correlation in combination with strategic interactions.

In addition, this result may account for the non-monotonicity in the investment speed with respect to industry concentration. [13] finds that investment in medium-concentration industries takes place earlier than in not only high-concentration industries but also in low-concentration industries. One firm is more likely to benefit from the failure of a specific rival in higher-concentration industries than in lower-concentration industries with numerous firms. Taking account of industry-wide uncertainty, the correlation among firm values tends to be high in low-concentration industries. This high correlation could mitigate preemptive competition and delay investment later than in medium-concentration industries. In our view, the option value depends not only

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on the numbers of both firms and projects but also on the correlation between project values.

We compare the option value $V_{1,2}^2$ in a duopoly with that of American min-option in a monopoly. The exercise of the min-option at time τ, unlike the max-option, yields the payoff $\min_{i=1,2}(X_i(\tau) - I_i)$. At time $t(\leq T)$ with $X_i(t) = x_i$, the option value of American min-option is the value function of the optimal stopping problem as follows:

$$V_{min}^1(x,t) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}_t^x[\mathrm{e}^{-r(\tau-t)} \min_{i=1,2}(X_i(\tau) - I_i)1_{\{\tau \leq T\}}]. \tag{24}$$

This type of option is investigated in [24] and [17]. We can show that $V_{min}^1(x,t) \leq V_{1,2}^2(x,t)$, where the equality holds for the symmetric projects with $\rho = 1$, as follows. Let $S_{min}^1(s)$ be the stopping region for problem (24). Consider the boundary of $S_{1,2}^2(s) \cup S_{min}^1(s)$. For $x \in \partial S_{1,2}^2(s) \setminus S_{min}^1(s)$, $V_{1,2}^2(x,s)$ is either $V_1^1(x,s)$ or $V_2^1(x,s)$ which is lager than $V_{min}^1(x,s)$. For $x \in \partial S_{min}^1(s) \setminus S_{1,2}^2(s)$, $V_{min}^1(x,s)$ is equal to $\min_{i=1,2}(x_i - I_i)$ which is smaller than $V_{1,2}^2(x,s)$ under Assumption (iv). Then, we have $V_{min}^1(x,s) \leq V_{1,2}^2(x,s)$ on the boundary. For the hitting time $\tilde{\tau}$ to the boundary, we have

$$\begin{aligned} V_{min}^1(x,t) &= \mathbb{E}_t^x[\mathrm{e}^{-r(\tilde{\tau}-t)} V_{min}^1(X(\tilde{\tau}), \tilde{\tau})1_{\{\tilde{\tau} \leq T\}}] \\ &\leq \mathbb{E}_t^x[\mathrm{e}^{-r(\tilde{\tau}-t)} V_{1,2}^2(X(\tilde{\tau}), \tilde{\tau})1_{\{\tilde{\tau} \leq T\}}] \\ &= V_{1,2}^2(x,t). \end{aligned}$$

Then, the option value $V_{1,2}^2$ in a duopoly is higher than the min-option value V_{min}^1.

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