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Sum of Sets of Integer Points of Common-Normal Faces of Integer Polyhedra

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***Abstract*—We consider the Minkowski sum of subsets of integer lattice, each of which is a set of integer points of a face of an extended submodular [Kashiwabara–Takabatake, *Discrete Appl. Math.* 131 (2003) 433] integer polyhedron supported by a common positive vector. We show a sufficient condition for the sum to contain all the integer points of its convex hull and a sufficient condition for the sum to include a specific subset of congruent integer points of its convex hull. The latter also gives rise to a subclass of extended submodular integer polyhedra, for which the sum of "copies" of a set of integer points of a face always contains all the integer points of its convex hull. We do this by using the properties of M-convex sets [Murota, *Discrete Convex Analysis* (2003)] and some logic from the elementary number theory. Our study has a direct significance for an economic problem of division of a bundle of indivisible goods.**

***Index Terms*—Minkowski sum of integer point sets, extended submodular polyhedron, M-convex set, indivisible goods**

I. Introduction

In this paper we consider the Minkowski sum of subsets of integer lattice, each of which is a set of integer points of a face of an integer polyhedron supported by a common vector. In particular, we consider the sum of sets of integer points of the faces of extended submodular [2] integer polyhedra supported by a common positive vector, and show a sufficient condition for the sum to contain all the integer points of its convex hull (Proposition III.1), and more weakly, a sufficient condition for the sum to include a specific subset of congruent integer points of its convex hull (Proposition III.2). The latter also gives rise to a subclass of extended submodular integer polyhedra, for which the sum of "copies" of a set of integer points of a face always contains all the integer points of its convex hull (Proposition III.4). We do this by using the properties of M-convex sets ([3]) and some logic from the elementary number theory.

The reason why we pay attention to the integer points of faces (not of the entire polyhedra) is application-oriented. Our study has a direct significance for an economic problem of division of a bundle of indivisible goods as follows. Let $\mathbf{Z}^d$ be the d-dimensional integer lattice and $X^r \subset \mathbf{Z}^d$ be a demand set of an agent $r = 1, 2$. When dealing with this problem we reverse the signs, and think a demand set to be a set of integer points of a face of an extended "supermodular" integer polyhedron supported by a common positive price vector. This amounts to say that agents have weakly convex and weakly monotone preference treating every pair of goods as net substitutes (see Section 4 for the detail). If $X := \{x^1 + x^2 \mid x^1 \in X^1,\ x^2 \in X^2\}$ then $X \subseteq \mathrm{conv}(X) \cap \mathbf{Z}^d$, where $\mathrm{conv}(X)$ is the convex hull of the sum X, and we call X hole-free if the inclusion holds with the equality. If X is hole-free then any $x \in \mathrm{conv}(X) \cap \mathbf{Z}^d$ is divided between the agents as $x = x^1 + x^2$, choosing some $x^r \in X^r$ for each $r = 1, 2$. When is X secured to be hole-free? If it is a hard-won property, then what sort of $x \in \mathrm{conv}(X) \cap \mathbf{Z}^d$ given what type of X^r's can be divided between the agents? Our propositions will answer to these questions. See Section 4 for the relevance of our results to this division problem.

The paper is organized as follows. Section 2 gives some preliminaries. Section 3 contains the results. Section 4 discusses about the economic application.

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II. Preliminaries

Let $\mathbf{R}^d$ be the d-dimensional Euclidean space with the standard basis $\{\mathbf{e}^1, \cdots, \mathbf{e}^d\}$. We denote by $\mathbf{0}$ and $\mathbf{1}$ the zero vector and the vector of all ones, respectively. We let $\mathbf{R}^d_+ := \{x \in \mathbf{R}^d \mid x_i \geq 0\ \forall i\}$, $\mathbf{R}^d_{++} := \{x \in \mathbf{R}^d \mid x_i > 0\ \forall i\}$, and denote the support and positive support of a vector $x \in \mathbf{R}^d$ by $\mathrm{supp}(x) := \{i \mid x_i \neq 0\}$ and $\mathrm{supp}^+(x) := \{i \mid x_i > 0\}$, respectively. The inner product of $x, y \in \mathbf{R}^d$ is written as $x \cdot y := \sum_{i=1}^d x_i y_i$, and the (Minkowski) sum of sets $X, Y \subseteq \mathbf{R}^d$ is written as $X + Y := \{x + y \mid x \in X,\ y \in Y\}$.

A finite intersection of closed half spaces is called a convex polyhedron, or simply a *polyhedron*. A polyhedron $P \subset \mathbf{R}^d$ is *down-monotone* if $x \in P$ and $y \leq x$ imply $y \in P$. For a down-monotone polyhedron $P \subset \mathbf{R}^d$, the *support function* $\delta^*_P\colon \mathbf{R}^d_+ \to \mathbf{R}$ of P is defined for each $a \in \mathbf{R}^d_+$ by the supremum of $a \cdot x$ over $x \in P$, namely by

$$\delta^*_P(a) := \sup\{a \cdot x \mid x \in P\}. \tag{1}$$

For each $a \in \mathbf{R}^d_+$, the set

$$P_a := \{x \in P \mid a \cdot x = \delta^*_P(a)\} \tag{2}$$

gives a *face* of P, and we say that a vector a *supports* P_a, or a is a normal vector of P_a. The faces with dimensionality 0, 1, and $d-1$ are called the *vertices*, the *edges*, and the *facets*, respectively.

A down-monotone polyhedron $P \subset \mathbf{R}^d$ is an *extended submodular polyhedron* [2] if its support function δ^*_P is *submodular* in that

$$\delta^*_P(a) + \delta^*_P(a') \geq \delta^*_P(a \vee a') + \delta^*_P(a \wedge a'), \quad a, a' \in \mathbf{R}^d_+, \tag{3}$$

where $a \vee a' := (\max\{a_1, a'_1\}, \cdots, \max\{a_d, a'_d\})$ and $a \wedge a' := (\min\{a_1, a'_1\}, \cdots, \min\{a_d, a'_d\})$. As proved in [2], the extended submodular polyhedra are characterized by a simultaneous exchangeability such that $x \in P$, $y \in P$, and $i \in \mathrm{supp}^+(x-y)$ imply either (i) $x - \epsilon\mathbf{e}^i \in P$ and $y + \epsilon\mathbf{e}^i \in P$ or (ii) $x - \epsilon(\mathbf{e}^i - s\mathbf{e}^j) \in P$ and $y + \epsilon(\mathbf{e}^i - s\mathbf{e}^j) \in P$, for some $j \in \mathrm{supp}^+(y - x)$ and real numbers $s > 0$ and $\epsilon > 0$. This in particular says that if P_a is a face of an extended submodular

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polyhedron supported by a *positive* vector $a \in \mathbf{R}^d_{++}$, then $x \in P_a$, $y \in P_a$, and $i \in \mathrm{supp}^+(x-y)$ imply

$$x - \epsilon(\mathbf{e}^i - s\mathbf{e}^j) \in P_a \text{ and } y + \epsilon(\mathbf{e}^i - s\mathbf{e}^j) \in P_a \quad (4)$$

for some $j \in \mathrm{supp}^+(y-x)$ and real numbers $s > 0$ and $\epsilon > 0$. Any edge of P_a supported by a positive a is parallel to some $\mathbf{e}^i - s\mathbf{e}^j$ ($i, j \in \{1, \cdots, d\}$, $i \neq j$, $s > 0$). As proved in [1], the class of extended submodular polyhedra is a subclass of *polybasic polyhedra*, which are defined to be the polyhedra with edge vectors of support size at most two, and characterized as having submodular support function or, alternatively, having faces with full-support normals that are reflections and axiwise scalings of base polyhedra (see [1]).

Let $\mathbf{Z}^d$ be the set of integer points of $\mathbf{R}^d$. In the following we assume that a polyhedron has a vertex. A polyhedron is *integral* if its vertices are all integer points. Let $P \subset \mathbf{R}^d$ be an extended submodular integer polyhedron. Then for any integer points x and y in an edge e of P supported by a vector $a \in \mathbf{R}^d_{++}$, we have $x - y = c(m_i(e)\mathbf{e}^i - m_j(e)\mathbf{e}^j)$ for some $i, j \in \{1, \cdots, d\}$ ($i \neq j$), an integer $c \geq 0$, and coprime positive integers $m_i(e)$ and $m_j(e)$ specific to e. Integers $z_1, \cdots, z_n$ are *coprime* if their greatest common divisor is one. They are *pairwise coprime* if any two of them are coprime. Note that the integer $m_i(e)$ may vary on the edges that involve $\mathbf{e}^i$ in their segmentation, even if the edges are taken from the same facet.

An M-convex set is a set of integer points characterized by a simultaneous exchangeability in 01-integers ([3]). A set $M \subset \mathbf{Z}^d$ is *M-convex* if $x \in M$, $y \in M$, and $i \in \mathrm{supp}^+(x-y)$ imply

$$x - \mathbf{e}^i + \mathbf{e}^j \in M \text{ and } y + \mathbf{e}^i - \mathbf{e}^j \in M \quad (5)$$

for some $j \in \mathrm{supp}^+(y-x)$. An M-convex set $M \subset \mathbf{Z}^d$ is *hole-free* in that $\mathrm{conv}(M) \cap \mathbf{Z}^d = M$. Note that the set of integer points of a convex set having integer vertices is always hole-free. An M-convex set coincides with the set of integer points of an integral base polyhedron. A hole-free set $E \subset \mathbf{Z}^d$ is M-convex if for any integer points x and y in an edge of $\mathrm{conv}(E)$ we have $x - y = c(e^i - e^j)$ for some $i, j \in \{1, \cdots, d\}$ ($i \neq j$) and an integer $c \geq 0$. A sum of M-convex sets is M-convex, and hence hole-free. The convex hull of an M-convex set is included in a hyperplane with normal vector $\mathbf{1}$, and can be regarded as an instance of a face of an extended submodular integer polyhedron supported by a positive vector.

III. The results

We begin with a structure of the set of integer solutions to the equation $a \cdot x = b$ with a certain d-dimensional positive rational vector a and a rational number b. Note that a can be expressed as a vector of unit fractions $(1/m_1, \cdots, 1/m_d)$, by suitably scaling a and b.

Lemma III.1. *Let $x^0 \in \mathbf{Z}^d$ be a solution to the equation $a \cdot x = b$, where a is a d-dimensional positive rational vector and b is a rational number. If $a = (1/m_1, \cdots, 1/m_d)$ with pairwise coprime positive integers $m_1, \cdots, m_d$, then, for any given $h \in \{1, \cdots, d\}$,*

$$\{x \mid a \cdot x = b,\ x \text{ integral}\} = \{x^0 + \sum_{i \neq h} \lambda_i(m_i\mathbf{e}^i - m_h\mathbf{e}^h) \mid \lambda_i \in \mathbf{Z},\ i = 1, \cdots, d,\ i \neq h\}. \quad (6)$$

Proof: Let $a \cdot y = b$, where $y \in \mathbf{R}^d$. Then $y = x^0 + \sum_{i \neq h} \lambda_i(m_i\mathbf{e}^i - m_h\mathbf{e}^h)$ for some $\lambda_i \in \mathbf{R}$ for each $i \neq h$, since $a \cdot (m_i\mathbf{e}^i - m_h\mathbf{e}^h) = 0$ for all $i \neq h$, and $m_i\mathbf{e}^i - m_h\mathbf{e}^h$ ($i = 1, \cdots, d$, $i \neq h$) are linearly independent. We show that $y \in \mathbf{Z}^d$ only if $\lambda_i \in \mathbf{Z}$ for all $i \neq h$ (the converse is clear). Let $z := y - x^0$ and assume $z \in \mathbf{Z}^d$ (viz., $y \in \mathbf{Z}^d$). Here $z_i = \lambda_i m_i$ for all $i \neq h$ and $z_h = -\sum_{i \neq h} \lambda_i m_h$, so $z_h = -\sum_{i \neq h}(z_i/m_i)m_h$, or, $(m_1 \cdots m_d/m_h)z_h = -\sum_{i \neq h}(m_1 \cdots m_d/m_i)z_i$, multiplying $m_1 \cdots m_d/m_h$ for both sides. To see that $\lambda_j \in \mathbf{Z}$ for all $j \neq h$, pick an arbitrary $j \neq h$ and assume $m_j > 1$ (if $m_j = 1$ then $\lambda_j = z_j/m_j \in \mathbf{Z}$, as desired). Then for an indeterminate equation

$$-(m_1 \cdots m_d/m_j)z_j = \sum_{i \neq j}(m_1 \cdots m_d/m_i)z_i \quad (7)$$

of $d-1$ unknowns z_i ($i = 1, \cdots, d$, $i \neq j$) to have integer solutions, it is necessary (and sufficient) that the greatest common divisor of the right hand side coefficients divides the left hand side. Since m_j divides all $m_1 \cdots m_d/m_i$ ($i = 1, \cdots, d$, $i \neq j$), m_j is a factor of the greatest common divisor and divides $-(m_1 \cdots m_d/m_j)z_j$. Since m_i ($i = 1, \cdots, d$) are pairwise coprime, this says that m_j divides z_j, i.e., $\lambda_j = z_j/m_j \in \mathbf{Z}$. ■

Thus, if $m_1, \cdots, m_d$ are pairwise coprime, the set of vectors

$$\{m_i\mathbf{e}^i - m_h\mathbf{e}^h \mid i = 1, \cdots, d,\ i \neq h\} \quad (8)$$

given any $h \in \{1, \cdots, d\}$ is a basis of the set of integer points of a hyperplane with normal $(1/m_1, \cdots, 1/m_d)$ containing an integer point, e.g., the origin. The pairwise coprimality condition is also necessary. To see this, let H_a and $H_{\mathbf{1}}$ be the sets of integer points of hyperplanes through the origin with normal vectors $a = (1/m_1, \cdots, 1/m_d)$ and $\mathbf{1}$, respectively, where $m_1, \cdots, m_d$ are not necessarily pairwise coprime positive integers. Then $H_a \supseteq TH_{\mathbf{1}}$ by a diagonal matrix $T := \mathrm{diag}(m_1, \cdots, m_d)$, and the set $\{\mathbf{e}^i - \mathbf{e}^d \mid i = 1, \cdots, d-1\}$, for example, is a basis of $H_{\mathbf{1}}$ by Lemma III.1. This basis is mapped by T to the set $\{m_i\mathbf{e}^i - m_d\mathbf{e}^d \mid i = 1, \cdots, d-1\}$, and if m_1 and m_2 are not coprime, for example, then letting $x^1 := m_1\mathbf{e}^1 - m_d\mathbf{e}^d \in H_a$ and $x^2 := m_2\mathbf{e}^2 - m_d\mathbf{e}^d \in H_a$, $x^2 - x^1 = m_2\mathbf{e}^2 - m_1\mathbf{e}^1 = c(m_2'\mathbf{e}^2 - m_1'\mathbf{e}^1)$ with some integer $c > 1$ and coprime integers m_2' and m_1', which says that there are $c - 1 > 0$ integer points between x^1 and x^2 not spanned by the vectors $m_i\mathbf{e}^i - m_d\mathbf{e}^d$ ($i \neq d$) resulting in $H_a \neq TH_{\mathbf{1}}$. Hence the pairwise coprimality is necessary for Eq. (8) to be a basis of H_a.

Using this lemma, we state and prove our first proposition. Recall that an edge e of a face of an extended submodular integer polyhedron supported by a positive vector is segmented by a vector of the form $m_i(e)\mathbf{e}^i - m_j(e)\mathbf{e}^j$, where $m_i(e)$ and $m_j(e)$ are coprime positive integers specific to e. For brevity, we denote by $X + x$ the sum $X + \{x\}$ of a set X and a point x. The set $X + (-x)$ is denoted by $X - x$.

Proposition III.1. *Let $P^1, \cdots, P^n$ be extended submodular integer polyhedra in $\mathbf{R}^d$, and $P^1_a, \cdots, P^n_a$ be their faces supported by a positive rational vector $a := (1/m_1, \cdots, 1/m_d)$. Let $E^r_a := P^r_a \cap \mathbf{Z}^d$, $r = 1, \cdots, n$. If $m_1, \cdots, m_d$ are pairwise coprime then $\sum_r E^r_a$ is hole-free.*

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Proof: Let b^r be a vertex of $\mathrm{conv}(E_a^r)$ for each $r = 1, \cdots, n$, and let $T := \mathrm{diag}(m_1, \cdots, m_d)$. Then each E_a^r lies in the set $TH_{\mathbf{1}} + b^r \subset \mathbf{Z}^d$, $r = 1, \cdots, n$, and $\sum_r E_a^r$ lies in $TH_{\mathbf{1}} + \sum_r b^r$, where $H_{\mathbf{1}} := \{x \in \mathbf{Z}^d \mid \mathbf{1} \cdot x = 0\}$. The set $TH_{\mathbf{1}} + \sum_r b^r$ is bijective to $H_{\mathbf{1}}$ and has a basis of the form Eq. (8). Now, the set $T^{-1}(E_a^r - b^r) \subset H_{\mathbf{1}}$ is M-convex, since $T^{-1}(E_a^r - b^r)$ is hole-free given hole-free $E_a^r - b^r$, and an edge of $\mathrm{conv}(E_a^r - b^r)$ segmented by a vector $m_i(e)\mathbf{e}^i - m_j(e)\mathbf{e}^j$ is mapped by T^{-1} to an edge of $\mathrm{conv}(T^{-1}(E_a^r - b^r))$ segmented by $\mathbf{e}^i - \mathbf{e}^j$ ($i, j \in \{1, \cdots, d\}$, $i \neq j$). Therefore $\sum_r T^{-1}(E_a^r - b^r) \subset H_{\mathbf{1}}$ is hole-free as a sum of M-convex sets, and so is $\sum_r E_a^r = T \sum_r T^{-1}(E_a^r - b^r) + \sum_r b^r$ by the bijectiveness of $TH_{\mathbf{1}} + \sum_r b^r$ and $H_{\mathbf{1}}$. ∎

Thus, for any $x \in \mathrm{conv}(\sum_r E_a^r) \cap \mathbf{Z}^d$, there is an $x^r \in E_a^r$ for each $r = 1, \cdots, n$ such that $x = x^1 + \cdots + x^n$, if $a = (1/m_1, \cdots, 1/m_d)$ with pairwise coprime $m_1, \cdots, m_d$. A simple example of such an a is where m_i are distinct primes or ones, which might be restrictive, though. Assuming that P^r are integral and extended submodular, what else can be said?

Proposition III.2. *Let P^r, P_a^r, and E_a^r be as in Proposition III.1, $r = 1, \cdots, n$, with an arbitrary $a \in \mathbf{R}_{++}^d$. Suppose there is an integer vector $m = (m_1, \cdots, m_d)$ common to all r such that all the points of E_a^r are congruent modulo m for each r. Let $x \in \mathrm{conv}(\sum_r E_a^r) \cap \mathbf{Z}^d$. If $x \equiv v$ (mod m) (x and v are congruent modulo m) for some vertex v of $\mathrm{conv}(\sum_r E_a^r)$, then $x \in \sum_r E_a^r$.*

Proof: Let b^r be a vertex of $\mathrm{conv}(E_a^r)$ for each $r = 1, \cdots, n$, and let $T := \mathrm{diag}(m_1, \cdots, m_d)$. Then each E_a^r lies in the set $TH_{\mathbf{1}} + b^r \subset \mathbf{Z}^d$, $r = 1, \cdots, n$, and $\sum_r E_a^r$ lies in $TH_{\mathbf{1}} + \sum_r b^r$, where $H_{\mathbf{1}} := \{x \in \mathbf{Z}^d \mid \mathbf{1} \cdot x = 0\}$. This time the set $TH_{\mathbf{1}} + \sum_r b^r$ is not bijective to $H_{\mathbf{1}}$, but the set $T^{-1}(E_a^r - b^r) \subset H_{\mathbf{1}}$ is M-convex, as can be shown similarly to the proof of Proposition III.1. Therefore $\sum_r T^{-1}(E_a^r - b^r) \subset H_{\mathbf{1}}$ is hole-free as a sum of M-convex sets, and the set $\sum_r E_a^r = T \sum_r T^{-1}(E_a^r - b^r) + \sum_r b^r$ is a subset of the set of integer points of $\mathrm{conv}(\sum_r E_a^r)$ that are congruent modulo m, including the set of all vertices. Hence, for $x \in \mathrm{conv}(\sum_r E_a^r) \cap \mathbf{Z}^d$, that $x \equiv v$ (mod m) for some vertex v of $\mathrm{conv}(\sum_r E_a^r)$ implies that $x \in \sum_r E_a^r$. ∎

Thus, if each E_a^r consists of points that are congruent modulo m (to be precise modulo m_i for all $i = 1, \cdots, d$), and if m is the same for all $r = 1, \cdots, n$, then $x \in \mathrm{conv}(\sum_r E_a^r) \cap \mathbf{Z}^d$ congruent modulo m with a vertex is expressed as $x = x^1 + \cdots + x^n$, $x^r \in E_a^r$ for each $r = 1, \cdots, n$. This may be quite obvious if E_a^r were not hyperplaner but rectangular, integer intervals in $\mathbf{Z}^d$. Proposition III.2 says that similar thing can be said for our hyperplaner environment. Note that if some E_a^r is taken from a facet (i.e. a $(d-1)$-dimensional face) of P^r then $m_1, \cdots, m_d$ are pairwise coprime and we are back in the situation of Proposition III.1. Also note that, while m_i and m_j are coprime if the pair (i, j) appears on an edge of some $\mathrm{conv}(E_a^r)$ (for E_a^r is hole-free), m_i and m_k may not be coprime if the pair (i, k) appears only on an edge of $\mathrm{conv}(\sum_r E_a^r)$, which is possible. For example, for $d = 3$, if $E_a^1 = \{(0, 0, 1), (2, 0, 0)\}$ and $E_a^2 = \{(0, 0, 1), (0, 2, 0)\}$ with $a = (1/2, 1/2, 1)$, then $E_a^1 + E_a^2 = \{(0, 0, 2), (0, 2, 1), (2, 0, 1), (2, 2, 0)\}$. We have $(m_1, m_2, m_3) = (2, 2, 1)$, where m_1 and m_2 are not coprime for the pair $(1, 2)$ appears only on an edge of $\mathrm{conv}(E_a^1 + E_a^2)$. We note that the sum $E_a^1 + E_a^2$ has a hole $(1, 1, 1)$, but this is not congruent modulo $(2, 2, 1)$ with the vertices (the first two components are not).

The existence of such an integer vector m as in Proposition III.2 will certainly impose restrictions for each set $E_a^r := P_a^r \cap \mathbf{Z}^d$, $r = 1, \cdots, n$. As can be seen from the proof of Proposition III.2 (and also from the argument of Proposition III.1 and after), the points of E_a^r are congruent modulo m if and only if it is an affine transformation of an M-convex set M such that $E_a^r = TM + b^r$, with $T := \mathrm{diag}(m_1, \cdots, m_d)$, where $m_1, \cdots, m_d$ are pairwise coprime positive integers, and a vertex b^r of $\mathrm{conv}(E_a^r)$. We therefore propose the following subclass of extended submodular integer polyhedra.

Definition III.1. An extended submodular integer polyhedron P satisfies a *simultaneous exchangeability in pairwise coprime integers on facets* if, for any facet f of P, the set of integer points $E := f \cap \mathbf{Z}^d$ satisfies the property that $x \in E$, $y \in E$, and $i \in \mathrm{supp}^+(x - y)$ imply

$$x - m_i(f)\mathbf{e}^i + m_j(f)\mathbf{e}^j \in E \text{ and } y + m_i(f)\mathbf{e}^i - m_j(f)\mathbf{e}^j \in E \tag{9}$$

for some $j \in \mathrm{supp}^+(y - x)$, with pairwise coprime positive integers $m_1(f), \cdots, m_d(f)$ specific to f.

This says that each facet f is written as $T\,\mathrm{conv}(M) + b$, with $T := \mathrm{diag}(m_1(f), \cdots, m_d(f))$, where $m_1(f), \cdots, m_d(f)$ are pairwise coprime, an M-convex set $M \subset \mathbf{Z}^d$, and a vector $b \in \mathbf{Z}^d$. This can be seen as a special form of axiwise scaling used when constructing the faces of polybasic polyhedra from base polyhedra ([1]).

Now, if every P^r belongs to this class of polyhedra, then every $E_a^r := P_a^r \cap \mathbf{Z}^d$ is a set of points congruent modulo $m(f^r)$ given any positive normal vector a. The hole-freeness of the sum is secured if all the summands are identical.

Proposition III.4. *Let $P \subset \mathbf{R}^d$ be an extended submodular integer polyhedron satisfying the simultaneous exchangeability in pairwise coprime integers on facets. Then, letting $a \in \mathbf{R}_{++}^d$ be arbitrary, the n-fold sum $E_a + \cdots + E_a$ of $E_a := P_a \cap \mathbf{Z}^d$ is hole-free.*

Proof: Let f be a facet of P including P_a and let $T := \mathrm{diag}(m_1(f), \cdots, m_d(f))$, where $m_i(f)$ are pairwise coprime positive integers specific to f. Then, choosing an arbitrary vertex b of $P_a = \mathrm{conv}(E_a)$, the set $M := T^{-1}(E_a - b)$ and its n-fold sum are M-convex sets lying in $H_{\mathbf{1}} := \{x \in \mathbf{Z}^d \mid \mathbf{1} \cdot x = 0\}$. Since $M + \cdots + M$ (n-fold) is hole-free, $E_a + \cdots + E_a = T(M + \cdots + M) + nb$ is hole-free due to the bijectiveness of $TH_{\mathbf{1}} + nb$ and $H_{\mathbf{1}}$ under the pairwise coprimality of the diagonal elements of T. ∎

IV. Discussion

In this section we discuss the implication of our results to the division problem as introduced in Section 1. Suppose there are d types of indivisible commodities $i = 1, \cdots, d$ and n agents $r = 1, \cdots, n$. The commodity space is $\mathbf{Z}_+^d := \{x \in \mathbf{Z} \mid x_i \geq 0 \ \forall i\}$. We express each agent's preference by a family of upper contour sets $E^r(z^r) \subset \mathbf{Z}_+^d$ consisting of $x \in \mathbf{Z}_+^d$ at least good as $z^r \in \mathbf{Z}_+^d$ for r. We assume that preferences are weakly convex in that $E^r(z^r)$ are hole-free

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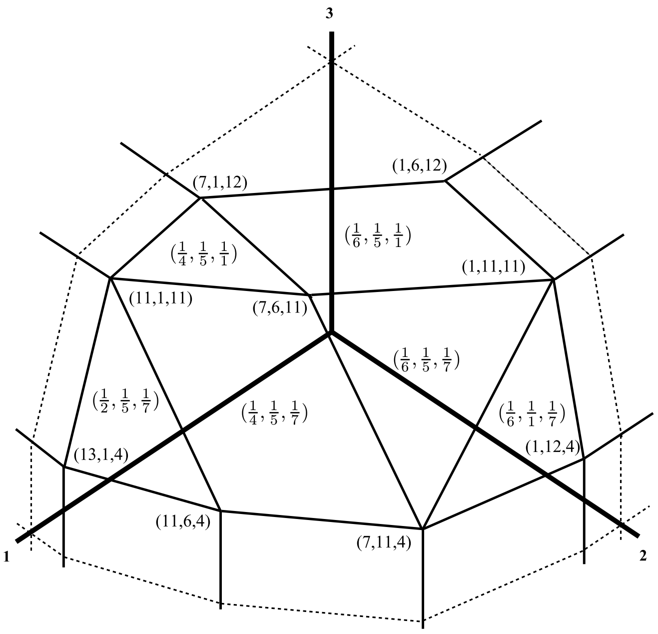

Fig. 1. An example of extended submodular integer polyhedron satisfying the simultaneous exchangeability in pairwise coprime integers on facets (the vectors of unit fractions are the normal vectors of the facets)

and weakly monotone in that $x \in E^r(z^r)$ and $y \geq x$ imply $y \in E^r(z^r)$ if $y \in \mathbf{Z}^d$. Let $P^r(z^r) := \mathrm{conv}(E^r(z^r))$. Then $P^r(z^r)$ are *up-monotone* in that $x \in P^r(z^r)$ and $y \geq x$ imply $y \in P^r(z^r)$. A face $P^r_a(z^r)$ of an up-monotone $P^r(z^r)$ is determined by the *inf-support function* $\delta^\circ_{P^r(z^r)} \colon \mathbf{R}^d_+ \to \mathbf{R}$ of $P^r(z^r)$ defined by

$$\delta^\circ_{P^r(z^r)}(a) := \inf\{a \cdot x \mid x \in P^r(z^r)\} \tag{10}$$

in such a way that

$$P^r_a(z^r) := \{x \in P^r(z^r) \mid a \cdot x = \delta^\circ_{P^r(z^r)}(a)\}. \tag{11}$$

Let $E^r_a(z^r) := P^r_a(z^r) \cap \mathbf{Z}^d$, where $P^r_a(z^r)$ is a face of $P^r(z^r)$ supported by a positive vector $a \in \mathbf{R}^d_{++}$. We interpret a as a vector of prices. Without loss of generality we assume that $E^r_a(z^r)$ consists of indifferent vectors of goods, and call it a *demand set* of agent r. For brevity let us denote $E^r(z^r)$, $P^r(z^r)$, $P^r_a(z^r)$, and $E^r_a(z^r)$ by E^r, P^r, P^r_a, and E^r_a, respectively. Our problem is written as follows.

> Let E^r_a be a demand set of agent r under a price vector $a \in \mathbf{R}^d_{++}$, $r = 1, \cdots, n$. Is a bundle of goods $x \in \mathrm{conv}(\sum_r E^r_a) \cap \mathbf{Z}^d$ divisible among the agents?

Analogously to [2], let us call an up-monotone polyhedron P^r an *extended supermodular polyhedron* if its *inf-support function* $\delta^\circ_{P^r}$ is *supermodular* in that

$$\delta^\circ_{P^r}(a) + \delta^\circ_{P^r}(a') \leq \delta^\circ_{P^r}(a \vee a') + \delta^\circ_{P^r}(a \wedge a'), \quad a, a' \in \mathbf{R}^d_+. \tag{12}$$

These are just a change in the signs and do not alter our results. The inf-support function δ° is thought to be the (cost-minimizing) expenditure function and its supermodularity in prices is interpreted that goods are net substitutes each other for the agent r.

Under this setting, Proposition III.1 tells us that if the price vector $a \in \mathbf{R}^d_{++}$ is such that $a = (1/m_1, \cdots, 1/m_d)$ with pairwise coprime integers $m_1, \cdots, m_d$, then any bundle of goods $x \in \mathrm{conv}(\sum_r E^r_a) \cap \mathbf{Z}^d$ is divisible among the agents. Here Proposition III.2 will assist in revealing the mechanism behind, that is, all the elements in the sum $\sum_r E^r_a$ are then congruent modulo m (note that $x_i = y_i$ are congruent modulo for any m_i). However, imposing the pairwise coprimality condition for a price vector is certainly restrictive.

Proposition III.4, which exploits Definition III.1, is imposing a similar condition for the facets of P^r, $r = 1, \cdots, n$. The simultaneous exchangeability in pairwise coprime integers on facets can be interpreted that *the marginal rates of substitution are well-defined on each demand set*. Under this setting, any $x \in \mathrm{conv}(\sum_r E^r_a) \cap \mathbf{Z}^d$ is divisible among the agents if their preferences are identical, as Proposition III.4 suggests.

For a possible scenario for the weakly convex and weakly monotone not necessarily identical preferences with well-defined marginal rates of substitution, let $I_r := \{i \mid i \in \mathrm{supp}(x-y),\ x, y \in E^r_a\}$, $r = 1, \cdots, n$. If I_r are linearly ordered in such a way that $I_{r_1} \subseteq \cdots \subseteq I_{r_n}$, where $(r_1, \cdots, r_n)$ is a permutation of $(1, \cdots, n)$, then we can show the hole-freeness of $\sum_r E^r_a = T \sum_{r=1}^n T^{-1}(E^r_a - b^r) + \sum_r b^r$ by letting $T := \mathrm{diag}(m_1(f^{r_n}), \cdots, m_d(f^{r_n}))$, where f^{r_n} is the facet of P^{r_n} that includes the maximum dimensional $E^{r_n}_a$ of agent r_n, and by letting b^r be a vertex of $\mathrm{conv}(E^r_a)$, for each $r = 1, \cdots, n$. This may be interpreted as a situation where *the agents are ranked in the order of the scope of substitutability*, though it is not clear how this is obtained.

References

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(Revised on May 12, 2011: Proposition III.3 was deleted.)