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Wavelet Method for Numerical Solution of Wave Equation with Neumann Boundary Conditions

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Abstract—In this paper, we derive a highly accurate numerical method for the solution of one-dimensional wave equation with Neumann boundary conditions. This hyperbolic problem is solved by using semidiscrete approximations. The space direction is discretized by wavelet-Galerkin method and the time variable is discretized by using various classical finite difference schemes. The numerical results show that this method gives high favourable accuracy while compared with the exact solution.

Index Terms—hyperbolic problem, semidiscrete approximations, stability, wavelet-Galerkin method.

I. INTRODUCTION

In this paper, we consider numerical solution of one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} = f, \quad a < x < b, \quad t > 0 \tag{1}$$

with initial conditions

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x), \quad a < x < b \tag{2}$$

and boundary conditions

$$\frac{\partial u}{\partial x}(a,t) = c(t), \quad \frac{\partial u}{\partial x}(b,t) = d(t), \quad t > 0, \tag{3}$$

where α is a positive constant and f is a constant, or a function of any or of both the independent variables x and t. Second order hyperbolic partial differential equations (PDE) like (1) appear in connection with structural dynamics. Several methods exist for the solution of second order hyperbolic equations with Dirichlet and other boundary conditions, for example, [2], [3], [7]. But their solution with Neumann boundary conditions is hardly available in the literature.

Usually, hyperbolic problems are solved by using semidiscrete approximations. For the solution of problem (1)-(3) in the present paper, the space direction is discretized by using wavelet-Galerkin method and the time variable is discretized by using various classical schemes originating from finite difference methods. Wavelets in consideration here are Daubechies compactly supported wavelets [6] which are differentiable.

Wavelet applications to the solution of PDE problems are relatively new. Some recent applications are [1], [4], [5], [9] among many more. To discretize a PDE problem by wavelet-Galerkin method, the Galerkin bases are constructed from orthonormal bases of compactly supported wavelets. This can be done in a number of ways. In this paper, we construct these basis functions as in Choudhury and Deka [4], which is a variant of Glowinski et al. [8]. .

In Section 2, we explain the approximation of the Sobolev space $H^m(a,b)$ using Daubechies scaling functions. Section 3 elaborates the method for the solution of problem (1)-(3). In Section 4, we demonstrate the method with the help of a numerical example. Section 5 concludes the paper.

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II. APPROXIMATION OF SOBOLEV SPACES IN DAUBECHIES BASES

For a positive integer N, consider two functions $\phi, \psi \in L^2(R)$ defined by

$$\phi(x) = \sum_k a_k \phi(2x-k), \ \psi(x) = \sum_k b_k \phi(2x-k), \tag{4}$$

where $\{a_k\}_{k \in Z}$ and $\{b_k\}_{k \in Z}$ are two specific sequences [6] such that $a_k = b_k = 0$ for k $\notin \{0, 1, \dots, S\}, \ S = 2N-1$. The functions ϕ and ψ are called dbN scaling function and dbN wavelet function respectively, where N is called their order. These functions are compactly supported with $\text{supp}(\phi) = \text{supp}(\psi) = [0, S]$. They are available in wavelet toolbox of MATLAB 6 for $1 \le N \le 45$. They satisfy the properties (3.9)-(3.12) in [4].

The integer translates and dilates of ϕ and ψ are defined as

$$\phi_{n,k}(x) = 2^{\frac{n}{2}} \phi(2^n x - k), \ \psi_{n,k}(x) = 2^{\frac{n}{2}} \psi(2^n x - k), \tag{5}$$

for $n, k \in Z$.
Now, for all $n \in Z$, we define

$$V_n = L^2\text{-closure}(\text{span}\{\phi_{n,k} : k \in Z\}). \tag{6}$$

We recall here that for an open interval (a,b) and for an integer $m \ge 1$, the space

$$H^m(a,b) = \{u \in H^{m-1}(a,b) : u' \in H^{m-1}(a,b)\} \tag{7}$$

is called the Sobolev space of order m, which is a Hilbert space with inner product $\langle u, v \rangle_m = \sum_{i=0}^m \int_a^b u^{(i)} v^{(i)} dx$ and associated norm $\|.\|_m$. It may be noted here that $H^0(a,b) = L^2(a,b)$.

Let N be any positive integer and let ϕ and ψ be the dbN scaling function and wavelet function respectively. Then by Theorem 1.1 in [8], there exists an integer m, $0 \le m < N$, such that the Sobolev space $H^m(a,b)$

can be approximated by the restrictions of translates and dilates of ϕ to (a,b).

We shift the support of ϕ from $[0,S]$ to $[a,b]$ by using the transformation $y=\frac{b-a}{S}x+a$ and let

$$I_n=\{k\in Z:\operatorname{supp}(\phi_{n,k})\cap(a,b)\neq\emptyset\}. \tag{8}$$

Considering V_n as defined in (6), we define the space $V_n(a,b)$ to be the set of restrictions of all functions in V_n to (a,b). In fact, we take

$$V_n(a,b)=\operatorname{span}\{\phi_{n,k}|_{(a,b)}:k\in I_n\}. \tag{9}$$

Since (a,b) is bounded, the space $V_n(a,b)$ is finite dimensional and is a closed subspace of $H^m(a,b)$. By Proposition 4.1 in [4], $\dim(V_n(a,b))=2^nS+S-1$ and a basis for $V_n(a,b)$ is given by

$$\{\phi_{n,k}\in V_n(a,b):1-S\leq k\leq 2^nS-1\}. \tag{10}$$

III. SOLUTION METHODOLOGY

Since PDE (1) is of second order in space with Neumann boundary conditions (3), the solution space for spatial direction for problem (1)-(3) is $H^1(a,b)$. Multiplying equation (1) by a function $v\in H^1(a,b)$ and integrating by parts with respect to x in (a,b), we get

$$\int_a^b\left(\frac{\partial^2u}{\partial t^2}v+\alpha\frac{\partial u}{\partial x}\frac{dv}{dx}\right)dx$$

$$=\int_a^b fvdx+\alpha[d(t)v(b)-c(t)v(a)], \tag{11}$$

which is the variational (weak) form of problem (1)-(3).

In Glowinski et al. [8], it is established that $N\geq 3$ is sufficient for the solution of problems of second order (in space). So, we let $N\geq 3$ be any integer and let ϕ be the *dbN* scaling function. Considering the basis $\{\phi_{n,j}\}$ of $V_n(a,b)$ in Section 2, the approximate solution of the variational problem (11) can be taken as

$$u_n(x,t)=\sum_j z_{n,j}(t)\phi_{n,j}(x). \tag{12}$$

Applying Galerkin method to problem (11) with the approximate solution (12), we get a system of second order ordinary differential equations in $z=[z_{n,j}]$:

$$M\ddot{z}+Az=F, \tag{13}$$

where A, M and F are the stiffness matrix, the mass matrix and the force vector respectively whose elemments are given by

$$\left\{\begin{array}{lcl} A_{ij} & = & \int_a^b\alpha\phi'_{n,j}\phi'_{n,i}dx,\\ M_{ij} & = & \int_a^b\phi_{n,j}\phi_{n,i}dx,\\ F_i & = & \int_a^b f\phi_{n,i}dx+\alpha[d(t)\phi_{n,i}(b)\\ & & -c(t)\phi_{n,i}(a)]. \end{array}\right. \tag{14}$$

There are several methods available to integrate equation (13). The most widely used method in structural dynamics is the Newmark family of time integration schemes [10]:

$$\left\{\begin{array}{l} z_{s+1}=z_s+\Delta t\dot{z}_s+\Delta t^2\left[\left(\frac{1}{2}-\mu\right)\ddot{z}_s+\mu\ddot{z}_{s+1}\right],\\ \dot{z}_{s+1}=\dot{z}_s+\Delta t\left[(1-\lambda)\ddot{z}_s+\lambda\ddot{z}_{s+1}\right], \end{array}\right. \tag{15}$$

where λ and μ are parameters that control the accuracy and stability of the scheme and z_s refers to the value of z at time $t=t_s=s\Delta t$.

The following schemes are special cases of (15):

1. The linear acceleration scheme: $\lambda=\frac{1}{2},\ \mu=\frac{1}{6}$; conditionally stable;
2. The constant-average acceleration scheme: $\lambda=\frac{1}{2},\ \mu=\frac{1}{4}$; unconditionally stable;
3. The Galerkin scheme: $\lambda=\frac{3}{2},\ \mu=\frac{4}{5}$; unconditionally stable;
4. The backward difference scheme: $\lambda=\frac{3}{2},\ \mu=1$; unconditionally stable.

For all schemes in which $\mu<\frac{\lambda}{2}$ and $\lambda\geq\frac{1}{2}$, the stability requirement is

$$\Delta t\leq\frac{1}{\omega}\sqrt{\frac{2}{\lambda-2\mu}}, \tag{16}$$

where ω is the maximum natural frequency of the associated problem.

The use of (15) in (13) gives the system of linear equations

$$\hat{M}_{s+1}z_{s+1}=\hat{F}_{s,s+1}, \tag{17}$$

where

$$\left\{\begin{array}{l} \hat{M}_{s+1}=A_{s+1}+a_0M,\\ \hat{F}_{s,s+1}=F_{s+1}+M(a_0z_s+a_1\dot{z}_s+a_2\ddot{z}_s);\\ a_0=\frac{1}{\mu(\Delta t)^2},\ a_1=a_0\Delta t,\ a_2=\frac{1}{2\mu}-1. \end{array}\right. \tag{18}$$

The vectors z_0 and $\dot{z}_0$ can be obtained by multiplying the initial conditions (2) by v, integrating and approximating with (12). $\ddot{z}_0$ can be calculated as an approximation from (13) given by

$$\ddot{z}_0=M^{-1}(F_0-Az_0). \tag{19}$$

At the end of each time step, the velocity vector $\dot{z}_{s+1}$ and the acceleration vector $\ddot{z}_{s+1}$ are computed using the relations

$$\left\{\begin{array}{l} \ddot{z}_{s+1}=a_0(z_{s+1}-z_s)-a_1\dot{z}_s-a_2\ddot{z}_s,\\ \dot{z}_{s+1}=\dot{z}_s+a_3\ddot{z}_s+a_4\ddot{z}_{s+1}, \end{array}\right. \tag{20}$$

where

$$a_3=(1-\lambda)\Delta t,\ a_4=\lambda\Delta t. \tag{21}$$

IV. NUMERICAL RESULTS

Here the methodology for the solution of the hyperbolic IBVP (1)-(3) described above is demonstrated with an example. The solution is performed using all the four time discretization schemes. The computations are performed by using MATLAB 6.5. The problem is:

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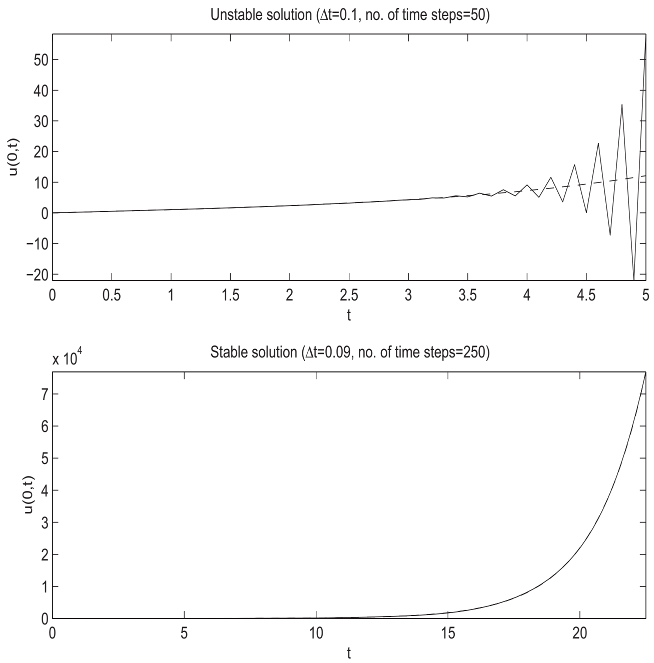

Fig. 1. $\lambda = \frac{1}{2},\ \mu = \frac{1}{6}$: db3(n=0) wavelet solution $(-)$ and exact solution $(--)$ at x=0

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1,\ t > 0; \\ u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = e^x, \quad 0 < x < 1; \\ \frac{\partial u}{\partial x}(0,t) = 2\sinh\left(\frac{t}{2}\right) + 1, \quad t > 0, \\ \frac{\partial u}{\partial x}(1,t) = 2e\sinh\left(\frac{t}{2}\right) + 1, \quad t > 0. \end{cases} \tag{22}$$

The problem is so constructed that its exact solution is

$$u(x,t) = 2e^x \sinh\left(\frac{t}{2}\right) + x. \tag{23}$$

For $\lambda = \frac{1}{2},\ \mu = \frac{1}{6}$, $db3$ scaling function is used for spatial discretization. As this scheme is conditionally stable, we have to find an upper limit of the time step Δt using the stability condition (16). The square of the maximum natural frequency of the associated problem for $n = 0$ is 1298.2. Therefore, the maximum time step is $\frac{1}{\sqrt{1298.2}}\sqrt{\frac{2}{\frac{1}{2}-\frac{1}{3}}} \approx 0.096$. Figure 1 shows the exact, unstable and stable solutions due to $db3(n = 0)$ scaling function at $x = 0$.

For $\lambda = \frac{3}{2},\ \mu = \frac{4}{5}$ and $\lambda = \frac{3}{2},\ \mu = 1$, we also use $db3$ scaling function for spatial discretization. As this scheme is unconditionally stable, there is no restriction on Δt. Table 1 shows the decay of maximum absolute error for both the schemes with decreasing time step due to $db3(n = 0, 1, 2)$ scaling functions at $t = 1$.

For $\lambda = \frac{1}{2},\ \mu = \frac{1}{4}$, Table 2 shows the maximum absolute errors between the exact and the computed solutions at $t = 1$ due to $db3(n = 0, 1, 2, 3)$ and $db4(n = 0, 1, 2, 3)$ scaling functions for $\Delta t = 0.01$ and $\Delta t = 0.001$ respectively.

V. Conclusion

In this paper, we have analysed a method for numerical solution of one dimensional wave equation with Neumann boundary conditions. The space direction is discretized by using wavelet-Galerkin method and the time variable is discretized by using classical finite difference schemes. The main advantages of this method are that the schemes are unconditionally stable (except one) and are useful for problems with time-dependent boundary conditions and with time-dependent source term. The method gives high favourable accuracy. The efficiency of the developed algorithm has been illustrated by a test problem.

Table 1: $\lambda = \frac{3}{2},\ \mu = \frac{4}{5}$ and $\lambda = \frac{3}{2},\ \mu = 1$
Maximum absolute errors at t=1 due to db3 scaling functions

Scheme	Time step	Maximum absolute error		
$(\lambda,\ \mu)$	(Δt)	n=0	n=1	n=2
$\lambda = \frac{3}{2}$	$\frac{1}{5}$	0.0608	0.0599	0.0598
$\mu = \frac{4}{5}$	$\frac{1}{10}$	0.0312	0.0299	0.0298
	$\frac{1}{20}$	0.0166	0.0151	0.0149
	$\frac{1}{40}$	0.0093	0.0077	0.0074
	$\frac{1}{80}$	0.0057	0.0040	0.0038
$\lambda = \frac{3}{2}$	$\frac{1}{5}$	0.0649	0.0641	0.0640
$\mu = 1$	$\frac{1}{10}$	0.0323	0.0310	0.0308
	$\frac{1}{20}$	0.0169	0.0154	0.0151
	$\frac{1}{40}$	0.0094	0.0078	0.0075
	$\frac{1}{80}$	0.0057	0.0040	0.0038

Table 2: $\lambda = \frac{1}{2},\ \mu = \frac{1}{4}$
Maximum absolute errors at t=1 due to db3 and db4 scaling functions

Scaling funcs.	n	Δt	Maxi. absolute error
db3	0	10^{-2}	2.0939×10^{-3}
	1	10^{-2}	3.4757×10^{-4}
	2	10^{-2}	5.4535×10^{-5}
	3	10^{-2}	1.1523×10^{-5}
db4	0	10^{-3}	4.2155×10^{-5}
	1	10^{-3}	2.9833×10^{-6}
	2	10^{-3}	2.4169×10^{-7}
	3	10^{-3}	6.1729×10^{-8}

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