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Two Edge-Disjoint Hamiltonian Cycles and Two-Equal Path Partition in Augmented Cubes

Ruo-Wei Hung[‡§] and Chien-Chih Liao[‡]

***Abstract*—The n-dimensional hypercube network Q_n is one of the most popular interconnection networks since it has simple structure and is easy to implement. The n-dimensional augmented cube, denoted by AQ_n, an important variation of the hypercube, possesses several embedding properties that hypercubes and other variations do not possess. The advantages of AQ_n are that the diameter is only about half of the diameter of Q_n and they are node-symmetric. Recently, some interesting properties of AQ_n were investigated. A graph G contains two-equal path partition if for any two distinct pairs of nodes (u_s, u_t) and (v_s, v_t) of G, there exist two node-disjoint paths P and Q satisfying that (1) P joins u_s and u_t, and Q joins v_s and v_t, (2) $|P| = |Q|$, and (3) every node of G appears in one path exactly once. In this paper, we first use a simple recursive method to construct two edge-disjoint Hamiltonian cycles in AQ_n for any integer $n \geqslant 3$. We then show that the n-dimensional augmented cube AQ_n, with $n \geqslant 2$, contains two-equal path partition.**

***Index Terms*—edge-disjoint Hamiltonian cycles, two-equal path partition, augmented cubes, hypercubes, parallel computing**

I. Introduction

PARALLEL computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [3], [7], [8], [9], [10], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [24]. The architecture of an interconnection network is usually modeled by a graph, where the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graphs and networks interchangeably.

The n-dimensional augmented cube AQ_n was first proposed by Choudum et al. [6] and possesses some properties superior to the hypercube. The diameter of augmented cubes is only about half of the diameter of hypercubes and augmented cubes are node-symmetric [6]. Recently, some interesting properties, such as conditional link faults, of the augmented cube AQ_n were investigated. Choudum and Sunitha proved AQ_n, with $n \geqslant 2$, is pancyclic, that is, AQ_n contains cycles of arbitrary length [6]. Hsu et al. considered the fault hamiltonicity and the fault hamiltonian connectivity of the augmented cube AQ_n [13]. Wang et al. showed that AQ_n, with $n \geqslant 4$, remains pancyclic provided faulty vertices and/or edges do not exceed $2n - 3$ [26]. Hsieh and Shiu proved that AQ_n is node-pancyclic, in which for every node u and any integer $l \geqslant 3$, the graph contains a cycle C of length l such that u is in C [11]. Hsu et al. proved that AQ_n is geodesic pancyclic and balanced pancyclic [14]. Recently, Chan et al. [5] improved the results in [14] to obtain a stronger result for geodesic-pancyclic and fault-tolerant panconnectivity of the augmented cube AQ_n. In [19], Ma et al. proved that AQ_n contains paths between any two distinct vertices of all lengths from their distance to $2^n - 1$; and that AQ_n still contains cycles of all lengths from 3 to 2^n when any $(2n - 3)$ edges are removed from AQ_n. Xu et al. determined the vertex and the edge forwarding indices of AQ_n as $2^n/9 + (-1)^{n+1}/9 + n2^n/3 - 2^n + 1$ and 2^{n-1}, respectively [27]. Chan computed the distinguishing number of the augmented cube AQ_n [4]. Lee et al. studied the Hamiltonian path problem on AQ_n with a required node being the end node of a Hamiltonian path [18].

Two Hamiltonian cycles in a graph are said to be *edge-disjoint* if they do not share any common edge. The edge-disjoint Hamiltonian cycles can provide advantage for algorithms that make use of a ring structure [25]. The following application about edge-disjoint Hamiltonian cycles can be found in [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an N-node ring that requires $N - 1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ warmhole or cut-through routing [17].

The edge-disjoint Hamiltonian cycles in k-ary n-cubes has been constructed in [1]. Barden et al. constructed the maximum number of edge-disjoint spanning trees in a hypercube [2]. Petrovic et al. characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments [23]. Hsieh et al. constructed edge-disjoint spanning trees in locally twisted cubes [12]. The existence of a Hamiltonian cycle in augmented cubes has been shown [6], [13]. However, there has been little work reported so far on edge-disjoint properties in the augmented cubes. In this paper, we use a recursive construction to show that, for any integer $n \geqslant 3$, there are two edge-disjoint Hamiltonian cycles in the n-

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dimensional augmented cube AQ_n.

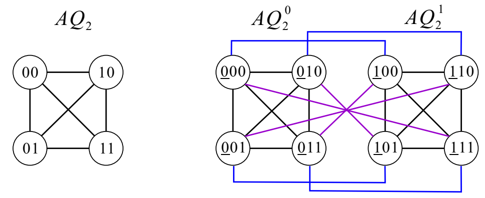
Finding node-disjoint paths is one of the important issues of routing among nodes in various interconnection networks. Node-disjoint paths can be used to avoid communication congestion and provide parallel paths for an efficient data routing among nodes. Moreover, multiple node-disjoint paths can be more fault-tolerant of nodes or link failures and greatly enhance the transmission reliability. A *path partition* of a graph G is a family of node-disjoint paths that contains all nodes of G. For an embedding of linear arrays in a network, the partition implies every node can be participated in a pipeline computation. Finding a path partition and its variants of a graph has been investigated [15], [16], [20], [21], [22]. In this paper, we study a variation of path partition, called *two-equal path partition*. A graph G contains *two-equal path partition* if for any two distinct pairs of nodes (u_s, u_t) and (v_s, v_t) of G, there exists a path partition $\{P, Q\}$ of G such that (1) P joins u_s and u_t, (2) Q joins v_s and v_t, and (3) $|P| = |Q|$. In this paper, we will show that the augmented cube AQ_n, with $n \geqslant 2$, contains two-equal path partition.

The rest of the paper is organized as follows. In Section II, the structure of the augmented cube is introduced, and some definitions and notations used in this paper are given. Section III shows the construction of two edge-disjoint Hamiltonian cycles in the augmented cubes. In Section IV, we show that augmented cubes contain two-equal path partition. Finally, we conclude this paper in Section V.

II. Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph $G = (V, E)$ is a pair of the node set V and the edge set E, where V is a finite set and E is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We will use $V(G)$ and $E(G)$ to denote the node set and the edge set of G, respectively. If (u, v) is an edge in a graph G, we say that u *is adjacent to* v. A *neighbor* of a node v in a graph G is any node that is adjacent to v. Moreover, we use $N_G(v)$ to denote the set of neighbors of v in G. The subscript 'G' of $N_G(v)$ can be removed from the notation if it has no ambiguity.

A path P, represented by $\langle v_0 \to v_1 \to \cdots \to v_{t-1}\rangle$, is a sequence of distinct nodes such that two consecutive nodes are adjacent. The first node v_0 and the last node v_{t-1} visited by P are called the *path-start* and *path-end* of P, denoted by $start(P)$ and $end(P)$, respectively, and they are called the *end nodes* of P. Path $\langle v_{t-1} \to \cdots \to v_1 \to v_0\rangle$ is called the *reversed path*, denoted by P_{rev}, of P. That is, P_{rev} visits the nodes of P from $end(P)$ to $start(P)$ sequently. In addition, P is a cycle if $|V(P)| \geqslant 3$ and $end(P)$ is adjacent to $start(P)$. A path $\langle v_0 \to v_1 \to \cdots \to v_{t-1}\rangle$ may contain other subpath Q, denoted as $\langle v_0 \to v_1 \to \cdots \to v_i \to Q \to v_j \cdots \to v_{t-1}\rangle$, where $Q = \langle v_{i+1} \to v_{i+2} \to \cdots \to v_{j-1}\rangle$. A path (or cycle) in G is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every node of G exactly once. A graph G is *Hamiltonian connected* if, for any two distinct nodes u, v, there exists a Hamiltonian path with end nodes u, v. Two paths (or cycles) P_1 and P_2 connecting a node u to a node v are said to be *edge-disjoint* iff $E(P_1) \cap E(P_2) = \emptyset$. Two paths (or cycles) Q_1 and Q_2 of graph G are called *node-disjoint* iff $V(Q_1) \cap V(Q_2) = \emptyset$. Two node-disjoint paths Q_1 and Q_2 can be *concatenated* into a path, denoted by $Q_1 \Rightarrow Q_2$, if $end(Q_1)$ is adjacent to $start(Q_2)$.



(a) (b)

Fig. 1. (a) The 2-dimensional augmented cube AQ_2, and (b) the 3-dimensional augmented cube AQ_3 containing AQ_2^0, AQ_2^1

Definition 1. A graph G contains *two-equal path partition* if for any two distinct pairs of nodes (u_s, u_t) and (v_s, v_t) of G, there exist two node-disjoint paths P and Q satisfying that (1) $start(P) = u_s$ and $end(P) = u_t$, (2) $start(Q) = v_s$ and $end(Q) = v_t$, (3) $|P| = |Q|$, and (4) $V(P) \cup V(Q) = V(G)$.

Now, we introduce augmented cubes. The node set of the n-dimensional augmented cube AQ_n is the set of binary strings of length n. A binary string b of length n is denoted by $b_{n-1}b_{n-2}\cdots b_1b_0$, where b_{n-1} is the most significant bit. We denote the complement of bit b_i by $\overline{b}_i = 1 - b_i$ and the leftmost bit complement of binary string b by $\overline{b} = \overline{b}_{n-1}b_{n-2}\cdots b_1b_0$. We then give the recursive definition of the n-dimensional augmented cube AQ_n, with integer $n \geqslant 1$, as follows.

Definition 2. [6] Let $n \geqslant 1$. The n-dimensional augmented cube, denoted by AQ_n, is defined recursively as follows.
(1) AQ_1 is a complete graph K_2 with the node set $\{0, 1\}$.
(2) For $n \geqslant 2$, AQ_n is built from two disjoint copies AQ_{n-1} according to the following steps. Let AQ_{n-1}^0 denote the graph obtained by prefixing the label of each node of one copy of AQ_{n-1} with 0, let AQ_{n-1}^1 denote the graph obtained by prefixing the label of each node of the other copy of AQ_{n-1} with 1. Then, adding 2^n edges between AQ_{n-1}^0 and AQ_{n-1}^1 by the following rule. A node $b = 0b_{n-2}b_{n-3}\cdots b_1b_0$ of AQ_{n-1}^0 is adjacent to a node $a = 1a_{n-2}a_{n-3}\cdots a_1a_0$ of AQ_{n-1}^1 iff either
(i) $a_i = b_i$ for all $n-2 \geqslant i \geqslant 0$ (in this case, (b, a) is called a *hypercube edge*), or
(ii)$a_i = \overline{b}_i$ for all $n-2 \geqslant i \geqslant 0$ (in this case, (b, a) is called a *complement edge*).

It was proved in [6] that AQ_n is node transitive, $(2n-1)$-regular, and has diameter $\lceil \frac{n}{2} \rceil$. According to Definition 2, AQ_n contains 2^n nodes. Further, AQ_n is decomposed into two sub-augmented cubes AQ_{n-1}^0 and AQ_{n-1}^1, where AQ_{n-1}^i consists of those nodes b with $b_{n-1} = i$. For each $i \in \{0, 1\}$, AQ_{n-1}^i is isomorphic to AQ_{n-1}. For example, Fig. 1(a) shows AQ_2 and Fig. 1(b) depicts AQ_3 consisting of two sub-augmented cubes AQ_2^0, AQ_2^1. The following proposition can be easily verified from Definition 2.

Proposition 1. *Let AQ_n be the augmented cube decomposed into AQ_{n-1}^0 and AQ_{n-1}^1. For any $b \in V(AQ_{n-1}^i)$ and $i \in \{0, 1\}$, $\overline{b} \in V(AQ_{n-1}^{1-i})$ and $\overline{b} \in N(b)$.*

Let b is a binary string $b_{t-1}b_{t-2}\cdots b_1b_0$ of length t. We

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denote b^i the new binary string obtained by repeating b string i times. For instance, $(10)^2 = 1010$ and $0^3 = 000$.

The following Hamiltonian connected property of the augmented cube can be proved by induction.

Lemma 2. *For any integer $n \geqslant 2$, AQ_n is Hamiltonian connected.*

Proof: We prove this lemma by induction on n, the dimension of the augmented cube AQ_n. Obviously, AQ_2 is Hamiltonian connected since it is a complete graph with 4 nodes. Assume that AQ_k, with $k \geqslant 2$, is Hamiltonian connected. We will prove that AQ_{k+1} is Hamiltonian connected. We first decompose AQ_{k+1} into two sub-augmented cubes AQ_k^0 and AQ_k^1. Let u, v be any two distinct nodes of AQ_{k+1}. There are two cases:

Case 1: $u, v \in V(AQ_k^i)$, for $i \in \{0, 1\}$. By inductive hypothesis, there is a Hamiltonian path P in AQ_k^i with end nodes u, v. Let $P = u \to P'$ and let $start(P') = w$. By inductive hypothesis, there is a Hamiltonian path Q in AQ_k^{1-i} such that $start(Q) = \overline{u}$ and $end(Q) = \overline{w}$. By Proposition 1, $\overline{u} \in N(u)$ and $\overline{w} \in N(w)$. Then, $u \Rightarrow Q \Rightarrow P'$ is a Hamiltonian path of AQ_{k+1} with end nodes u, v.

Case 2: $u \in V(AQ_k^i)$ and $v \in V(AQ_k^{1-i})$, for $i \in \{0, 1\}$. Let w be a node in AQ_k^i such that $w \neq u$ and $\overline{w} \neq v$. By inductive hypothesis, there is a Hamiltonian path P in AQ_k^i such that $start(P) = u$ and $end(P) = w$. In addition, there is a Hamiltonian path Q in AQ_k^{1-i} such that $start(Q) = \overline{w}$ and $end(Q) = v$. By Proposition 1, $\overline{w} \in N(w)$. Then, $P \Rightarrow Q$ is a Hamiltonian path of AQ_{k+1} with end nodes u, v.

By the above cases, AQ_{k+1} is Hamiltonian connected. By induction, AQ_n, with $n \geqslant 2$, is Hamiltonian connected. ■

III. Two Edge-Disjoint Hamiltonian Cycles

Obviously, AQ_2 has no two edge-disjoint Hamiltonian cycles since each node is incident to three edges. For any integer $n \geqslant 3$, we will construct two edge-disjoint Hamiltonian paths, P and Q, in AQ_n such that $start(P) = 0(0)^{n-3}00$, $end(P) = 1(0)^{n-3}00$, $start(Q) = 0(0)^{n-3}10$, and $end(Q) = 1(0)^{n-3}10$. By Proposition 1, $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. Thus, P and Q are two edge-disjoint Hamiltonian cycles.

Now, we show that AQ_3 contains two edge-disjoint Hamiltonian paths in the following lemma.

Lemma 3. *There are two edge-disjoint Hamiltonian paths P and Q in AQ_3 such that $start(P) = 000$, $end(P) = 100$, $start(Q) = 010$, and $end(Q) = 110$.*

Proof: We prove this lemma by constructing such two paths. Let

$P = \langle 000 \to 010 \to 011 \to 001 \to 101 \to 111 \to 110 \to 100\rangle$, and let

$Q = \langle 010 \to 001 \to 000 \to 011 \to 111 \to 100 \to 101 \to 110\rangle$.

Fig. 2 depicts the constructions of P and Q. Clearly, P and Q are edge-disjoint Hamiltonian paths in AQ_3. ■

By Proposition 1, nodes 000 and 100 are adjacent, and nodes 010 and 110 are adjacent. Thus, we have the following corollary.

Corollary 4. *There are two edge-disjoint Hamiltonian cycles in AQ_3.*

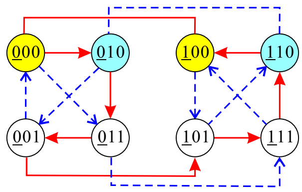

Fig. 2. Two edge-disjoint Hamiltonian paths in AQ_3, where solid arrow lines indicate a Hamiltonian path P and dotted arrow lines indicate the other edge-disjoint Hamiltonian path Q

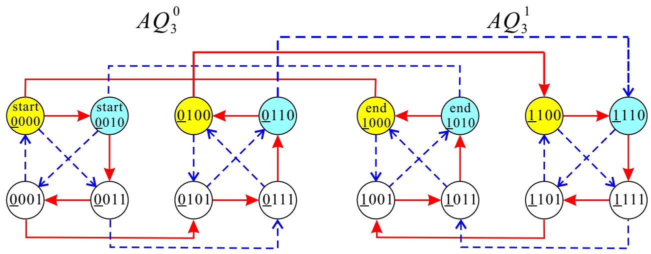

Fig. 3. Two edge-disjoint Hamiltonian paths in AQ_4, where solid arrow lines indicate a Hamiltonian path P and dotted arrow lines indicate the other edge-disjoint Hamiltonian path Q

Using Lemma 3, we prove the following lemma.

Lemma 5. *There are two edge-disjoint Hamiltonian paths P and Q in AQ_4 such that $start(P) = 0000$, $end(P) = 1000$, $start(Q) = 0010$, and $end(Q) = 1010$.*

Proof: We first decompose AQ_4 into two sub-augmented cubes AQ_3^0 and AQ_3^1. By Lemma 3, there are two edge-disjoint Hamiltonian paths P^i and Q^i, for $i \in \{0, 1\}$, in AQ_3^i such that $start(P^i) = i000$, $end(P^i) = i100$, $start(Q^i) = i010$, and $end(Q^i) = i110$. By Proposition 1, we have that

$$end(P^0) \in N(end(P^1)) \text{ and } end(Q^0) \in N(end(Q^1)).$$

Let $P = P^0 \Rightarrow P_{\text{rev}}^1$ and let $Q = Q^0 \Rightarrow Q_{\text{rev}}^1$, where P_{rev}^1 and Q_{rev}^1 are the reversed paths of P^1 and Q^1, respectively. Then, P and Q are two edge-disjoint Hamiltonian paths in AQ_4 such that $start(P) = 0000$, $end(P) = 1000$, $start(Q) = 0010$, and $end(Q) = 1010$. Fig. 3 shows the constructions of such two edge-disjoint Hamiltonian paths in AQ_4. Thus, the lemma hods true. ■

By Proposition 1, nodes 0000 and 1000 are adjacent, and nodes 0010 and 1010 are adjacent. The following corollary immediately holds true from Lemma 5.

Corollary 6. *There are two edge-disjoint Hamiltonian cycles in AQ_4.*

Based on Lemma 3, we prove the following lemma by the same arguments in proving Lemma 5.

Lemma 7. *For any integer $n \geqslant 3$, there are two edge-disjoint Hamiltonian paths P and Q in AQ_n such that $start(P) = 0(0)^{n-3}00$, $end(P) = 1(0)^{n-3}00$, $start(Q) = 0(0)^{n-3}10$, and $end(Q) = 1(0)^{n-3}10$.*

Proof: We prove this lemma by induction on n, the dimension of the augmented cube. It follows from Lemma 3 that the lemma holds for $n = 3$. Suppose that the lemma is true for the case $n = k$ ($k \geqslant 3$). Assume that $n = k + 1$. We will prove the lemma holds when

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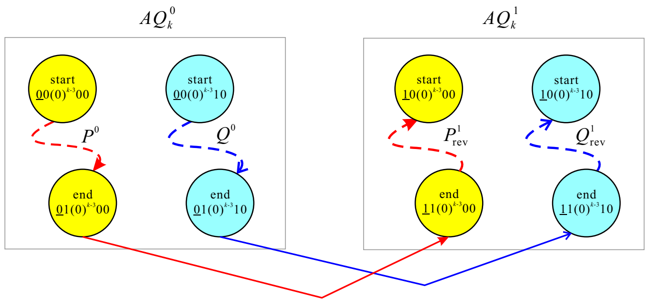

Fig. 4. The constructions of two edge-disjoint Hamiltonian paths in AQ_{k+1}, with $k \geqslant 3$, where dotted arrow lines indicate the paths and solid arrow lines indicate concatenated edges

$n = k + 1$. The proof is the same as that of Lemma 5. We first partition AQ_{k+1} into two sub-augmented cubes AQ_k^0 and AQ_k^1. By the induction hypothesis, there are two edge-disjoint Hamiltonian paths P^i and Q^i, for $i \in \{0, 1\}$, in AQ_k^i such that $start(P^i) = i0(0)^{k-3}00$, $end(P^i) = i1(0)^{k-3}00$, $start(Q^i) = i0(0)^{k-3}10$, and $end(Q^i) = i1(0)^{k-3}10$. By Proposition 1, we get that

$$end(P^0) \in N(end(P^1)) \text{ and } end(Q^0) \in N(end(Q^1)).$$

Let $P = P^0 \Rightarrow P^1_{\text{rev}}$ and let $Q = Q^0 \Rightarrow Q^1_{\text{rev}}$, where P^1_{rev} and Q^1_{rev} are the reversed paths of P^1 and Q^1, respectively. Then, P and Q are two edge-disjoint Hamiltonian paths in AQ_{k+1} such that $start(P) = 0(0)^{k-2}00$, $end(P) = 1(0)^{k-2}00$, $start(Q) = 0(0)^{k-2}10$, and $end(Q) = 1(0)^{k-1}10$. Fig. 4 depicts the constructions of such two edge-disjoint Hamiltonian paths in AQ_{k+1}. Thus, the lemma hods true when $n = k + 1$. By induction, the lemma holds true. ■

By Proposition 1, nodes $start(P) = 0(0)^{n-3}00$ and $end(P) = 1(0)^{n-3}00$ are adjacent, and nodes $start(Q) = 0(0)^{n-3}10$ and $end(Q) = 1(0)^{n-3}10$ are adjacent. It immediately follows from Lemma 7 that the following theorem holds true.

Theorem 8. *For any integer $n \geqslant 3$, there are two edge-disjoint Hamiltonian cycles in AQ_n.*

IV. Two-Equal Path Partition

In this section, we will show that, for any $n \geqslant 2$, the n-dimensional augmented cube AQ_n contains two-equal path partition. That is, for any two distinct pairs of nodes (u_s, u_t) and (v_s, v_t) of AQ_n, there exist two node-disjoint paths P and Q of AQ_n satisfying that (1) $start(P) = u_s$ and $end(P) = u_t$, (2) $start(Q) = v_s$ and $end(Q) = v_t$, (3) $|P| = |Q|$, and (4) $V(P) \cup V(Q) = V(AQ_n)$. We will prove it by induction on n, the dimension of AQ_n. Initially, AQ_2 clearly contains two-equal path partition since it is a complete graph with four nodes.

Lemma 9. *AQ_2 contains two-equal path partition.*

Now, suppose that AQ_k, with $k \geqslant 2$, contains two-equal path partition. We will prove that AQ_{k+1} contains two-equal path partition. First, we decompose AQ_{k+1} into two sub-augmented cubes AQ_k^0 and AQ_k^1. Let (u_s, u_t) and (v_s, v_t) be any two pairs of distinct nodes in AQ_{k+1}. We will construct two node-disjoint paths P and Q of AQ_{k+1} such that P joins u_s and u_t, Q joins v_s and v_t, and $|P| = |Q| = 2^k$. The are four cases:

Case 1: u_s, u_t, v_s, v_t are in the same sub-augmented cube. Without loss of generality, assume that u_s, u_t, v_s, v_t are in AQ_k^0. By inductive hypothesis, there is a path partition $\{P^0, Q^0\}$ of AQ_k^0 such that $|P^0| = |Q^0|$, $start(P^0) = u_s$, $end(P^0) = u_t$, $start(Q^0) = v_s$, and $end(Q^0) = v_t$. Let $P^0 = u_s \rightarrow P'$ and $Q^0 = v_s \rightarrow Q'$. Let $w_P = start(P')$ and let $w_Q = start(Q')$. Let $(\overline{u}_s, \overline{w}_P)$ and $(\overline{v}_s, \overline{w}_Q)$ be two pairs of distinct nodes in AQ_k^1. By inductive hypothesis, there are two node-disjoint paths P^1 and Q^1 of AQ_k^1 such that $|P^1| = |Q^1| = 2^{k-1}$, $start(P^1) = \overline{u}_s$, $end(P^1) = \overline{w}_P$, $start(Q^1) = v_s$, and $end(Q^1) = \overline{w}_Q$. By Proposition 1, $\overline{u}_s \in N(u_s)$, $\overline{w}_P \in N(w_P)$, $\overline{v}_s \in N(v_s)$, and $\overline{w}_Q \in N(w_Q)$. Let $P = u_s \Rightarrow P^1 \Rightarrow P'$ and let $Q = v_s \Rightarrow Q^1 \Rightarrow Q'$. Then, $\{P, Q\}$ is a path partition of AQ_{k+1} such that P joins u_s and u_t, Q joins v_s and v_t, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(a).

Case 2: u_s, u_t, v_s are in the same sub-augmented cube, and v_t is in another sub-augmented cube. Without loss of generality, assume that u_s, u_t, v_s are in AQ_k^0 and that v_t is in AQ_k^1. Let x be a node in AQ_k^0 such that $x \notin \{u_s, u_t, v_s\}$ and $\overline{x} \neq v_t$. By inductive hypothesis, there is a path partition $\{P^0, Q^0\}$ of AQ_k^0 such that $|P^0| = |Q^0|$, $start(P^0) = u_s$, $end(P^0) = u_t$, $start(Q^0) = v_s$, and $end(Q^0) = x$. Let $P^0 = u_s \rightarrow P'$ and let $w = start(P')$. Consider that $\overline{w} \notin \{\overline{x}, v_t\}$. Let $(\overline{u}_s, \overline{w})$ and $(\overline{x}, v_t)$ be two pairs of distinct nodes in AQ_k^1. By inductive hypothesis, there are two node-disjoint paths P^1 and Q^1 of AQ_k^1 such that $|P^1| = |Q^1| = 2^{k-1}$, $start(P^1) = \overline{u}_s$, $end(P^1) = \overline{w}$, $start(Q^1) = x$, and $end(Q^1) = v_t$. By Proposition 1, $\overline{u}_s \in N(u_s)$, $\overline{w} \in N(w)$, and $\overline{x} \in N(x)$. Let $P = u_s \Rightarrow P^1 \Rightarrow P'$ and let $Q = Q^0 \Rightarrow Q^1$. Then, $\{P, Q\}$ is a path partition of AQ_{k+1} such that P joins u_s and u_t, Q joins v_s and v_t, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(b). On the other hand, consider that $\overline{w} \in \{\overline{x}, v_t\}$. Since $|V(AQ_k^0)| = |V(AQ_k^1|) = 2^k \geqslant 4$, we can easily choose w and x such that $\overline{w} \notin \{\overline{x}, v_t\}$. Then, we can build two node-disjoint paths P and Q of AQ_{k+1} by the same technique.

Case 3: u_s, u_t are in the same sub-augmented cube, and v_s, v_t are in another sub-augmented cube. Without loss of generality, assume that u_s, u_t are in AQ_k^0 and that v_s, v_t are in AQ_k^1. By Lemma 2, there are Hamiltonian paths P and Q of AQ_k^0 and AQ_k^1, respectively, such that P joins u_s, u_t and Q joins v_s, v_t. Thus, $\{P, Q\}$ is a path partition of AQ_{k+1} with $|P| = |Q| = 2^k$. Fig. 5(c) depicts the construction of the two paths in this case.

Case 4: u_s, v_s are in the same sub-augmented cube, and u_t, s_t are in another sub-augmented cube. Without loss of generality, assume that u_s, v_s are in AQ_k^0 and that u_t, v_t are in AQ_k^1. Let x, y be two distinct nodes of AQ_k^0 such that $x, y \notin \{u_s, v_s\}$ and $\overline{x}, \overline{y} \notin \{u_t, v_t\}$. Let (u_s, x) and (v_s, y) be two pairs of distinct nodes in AQ_k^0, and let $(\overline{x}, u_t)$ and $(\overline{y}, v_t)$ be two pairs of distinct nodes in AQ_k^1. By inductive hypothesis, there are two node-disjoint paths P^0 and Q^0 of AQ_k^0 such that $|P^0| = |Q^0| = 2^{k-1}$, $start(P^0) = u_s$, $end(P^0) = x$, $start(Q^0) = v_s$, and $end(Q^0) = y$. In addition, there are two node-disjoint paths P^1 and Q^1 of AQ_k^1 such that $|P^1| = |Q^1| = 2^{k-1}$, $start(P^1) = \overline{x}$, $end(P^1) = u_t$, $start(Q^1) = \overline{y}$, and $end(Q^1) = v_t$. By Proposition 1, $\overline{x} \in N(x)$ and $\overline{y} \in N(y)$. Let $P = P^0 \Rightarrow P^1$ and let $Q = Q^0 \Rightarrow Q^1$. Then, $\{P, Q\}$ is a path partition of

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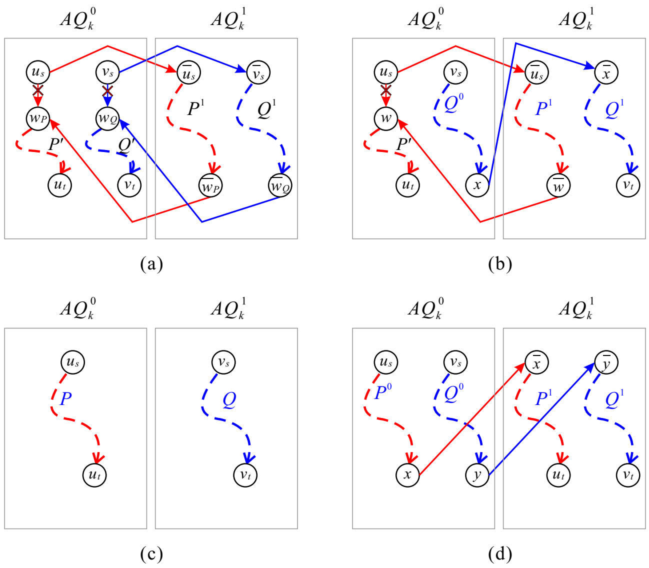

Fig. 5. The constructions of two node-disjoint paths in AQ_{k+1}, with $k \geqslant 2$, for (a) $u_s, u_t, v_s, v_t \in AQ_k^0$, (b) $u_s, u_t, v_s \in AQ_k^0$ and $v_t \in AQ_k^1$, (c) $u_s, u_t \in AQ_k^0$ and $v_s, v_t \in AQ_k^1$, and (d) $u_s, v_s \in AQ_k^0$ and $u_t, v_t \in AQ_k^1$, where dotted arrow lines indicate the paths, solid arrow lines indicate concatenated edges, and the symbol '$\times$' denotes the destruction to an edge in a path

AQ_{k+1} such that P joins u_s and u_t, Q joins v_s and v_t, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(d).

By the above cases, we have that AQ_{k+1} contains two-equal path partition. By induction, AQ_n, with $n \geqslant 2$, contains two-equal path partition. Thus, we conclude the following theorem.

Theorem 10. *For any integer $n \geqslant 2$, AQ_n contains two-equal path partition.*

V. Concluding Remarks

In this paper, we construct two edge-disjoint Hamiltonian cycles (paths) of a n-dimensional augmented cube AQ_n, for any integer $n \geqslant 3$. In addition, we prove that AQ_n, with $n \geqslant 2$, contains two-equal path partition. In the construction of two edge-disjoint Hamiltonian cycles (paths) of AQ_n, some edges are not used. It is interesting to see if there are more edge-disjoint Hamiltonian cycles of AQ_n for $n \geqslant 4$. We would like to post it as an open problem to interested readers.

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