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Optimal Edge-Fault-Tolerant Vertex-Pancyclicity of Augmented Cubes

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Abstract—**The *n*-dimensional augmented cube, denoted as AQ_n, a variation of the hypercube, possesses some properties superior to those of the hypercube. In this paper, we show that every vertex in AQ_n lies on a fault-free cycle of every length from 4 to 2^n, even if there are up to $2n - 3$ link faults. We also show that this result is optimal.**

Index Terms—**hypercubes, augmented hypercubes, fault-tolerant, pancyclic, vertex-pancyclic, interconnection network.**

I. Introduction

A *graph G* is a triple consisting of a *vertex set V*(G), an *edge set E*(G), and a relation that associates with each edge two vertices called its endpoints [26]. We usually use a graph to represent the topology of an interconnection network (network for short). The hypercube is one of the most versatile and efficient interconnection networks discovered to date for parallel computation. The hypercube is ideally suited to both special-purpose and general-purpose tasks, and can efficiently simulate many other same sized networks [18]. We usually use Q_n to denote an *n*-dimensional hypercube. Many variants of the hypercube have been proposed. The augmented cube, recently proposed by Choudum and Sunitha [5], is one of such variations. An *n*-dimensional augmented cube AQ_n can be formed as an extension of Q_n by adding some links. For any positive integer n, AQ_n is a vertex transitive, $(2n - 1)$-regular, and $(2n - 1)$-connected graph with 2^n vertices. AQ_n retains all favorable properties of Q_n since $Q_n \subset AQ_n$. Moreover, AQ_n possesses some embedding properties that Q_n does not. Previous works relating to the augmented hypercube can be found in [2], [5], [14], [15], [16], [19], [20], [23], [25].

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Many efficient algorithms designed based on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in [18]. The *pancyclicity* of a network represents its power of embedding rings of all possible lengths. A graph G is called *m-pancyclic* whenever G contains a cycle of each length l for $m \le l \le |V(G)|$. A 3-pancyclic graph is called pancyclic. The arrangement graph [7], the hypercomplete network [6], the WK-recursive network [10], the alternating group graph [17], and the hyper-de Bruijn networks [11] are all pancyclic. A graph G is *m-vertex-pancyclic* (respectively, *m-edge-pancyclic*) if every vertex (respectively, edge) lies on a cycle of every length from m to $|V(G)|$. In addition, a 3-vertex-pancyclic graph (respectively, 3-edge-pancyclic graph) is called vertex-pancyclic (edge-pancyclic). It is clear that if a graph G is m-edge-pancyclic, then it is m-vertex-pancyclic. The crossed cube [8], the twisted cube [9], and the möbius cube [24] are 4-edge-pancyclic. The recursive circulant graphs with some condition [1], the alternating group graph [3], and the augmented cube [20] are edge-pancyclic.

Since faults may occur to networks, the fault tolerance of networks is an important issue in designing network topologies. Let $F_e \subset E(G)$ (respectively, $F_v \subset V(G)$) denote the faulty edges (respectively, the faulty vertices) in a graph G and let $F = F_e \cup F_v$. Suppose that $G - F$ is P, where P is m-pancyclic, pancyclic, vertex-pancyclic, m-vertex-pancyclic, edge-pancyclic, or m-edge-pancyclic. Then, we call G $|F|$ fault-tolerant P. In addition, G is $|F|$-edge fault-tolerant P (respectively, $|F|$-vertex fault-tolerant P) if $F = F_e$ (respectively, if $F = F_v$). Note that if G is $|F|$ fault-tolerant P, then G is $|F|$-edge fault-tolerant P and $|F|$-vertex fault-tolerant P. Previously, the pancyclicity on various faulty networks was studied in [4], [12], [20], [21], [22], [23]. In [23], AQ_n has been shown to be $(2n - 3)$ fault-tolerant pancyclic, where $n \ge 4$. Up to now, there is no research to concern fault-tolerant vertex-pancyclicity or fault-tolerant edge-pancyclicity. In this paper, we show that $AQ_n - F_e$ is 4-vertex-pancyclic if $|F_e| \le 2n - 3$, where $n \ge 2$. That is, we show that AQ_n is $(2n - 3)$-edge fault-tolerant 4-vertex-pancyclic, where $n \ge 2$. In addition, we also show that this result is optimal.

II. Preliminaries

Let G be a graph and let $u, v \in V(G)$. The degree of vertex v in G, written as $\deg_G(v)$, is the number of edges incident to v in G. In addition, $\delta(G) = \min\{\deg_G(v) | v \in V(G)\}$. A *path* $P[x_0, x_t] = \langle x_0, x_1, \cdots, x_t \rangle$ is a sequence of nodes such that two consecutive nodes are adjacent. t is the distance between nodes x_0 and x_t if $P[x_0, x_t]$ is a shortest path in G. We use $d_G(x_0, x_t)$ to denote the distance between x_0 and x_t in G, and use (u, v) to denote an edge whose endpoints are u and v. Moreover, a path $\langle x_0, x_1, \cdots, x_t \rangle$ may contain other subpaths, denoted as $\langle x_0, x_1, \cdots, x_i, P[x_i, x_j], x_j, \cdots, x_t \rangle$, where $P[x_i, x_j] = \langle x_i, x_{i+1}, \cdots, x_{j-1}, x_j \rangle$. A *cycle* is a path with $x_0 = x_t$ and $t \ge 3$. A cycle (respectively, path) in G is called a *Hamiltonian cycle* (respectively, *Hamiltonian path*) if it contains every vertex of G exactly once.

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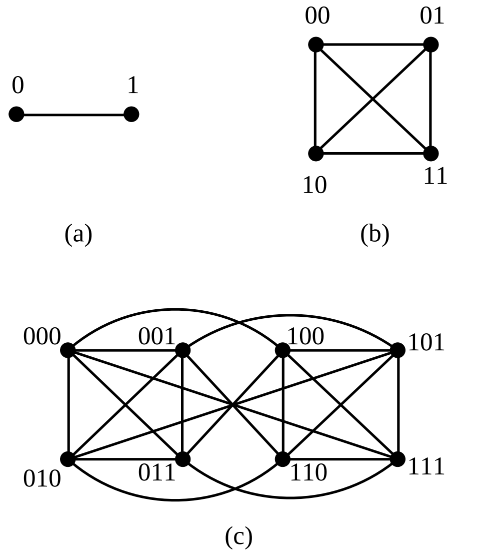

Figure 1. The structures of (a) AQ_1, (b) AQ_2, and (c) AQ_3

An n-dimensional hypercube (n-cube for short) Q_n is an undirected graph with 2^n nodes each labeled with a distinct binary string $b_1b_2\ldots b_n$. Nodes $b = b_1\ldots b_i\ldots b_n$ and $b^i = b_1\ldots \overline{b_i} \ldots b_n$ are joined by an edge along dimension i, where $1 \le i \le n$ and $\overline{b_i}$ represents the one complement of b_i.

An n-dimensional augmented hypercube AQ_n is Q_n augmented by adding more links among its nodes (thus, $V(AQ_n) = V(Q_n)$). For a node $b = b_1b_2\ldots b_n$, it has $n - 1$ more links to connect to nodes $b_a^i = b_1b_2\ldots b_{i-1}\overline{b_i}\,\overline{b_{i+1}}\ldots\overline{b_n}$ in addition to its original n links, for all $i \in \{1, 2, \ldots, n - 1\}$. AQ_n has $2^{n-1}(n - 1)$ more links than Q_n. Note that AQ_1 is isomorphic to Q_1. Let AQ_{n-1}^0 (respectively, AQ_{n-1}^1) be the subgraph of AQ_n induced by $\{0b_2b_3\ldots b_n | b_i = 0 \text{ or } 1 \text{ for } 2 \le i \le n\}$ (respectively, $\{1b_2b_3\ldots b_n \mid b_i = 0 \text{ or } 1 \text{ for } 2 \le i \le n\}$). It is easy to see that AQ_{n-1}^0 (respectively, AQ_{n-1}^1) is isomorphic to AQ_{n-1}. In addition, AQ_n can be recursively constructed by adding 2^n edges between AQ_{n-1}^0 and AQ_{n-1}^1. A vertex $b = 0b_2b_3\ldots b_n \in V(AQ_{n-1}^0)$ is jointed to two vertices in AQ_{n-1}^1, which are $b^h = b^1 = 1b_2b_3\ldots b_n$ and $b^a = b_a^1 = 1\overline{b_2}\,\overline{b_3}\ldots\overline{b_n}$. It is easy to see that an edge joins nodes b^h and b^a. The structures of AQ_1, AQ_2, and AQ_3 are shown in Figure 1. It is known that AQ_n is a vertex-transitive and $(2n - 1)$-regular graph [1].

III. Some important properties

In this section, we introduce some important properties of the augmented cube, which are needed to derive our main result.

Lemma 1. [20] Let $u, v \in V(AQ_n)$, where $n \ge 2$. There exists a path $P[u, v]$ of length l in AQ_n, where $d_{AQ_n}(u,v) \le l \le 2^n - 1$.

Lemma 2. [20] Let $u, v \in V(AQ_n)$, where $n \ge 2$. There exists a Hamiltonian path $P[u, v]$ in $AQ_n - F$, where $F \subset E(AQ_n)$, if $|F| \le 2n - 4$. Moreover, there exists a cycle of length l in $AQ_n - F$ if $|F| \le 2n - 3$, where $3 \le l \le 2^n$.

Lemma 3. Let $(x, y) \in E(AQ_{n-1}^0)$. Then (x^h, y^h) and (x^a, y^a) are also edges in AQ_{n-1}^1.

Proof. It is trivial if $y = x^i$, where $i \in \{2, 3, \ldots, n\}$. Now, consider that $y = x_a^i$ for some $i \in \{2, 3, \ldots, n - 1\}$. Suppose that $x = 0x_2x_3\ldots x_n$. Then, $y = 0x_2x_3\ldots\overline{x_i}\,\overline{x_{i+1}}\ldots\overline{x_n}$, $x^h = 1x_2x_3\ldots x_n$, $y^h = 1x_2x_3\ldots\overline{x_i}\,\overline{x_{i+1}}\ldots\overline{x_n}$, $x^a = 1\overline{x_2}\,\overline{x_3}\ldots\overline{x_n}$, $y^a = 1\overline{x_2}\,\overline{x_3}\ldots\overline{x_{i-1}}x_i\ldots x_n$. Thus, $y^h = (x^h)_a^i$ and $y^a = (x^a)_a^i$. The result follows.

Lemma 4. [20] Let u, v , x, y be any four distinct vertices in AQ_n where $n \ge 2$. There exist a path $P[u, v]$ and a path $P[x, y]$ such that $V(P[u, v]) \cap V(P[x, y]) = \varnothing$ and $V(P[u, v]) \cup V(P[x, y]) = V(AQ_n)$.

Lemma 5. [15] Let $F \subset AQ_n$ and $u, v \in V(AQ_n - F)$ where $n \ge 2$. Then there exists a Hamiltonian path $P[u, v]$ in $AQ_n - F$ if $|F| \le 2n - 4$ when $n \ne 3$ and $|F| = 1$ when $n = 3$.

Lemma 6. [15] Let $F \subset AQ_n$ where $n \ge 2$. Then $AQ_n - F$ is Hamiltonian if $|F| \le 2n - 3$ when $n \ne 3$ and $|F| = 2$ when $n = 3$.

Lemma 7. Let $F \subset E(AQ_n)$ and $(u, v) \in E(AQ_n)$, where $n \ge 2$ and $|F| = 1$. Then there exists a path $P[u, v]$ of length 2 in $AQ_n - F$.

Proof. We proceed by induction on n. It is very easy to see that the lemma holds for AQ_2. As per our induction hypothesis, assume that the lemma holds for AQ_{n-1} for some $n \ge 3$. If $(u, v) \in AQ_{n-1}^0$ or $(u, v) \in AQ_{n-1}^1$, then the lemma holds by the induction hypothesis. Assume that $u \in V(AQ_{n-1}^0)$ and $v \in V(AQ_{n-1}^1)$. First, consider that $v = u^h$. One of the paths $\langle u, u^a, u^h (= v)\rangle$ and $\langle u (= v^h), v^a, v\rangle$ will be in $AQ_n - F$ since these two paths are edge-disjoint. Then, consider that $v = u^a$. One of the paths $\langle u, u^h, u^a (= v)\rangle$ and $\langle u (= v^a), v^h, v\rangle$ will be in $AQ_n - F$ since these two paths are edge-disjoint.

Lemma 8. $AQ_n - F$ is Hamiltonian if $F \subset E(AQ_n)$ with $|F| \le 2n - 2$ and $\delta(AQ_n - F) \ge 2$, where $n \ge 2$.

Proof. The proof is omitted due to the page limitation.

IV. Edge-fault-tolerant 4-vertex-pancyclicity

In this section, by the aid of the lemmas in Section 3, we will show that AQ_n is $(2n - 3)$-edge fault-tolerant 4-vertex-pancyclic. We format the theorem as follows.

Theorem 1. Let $F \subset E(AQ_n)$ denote the faulty edge set of AQ_n, where $n \ge 2$. $AQ_n - F$ is 4-vertex-pancyclic if $|F| \le 2n - 3$.

Proof. We proceed by induction on n. It is very easy to see that the theorem holds for AQ_2. As per our induction hypothesis, assume that the result holds for AQ_{n-1} for some $n \ge 3$. Consider that AQ_n and $F \subset E(AQ_n)$, where $n \ge 3$ and $|F| \le 2n - 3$. For simplicity, we may assume $|F| = 2n - 3$. Since AQ_n is vertex-symmetric, we only need to show that $z = 0^n$ (n consecutive 0's) lies on a cycle of length l in $AQ_n - F$, where $4 \le l \le 2^n$. In addition, by Lemma 2, there exist a Hamiltonian cycle C in $AQ_n - F$ and clearly $z \in V(C)$ and $|V(C)| = 2^n$. As a result, we only need to show that $4 \le l \le 2^n - 1$. Let $F_0 = F \cap E(AQ_{n-1}^0)$, $F_1 = F \cap E(AQ_{n-1}^1)$, $F_c = F \cap (\{(b, b^h)| b \in$

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$V(AQ^0_{n-1})\} \cup \{(b, b^a) | b \in V(AQ^0_{n-1})\})$. Four cases are considered:

Case 1: $|F_0| = 0$. Three cases are further considered:

Case 1.1: $4 \le l \le 2^{n-1}$. Since $|F_0| = 0$, by the induction hypothesis, there exists a cycle C of the length $l \in \{4, \ldots, 2^{n-1}\}$ in $AQ^0_{n-1} - F_0$ such that $z \in V(C)$. Clearly, C is the desired cycle.

Case 1.2: $2^{n-1} + 1 \le l \le 2^{n-1} + 2$. We have two scenarios as follows:

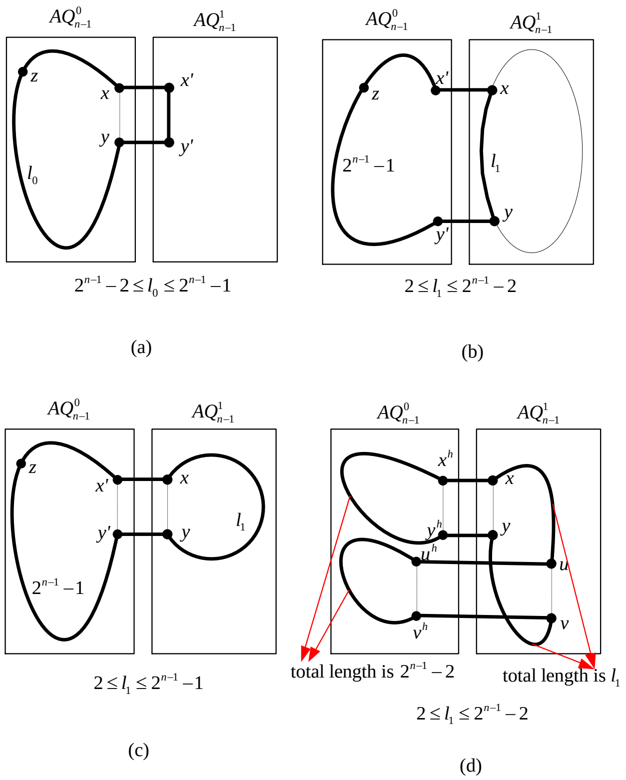

Figure 2. Construction of a cycle of length $l \in \{4, 5, \ldots, 2^n - 1\}$ in $AQ_n - F$ with $F \subset E(AQ_n)$ and $|F| = 2n - 3$

Case 1.2.1: $n = 3$. We have $l = 5$ or 6. First, consider that $l = 5$. Since $|F_c| \le |F| = 3$ and $|V(AQ^1_2)| = 4$, we can find a vertex $x \in AQ^1_2$ such that (x, x^a), $(x, x^h) \notin F_c$. In addition, by **Lemma 5**, there is a Hamiltonian path $P[x^h, x^a]$ (with length 3) in AQ^0_2 (thus, $z \in V(P[x^h, x^a])$). The desired cycle of length $l = 5$ can be constructed by $\langle x, x^h, P[x^h, x^a], x^a, x\rangle$.

Now consider that $l = 6$. Let $(x, y) \in E(AQ^1_2 - F)$ such that $|\{(x, x^a), (x, x^h)\} \cap F_c| \le 1$, $|\{(y, y^a), (y, y^h)\} \cap F_c| \le 1$, $x^a \ne y^h$, and $x^h \ne y^a$ [1]. Let (x, x'), $(y, y') \in E(AQ_3) - F_c$, where $x' \in \{x^a, x^h\}$ and $y' \in \{y^a, y^h\}$. Clearly, $x' \ne y'$. By Lemma **5**, there is a Hamiltonian path $P[y', x']$ (with length 3) in AQ^0_2. The desired cycle of length $l = 6$ can be constructed by $\langle x, y, y', P[y', x'], x', x\rangle$.

Case 1.2.2: $n \ge 4$. By the induction hypothesis, there exists a cycle C_0 in $AQ^0_{n-1} - F_0$ such that $z \in V(C_0)$, where $2^{n-1} - 1 \le |V(C_0)| \le 2^{n-1}$. Moreover, let $(x, y) \in E(C_0)$ such that (x, x^h), (y, y^h), $(x^h, y^h) \notin F$ or (x, x^a), (y, y^a), $(x^a, y^a) \notin F$ [2] (note that by Lemma 3, (x^h, y^h) and (x^a, y^a) are also edges in AQ^1_{n-1}). Additionally, let $P[x, y] = C_0 - \{(x, y)\}$ and $l_0 = |E(P[x, y])|$. (Thus, we have $2^{n-1} - 2 \le l_0 \le 2^{n-1} - 1$.) Let $(x', y') \in \{(x^h, y^h), (x^a, y^a)\}$ such that (x, x'), (y, y'), $(x', y') \in E(AQ_n) - F$. The desired cycle of length $l = l_0 + 3 \in \{2^{n-1} + 1, 2^{n-1} + 2\}$ can be constructed by $\langle x, P[x, y], y, y', x', x\rangle$ (see Figure 2(a)).

Case 1.3: $2^{n-1} + 3 \le l \le 2^n - 1$. We have three scenarios as follows:

Case 1.3.1: $|F_1| \le 2n - 5$. By Lemma 2, there exists a Hamiltonian cycle C_1 in $AQ^1_{n-1} - F_1$. Let $P[x, y]$ be the subpath of C_1 with length l_1 such that (x, x^h), $(y, y^h) \notin F_c$ or (x, x^a), $(y, y^a) \notin F_c$, where $2 \le l_1 \le 2^{n-1} - 2$ [3]. Let $x' \in \{x^h, x^a\}$, $y' \in \{y^h, y^a\}$, and (x, x'), $(y, y') \notin F_c$. Since $|F_0| = 0$, by Lemma 2, there exists a Hamiltonian path $P[y', x']$ in $AQ^0_{n-1} - F_0$ (certainly, $z \in V(P[y', x'])$). The desired cycle of length $l = 2^{n-1} - 1 + 2 + l_1 \in \{2^{n-1} + 3, 2^{n-1} + 4, \ldots, 2^n - 1\}$ can be constructed by $\langle x, P[x, y], y, y', P[y', x'], x', x\rangle$ (see Figure 2(b)).

Case 1.3.2: $|F_1| = 2n - 4$ (thus, $|F_c| = 1$). Let $(\tilde{x}, \tilde{y}) \in F_1$. Then $|F_1 - \{(\tilde{x}, \tilde{y})\}| = 2n - 5$. By Lemma 2, there exists a cycle C_1 in $AQ^1_{n-1} - (F_1 - \{(\tilde{x}, \tilde{y})\})$, where $3 \le |V(C_1)| \le 2^{n-1} - 1$. If $(\tilde{x}, \tilde{y}) \in E(C_1)$, then let $x = \tilde{x}$ and $y = \tilde{y}$. If $(\tilde{x}, \tilde{y}) \notin E(C_1)$, then randomly choose an edge $(x, y) \in E(C_1)$. Since $|F_c| = 1$, we have (x, x^h), $(y, y^h) \notin F_c$ or (x, x^a), $(y, y^a) \notin F_c$. Let $P[x, y] = C - \{(x, y)\}$ and let l_1 be the length of $P[x, y]$. (Thus, we have $2 \le l_1 \le 2^{n-1} - 2$.) Let $(x', y') \in \{(x^h, y^h), (x^a, y^a)\}$ such that (x, x'), $(y, y') \in E(AQ_n) - F_c$. Since $|F_0| = 0$, by Lemma 2, there exists a Hamiltonian path $P[y', x']$ in $AQ^0_{n-1} - F_0$ (certainly, $z \in V(P[y', x'])$). The desired cycle of length $l = (2^{n-1} - 1) + 2 + l_1 \in \{2^{n-1} + 3, 2^{n-1} + 4, \ldots, 2^n - 1\}$ can be constructed by $\langle x, P[x, y], y, y', P[y', x'], x', x\rangle$ (see Figure 2(c)).

Case 1.3.3: $|F_1| = 2n - 3$ (thus, $|F_c| = 0$). First, consider that $\delta(AQ^1_{n-1} - F_1) = 0$. Clearly, exactly one vertex s in $AQ^1_{n-1} - F_1$ has degree 0, $F_1 = \{(s, t) | t \in V(AQ^1_{n-1})\}$, and $F \cap (AQ^1_{n-1} - \{s\}) = \varnothing$. By Lemma 6, there exists a Hamiltonian cycle C_1 in $AQ^1_{n-1} - \{s\}$ (thus, $|V(C_1)| = 2^{n-1} - 1$). Let $P[x, y]$ be the subpath of C_1 with length l_1, where $2 \le l_1 \le 2^{n-1} - 2$. The construction is similar to that of Case 1.3.1.

Now, consider that $\delta(AQ^1_{n-1} - F_1) \ge 1$. Let x, y, u, v be distinct and (x, y), $(u, v) \in F_1$. Then, $|F_1 - \{(x, y), (u, v)\}| = 2n - 5$. By Lemma 2, there exists a cycle C_1 in $AQ^1_{n-1} - (F_1 - \{(x,$

[1] Note that if $(x, y) \notin S = \{(100, 111), (101, 110)\}$, then $x^a \ne y^h$ and $x^h \ne y^a$. The existence of such an edge can be reasoned as follows. If we have $|\{(v, v^a), (v, v^h)\} \cap F_c| \le 1$ for all $v \in V(AQ^1_2)$, then $|E(AQ^1_2 - F - S)| \ge 1$ and the result follows. If there is a vertex t with $|\{(t, t^a), (t, t^h)\} \cap F_c| = 2$, then we have $|\{(v, v^a), (v, v^h)\} \cap F_c| \le 1$ for all $v \in V(AQ^1_2 - \{t\})$ and $|F_1| \le |F| - |F_c| = 1$. Since $|E(AQ^1_2 - \{t\}) - S| = 2$, we have $|E(AQ^1_2 - \{t\}) - S - F_1| \ge 1$ and the result follows.

[2] Since $|E(C_0)| \ge 2^{n-1} - 1$, we have at least $2^{n-1} - 1$ choices. If such an edge does not exist, then $|F| \ge 2^{n-1} - 1 > 2n - 3$ when $n \ge 4$, which is a contradiction.

[3] Since $|E(C_1)| = 2^{n-1}$, we have at least 2^{n-1} choices. If such a path does not exist, then $|F| \ge 2^{n-1} > 2n - 3$ when $n \ge 3$, which is a contradiction.

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y), (u, v)}), where $3 \le V(C_1) \le 2^{n-1} - 1$. If $\{(x, y), (u, v)\} \not\subset E(C_1)$, then the construction is similar to that of Case 1.3.2. (Note that if $|V(C_1)| = 3$, then definitely $\{(x, y), (u, v)\} \not\subset E(C_1)$.) Otherwise, (i.e. $(x, y), (u, v) \in E(C_1)$ and $|V(C_1)| \ge 4$); let $P[x, u]$ and $P[v, y]$ be two subpaths of C_1 (the discussion of the case that $P[x, v]$ and $P[u, y]$ are two subpaths of C_1 is similar). By Lemma 4, there exist a path $P[u^h, v^h]$ and a path $P[y^h, x^h]$ such that $V(P[u^h, v^h]) \cap V(P[y^h, x^h]) = \varnothing$ and $V(P[u^h, v^h]) \cup V(P[y^h, x^h]) = V(AQ^1_{n-1})$. Let the total length of $P[x, u]$ and $P[v, y]$ be l_1. (Thus, we have $2 \le l_1 \le 2^{n-1} - 3$.) In addition, the total length of $P[u^h, v^h]$, and $P[y^h, x^h]$ is $2^{n-1} - 2$. The desired cycle of length $l = (2^{n-1} - 2) + 4 + l_1 \in \{2^{n-1}+ 4, 2^{n-1}+ 5, \ldots, 2^n - 1\}$ can be constructed by $\langle x, P[x, u], u, u^h, P[u^h, v^h], v^h, v, P[v, y], y, y^h, P[y^h, x^h], x^h, x\rangle$ (see Figure 2(d)). Note that the desired cycle of length $2^{n-1}+ 3$ can be constructed by using a method similar to that in Case 1.3.2. (Let $|V(C_1)| = 3$ and we have $\{(x, y), (u, v)\} \not\subset E(C_1)$.)

Case 2: $1 \le |F_0| \le 2n - 5$. Thus, $|F_1| + |F_c| \le 2n - 4$. Three cases are further considered:

Case 2.1: $4 \le l \le 2^{n-1}$. Since $|F_0| \le 2n - 5$, by the induction hypothesis, there exists a cycle C of the required length $l \in \{4, \ldots, 2^{n-1}\}$ in $AQ^0_{n-1} - F_0$ such that $z \in V(C)$. Clearly, C is the desired cycle.

Case 2.2: $2^{n-1} + 1 \le l \le 2^{n-1} + 2$. By the induction hypothesis, there exists a cycle C_0 in $AQ^0_{n-1} - F_0$ such that $z \in V(C_0)$, where $2^{n-1} - 1 \le |E(C_0)| \le 2^{n-1}$. Moreover, let $(x, y) \in E(C_0)$ such that $(x, x^h), (y, y^h), (x^h, y^h) \notin F$ or $(x, x^a), (y, y^a), (x^a, y^a) \notin F$ [4]. The construction is similar to that of Case 1.2.2.

Case 2.3: $2^{n-1}+ 3 \le l \le 2^n - 1$. If $|F_1| \le 2n - 5$, then the construction is similar to that of Case 1.3.1. If $|F_1| = 2n - 4$ (thus, $|F_c| = 0$), then the construction is similar to that of Case 1.3.2.

Case 3: $|F_0| = 2n - 4$. Thus, $|F_1| + |F_c| = 1$. Three cases are further considered:

Case 3.1: $l = 4$. Since AQ^0_{n-1} is $(2n - 3)$-regular, there exists an edge $(z, v) \in E(AQ^0_{n-1}) - F_0$. Let cycle $C_1 = \langle z, v, v^h, z^h, z\rangle$, cycle $C_2 = \langle z, v, v^a, z^a, z\rangle$, and cycle $C_3 = \langle z, z^h, u, z^a, z\rangle$, where $u = 1^2 0^{n-2}$. Since $E(AQ^0_{n-1}) \cap (E(C_1) \cup E(C_2) \cup E(C_3)) = \{(z, v)\}$ and $(z, v) \notin F_0$, we have $|F \cap (E(C_1) \cup E(C_2) \cup E(C_3))| = |(F_1 \cup F_c) \cap (E(C_1) \cup E(C_2) \cup E(C_3))| \le |F_1| + |F_c| = 1$. Moreover, we have $E(C_1) \cap E(C_2) \cap E(C_3) = \varnothing$[5]. As a result, we have $E(C_1) \cap F = \varnothing$, $E(C_2) \cap F = \varnothing$, or $E(C_3) \cap F = \varnothing$. Let $j \in \{1, 2, 3\}$ and $E(C_j) \cap F = \varnothing$. Clearly, C_j is the desired cycle of length 4 in $AQ_n - F$.

Case 3.2: $l = 5$. Since AQ^0_{n-1} is $(2n - 3)$-regular, there exists an edge $(z, v) \in E(AQ^0_{n-1} - F_0)$. First, consider that $v = z^i_a = 0^{i-1}1^{n-i+1}$ for some $i \in \{2, \ldots, n - 1\}$. Let cycle $C_1 = \langle 0^n, 0^{i-1}1^{n-i+1}, 10^{i-2}1^{n-i+1}, 10^{i-1}1^{n-i}, 10^{n-1}, 0^n\rangle$ and cycle $C_2 = \langle 0^n, 0^{i-1}1^{n-i+1}, 1^{i-1}0^{n-i+1}, 1^i0^{n-i}, 1^n, 0^n \rangle$. Note that $E(C_1) \cap E(C_2) = \{(z, v)\}$, $E(C_1) \cap F_0 = \varnothing$, and $E(C_2) \cap F_0 = \varnothing$[6]. Moreover, since $|F_1| + |F_c| = 1$, we have $E(C_1) \cap F = \varnothing$ or $E(C_2) \cap F = \varnothing$. Let $j \in \{1, 2\}$ and $E(C_j) \cap F = \varnothing$. Clearly, C_j is the desired cycle of length 5 in $AQ_n - F$.

Now consider that $v = z^i = 0^{i-1}10^{n-i}$ for some $i \in \{2, 3, \ldots, n\}$. If $n \ge 4$, let cycle $C_1 = \langle 0^n, 0^{i-1}10^{n-i}, 10^{i-2}10^{n-i}, 10^{n-1}, 1^n, 0^n\rangle$, cycle $C_2 = \langle 0^n, 0^{i-1}10^{n-i}, 1^{i-1}01^{n-i}, 1^n, 10^{n-1}, 0^n \rangle$, and cycle $C_3 = \langle 0^n, 10^{n-1}, 1^20^{n-2}, 1^30^{n-3}, 1^n, 0^n\rangle$. Since $E(AQ^0_{n-1}) \cap (E(C_1) \cup E(C_2) \cup E(C_3)) = \{(z, v)\}$ and $(z, v) \notin F_0$, we have $|F \cap (E(C_1) \cup E(C_2) \cup E(C_3))| = |(F_1 \cup F_c) \cap (E(C_1) \cup E(C_2) \cup E(C_3))| \le |F_1| + |F_c| = 1$. Moreover, $E(C_1) \cap E(C_2) \cap E(C_3) = \varnothing$. As a result, we have $E(C_1) \cap F = \varnothing$, $E(C_2) \cap F = \varnothing$, or $E(C_3) \cap F = \varnothing$. Let $j \in \{1, 2, 3\}$ and $E(C_j) \cap F = \varnothing$. Clearly, C_j is the desired cycle of length 5 in $AQ_n - F$. If $n = 3$, then v = 001 or 010. When v is 001, the desired cycles of length 5 are as listed below (the construction of the case that v = 010 is similar):

The edge in $F - F_0$	**Cycle of length 5**
00), (001, 101), (010, 101), (010, 110), (011, 100), (011, 111), (100, 101), (100, 110), (100, 111), or (110, 111).	01, 110, 101, 111, 000⟩
11), (001, 110), or (101, 110)	01, 101, 111, 100, 000⟩
11)	00, 101, 110, 111, 000⟩

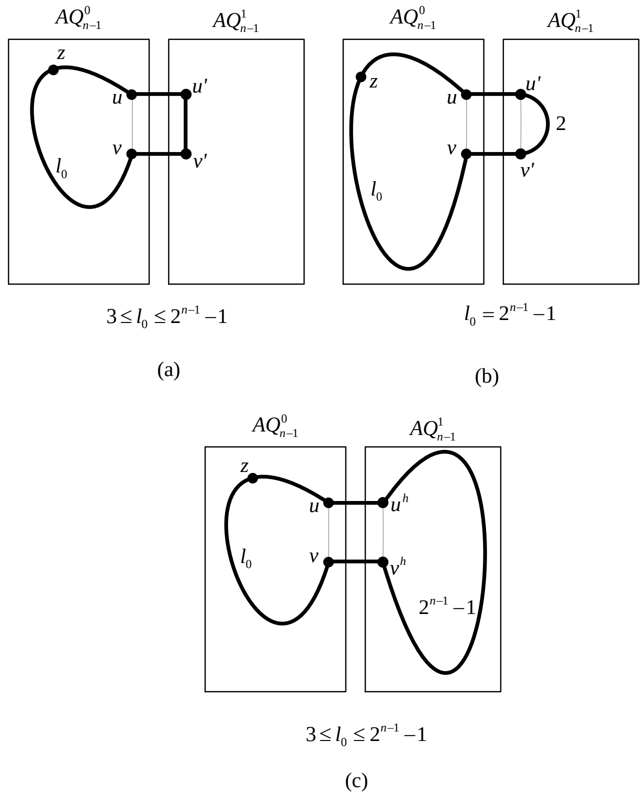

Figure 3. Construction of a cycle of length $l \in \{4, 5, \ldots, 2^n - 1\}$ in $AQ_n - F$ with $F \subset E(AQ_n)$ and $|F| = 2n - 3$

Case 3.3: $6 \le l \le 2^n - 1$. Since $|F_0| = 2n - 4 \ge 2$ and $|F_c|+|F_1| = 1$, we can find an edge $(x, y) \in F_0$ such that $(x, x^h), (y, y^h), (x^h, y^h) \notin F$ or $(x, x^a), (y, y^a), (x^a, y^a) \notin F$. Let $(x', y') \in \{(x^h, y^h), (x^a, y^a)\}$ such that $(x, x'), (y, y'), (x', y') \in E(AQ_n) - F$. Since $|F_0 - (x, y)| = 2n - 5$, by the induction hypothesis, there exists a cycle C_0 in $AQ^0_{n-1} - (F_0 - (x, y))$ such that $z \in V(C_0)$, where $4 \le |E(C_0)| \le 2^{n-1}$. If $(x, y) \in E(C_0)$ then let $u = x$, $v = y$, $u' = x'$,

[4] Since $|E(C_0)| \ge 2^{n-1} - 1$, we have at least $2^{n-1} - 1$ choices. If such an edge does not exist, then $|F_1| + |F_c| \ge 2^{n-1} - 1 > 2n - 4$, which is a contradiction.

[5] Note that if $v = 01^{n-1}$, then $z^a = v^h = 1^n$ and $z^h = v^a = 10^{n-1}$. As a result, $E(C_1) \cap E(C_2) = \{(z, v), (z^a, z^h)\}$.

[6] When $i \neq 2$, it is easy to see that $V(C_1) \cap V(C_2) = \{z, v\}$ and the result follows. When $i = 2$, $C_1 = \langle 0^n, 01^{n-1}, 1^n, 101^{n-2}, 10^{n-1}, 0^n \rangle$ and cycle $C_2 = \langle 0^n, 01^{n-1}, 10^{n-1}, 1^20^{n-2}, 1^n, 0^n \rangle$, which also can verify that the result is true.

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and $v' = y'$. Otherwise, let $(u, v) \in E(C_0)$ such that (u, u^h), (v, v^h), $(u^h, v^h) \notin F$, and let $u' = u^h$ and $v' = v^h$. Let $P[u, v] = C_0 - \{(u, v)\}$ and $l_0 = |E(P[u, v])|$. (Thus, we have $3 \le l_0 \le 2^{n-1} - 1$.) The desired cycle of length $l = l_0 + 3 \in \{6, 7, \dots, 2^{n-1} + 2\}$ can be constructed by $\langle u, P[u, v], v, v', u', u\rangle$ (see Figure 3(a)).

Since $|F_1| \le 1$, by Lemma 7, there exists a path $P[v', u']$ of length 2 in $AQ^1_{n-1} - F$. Let $l_0 = 2^{n-1} - 1$; the desired cycle of length $l = l_0 + 4 = 2^{n-1} + 3$ can be constructed by $\langle u, P[u, v], v, v', P[v', u'], u', u\rangle$ (see Figure 3(b)).

Also, since $|F_1| \le 1$, by Lemma 2, there exists a Hamiltonian path $P[v', u']$ in $AQ^1_{n-1} - F$. The desired cycle of length $l = l_0 + 2 + 2^{n-1} - 1 \in \{2^{n-1} + 4, 2^{n-1} + 5, \dots, 2^n - 1\}$ can be constructed by $\langle u, P[u, v], v, v', P[v', u'], u', u\rangle$ (see Figure 3(c)).

Case 4: $|F_0| = 2n - 3$. Thus, $|F_1| = |F_c| = 0$. Two cases are further considered:

Case 4.1: $4 \le l \le 2^{n-1} + 1$. Remember that z^h is a neighbor of z^a; that is, $d_{AQ^1_{n-1}}(z^h, z^a) = 1$. By Lemma 1, there exists a path $P[z^h, z^a]$ of length l_1 in AQ^1_{n-1}, where $1 \le l_1 \le 2^{n-1} - 1$. Therefore, the desired cycle of length $l = l_1 + 2 \in \{3, 4, \dots, 2^{n-1} + 1\}$ can be constructed by $\langle z, z^h, P[z^h, z^a], z^a, z\rangle$ (see Figure 4(a)).

Case 4.2: $2^{n-1} + 2 \le l \le 2^n - 1$. We have the following scenario:

Case 4.2.1: $\delta(AQ^0_{n-1} - F_0) = 0$. Clearly, exactly one vertex v in $AQ^0_{n-1} - F_0$ has degree 0, for otherwise, $|F_0| \ge 2(2n - 3) - 1 = 4n - 7 > 2n - 3$. Moreover, $F_0 = \{(v, u) | u \in V(AQ^0_{n-1})\}$ and $F \cap E(AQ^0_{n-1} - \{v\}) = \varnothing$. By Lemma 6, there exists a Hamiltonian cycle C in $AQ^0_{n-1} - \{v\}$. Thus, we have $|E(C)| = 2^{n-1} - 1$. Let $(x, y) \in E(C)$, and $P[x, y] = C - \{(x, y)\}$. Then, the length of $P[x, y]$ is $2^{n-1} - 2$. By Lemma 1, there exists a path $P[y^h, x^h]$ of length l_1 in AQ^1_{n-1}, where $2 \le l_1 \le 2^{n-1} - 1$. The desired cycle of length $l = l_1 + 2^{n-1} - 2 + 2 \in \{2^{n-1} + 2, 2^{n-1} + 3, \dots, 2^n - 1\}$ can be constructed by $\langle x, P[x, y], y, y^h, P[y^h, x^h], x^h, x\rangle$ (see Figure 4(b))

Case 4.2.2: $\delta(AQ^0_{n-1} - F_0) = 1$. First, consider that $n = 3$. It is easy to construct a path $P[x, y]$ of length 2 in $AQ^0_2 - F_0$ by hand such that $z \in V(P[x, y])$ for some vertices x, y in AQ^0_{n-1}. By

Lemma 1, there exists a path $P[y^h, x^h]$ of length l_1 in AQ^1_2, where $1 \le l_1 \le 3$. The desired cycle of length $l = l_1 + 2 + 2 \in \{5, 6, 7\}$ can be constructed by $\langle x, P[x, y], y, y^h, P[y^h, x^h], x^h, x\rangle$.

Now consider that $n \ge 4$. There is at most one vertex x with degree 1 in $AQ^0_{n-1} - F_0$, for otherwise, $|F_0| \ge 2(2n - 4) - 1 = 4n - 9 > 2n - 3$, which is a contradiction. Let $(x, y) \in F_0$. Then, we have $\delta(AQ^0_{n-1} - (F_0 - \{(x, y)\})) = 2$ and $|F_0 - \{(x, y)\}| = 2n - 4$. By Lemma 8, there exists a Hamiltonian cycle C in $AQ^0_{n-1} - (F_0 - \{(x, y)\})$. Obviously, $(x, y) \in E(C)$. Let $P[x, y] = C - \{(x, y)\}$. Then, the length of $P[x, y]$ is $2^{n-1} - 1$. By Lemma 1, there exists a path $P[y^h, x^h]$ of length l_1 in AQ^1_{n-1}, where $1 \le l_1 \le 2^{n-1} - 2$. The desired cycle of length $l = l_1 + 2^{n-1} - 1 + 2 \in \{2^{n-1} + 2, 2^{n-1} + 3, \dots, 2^n - 1\}$ can be constructed by $\langle x, P[x, y], y, y^h, P[y^h, x^h], x^h, x\rangle$ (see Figure 4(c)).

Case 4.2.3: $\delta(AQ^0_{n-1} - F_0) \ge 2$. We have $n \ge 4$. Let $(u, v) \in F_0$. Clearly, $\delta(AQ^0_{n-1} - (F_0 - \{(u, v)\})) \ge 2$ and $|F_0 - \{(u, v)\}| = 2n - 4$. By Lemma 8, there exists a Hamiltonian cycle C in $AQ^0_{n-1} - (F_0 - \{(u, v)\})$. If $(u, v) \in E(C)$, then let $x = u$ and $y = v$. If $(u, v) \notin E(C)$, then randomly choose an edge $(x, y) \in E(C)$. The rest of the construction is the same as that of Case 4.2.2. □

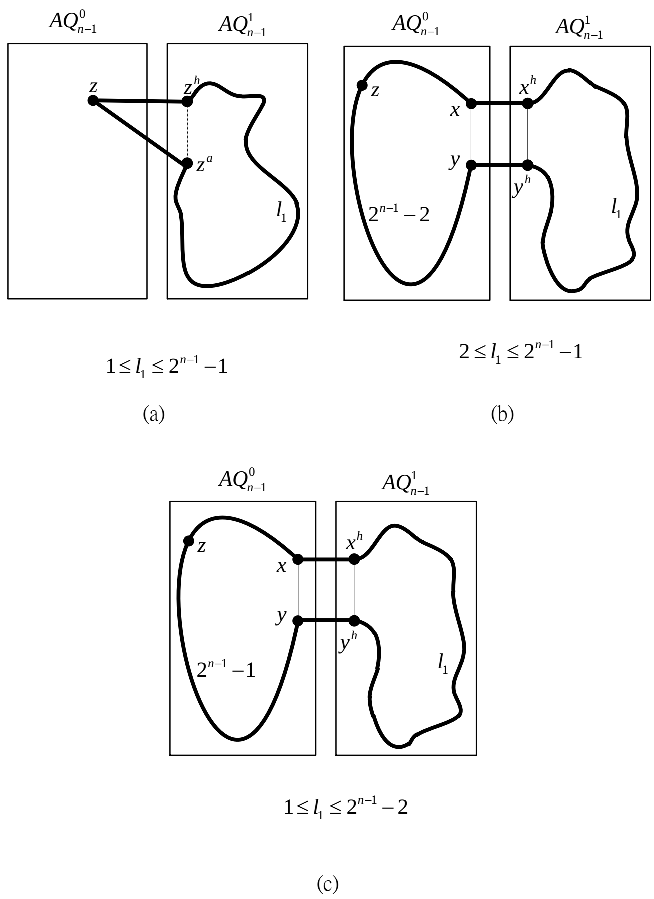

Figure 4. Construction of a cycle of length $l \in \{4, 5, \dots, 2^n - 1\}$ in $AQ_n - F$ with $F \subset E(AQ_n)$ and $|F| = 2n - 3$

Note that there are distributions of $2n - 2$ edge faults over a AQ_n such that no fault-free Hamiltonian cycle can be found in the faulty AQ_n, since AQ_n is $(2n - 1)$-regular. Moreover, there are distributions of n edge faults over an AQ_n such that no fault-free cycle of length three can be found. Consider that a vertex $u = 0^n$ (n consecutive 0's). Suppose that $F = \{(u, u^i_a) | i \in \{1, 2, \dots, n - 1\}\} \cup \{(u, u^n)\}$. Thus, $(u, u^i) \notin F$, for all $i \in \{1, 2, \dots, n - 1\}$. Since u^i is not a neighbor of u^j, for all $i, j \in \{1, 2, \dots, n - 1\}$ and $i \ne j$, u cannot lie on a cycle of length 3. Therefore, our result is optimal.

V. Discussion and Conclusion

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are

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suitable for developing simple algorithms with low communication costs. The *pancyclicity* of a network represents its power of embedding rings of all possible lengths. In this paper, using inductive proofs, we showed that AQ_n is $(2n-3)$-edge fault-tolerant 4-vertex-pancyclic. In other words, every vertex of an AQ_n with at most $2n-3$ faulty edges lies on a fault-free cycle of every length from 4 to 2^n. In addition, we also showed that our result is optimal.

AQ_n is $(2n-3)$ fault-tolerant pancyclic (thus, $(2n-3)$-edge fault-tolerant pancyclic), where $n \geq 4$ [23]. We have shown that AQ_n is $(2n-3)$-edge fault-tolerant 4-vertex-pancyclic, but not n-edge fault-tolerant vertex-pancyclic. Therefore, AQ_n is not n fault-tolerant vertex-pancyclic and not n-edge fault-tolerant edge-pancyclic. A topic for further research is to explore the vertex-pancyclicity and/or edge-pancyclicity of augmented cubes in the presence of hybrid faults.

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