N-Player Cutthroat Played on Stars is PSPACE-Complete

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***Abstract*—N-player Cutthroat is an n-player version of Cutthroat, a two-player combinatorial game played on graphs. Because of queer games, i.e., games where no player has a winning strategy, cooperation is a key-factor in n-player games and, as a consequence, n-player Cutthroat played on stars is PSPACE-complete.**

***Index Terms*—combinatorial games, cutthroat, n-player games, PSPACE-complete.**

I. Introduction

The game of Cutthroat [1] is a combinatorial game played on a graph where each vertex has been colored black, white or green. Two players, Left and Right, move alternately. Left removes a black or a green vertex and Right removes a white or a green vertex. After a move, or at the beginning of play, any monochromatic, connected component is also removed from the graph.

For example, if the graph were a single edge with one white and one black vertex then Left moving first would win since he would leave just the white vertex, a monochromatic component which would then be removed. Similarly, Right would win moving first. A winning strategy has been found for White-Black Cutthroat, i.e., Cutthroat where the vertices will only be colored white or black, played on stars [2].

N-player Cutthroat is the n-player version of Cutthroat played on graph where each vertex v has been labeled by $l(v) \subseteq \{1, 2, \ldots, n\}, l(v) \neq \emptyset$. The first player removes a vertex v such that $1 \in l(v)$. The second player removes a vertex v such that $2 \in l(v)$. The other players move in similar way. After a move, or at the beginning of play, any connected component, where all the vertices have the same label, is also removed from the graph. Players take turns making legal moves in cyclic fashion (1-st, 2-nd, …, n-th, 1-st, 2-nd, …). When one of the players is unable to move, that player leaves the game and the remaining $n-1$ players continue playing in the same mutual order as before. The remaining player is the winner.

We briefly recall the definition of *queer* game introduced by Propp [3]:

Definition 1: A position in a three-player combinatorial game is called queer if no player can force a win.

Such a definition is easily generalizable to n players:

Definition 2: A position in an n-player combinatorial game is called queer if no player can force a win.

In the game of n-player Cutthroat, it is not always possible to determine the winner because of queer games, as shown in Fig. 1. In this case, no player has a winning strategy because if the first player removes the vertex on the left, then the third player wins but if the first player removes the vertex on the right, then the second player wins.

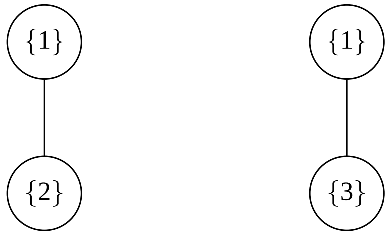

Fig. 1. A simple example of queer game in three-player Cutthroat.

In two player games [4], [5], [6] players are in conflict to each other and coalitions are not allowed but in n-player games [7], [8], [9], [10], [11], when the game is queer, only cooperation between players can guarantee a winning strategy, i.e., one player of the coalition is always able to make the last move. As a consequence, to establish whether or not a coalition has a winning strategy is a crucial point.

In previous works, we analyzed the complexity of n-player Hackenbush [12], n-player Toppling Dominoes [13], n-player Cherries [14], and some map-coloring multi-player games [15]. In this paper we show that, in Cutthroat, cooperation between a group of players can be much more difficult than competition and, as a consequence, n-player Cutthroat played on stars is $\mathcal{PSPACE}$-complete.

II. The Complexity of n-player Cutthroat

In this section we show that the $\mathcal{PSPACE}$-complete problem of *Quantified Boolean Formulas* [16], QBF for short, can be reduced by a polynomial time reduction to n-player Cutthroat.

Let $\varphi \equiv \exists x_1 \forall x_2 \exists x_3 \ldots Q x_n \psi$ be an instance of QBF, where Q is $\exists$ for n odd and $\forall$ otherwise, and ψ is a quantifier-free Boolean formula in conjunctive normal form. We recall that QBF asks if there exists an assignment to the variables $x_1, x_3, \ldots, x_{2\lceil n/2 \rceil - 1}$ such that the formula evaluates to true.

Definition 3: A star S_m is a tree with one internal node and m leaves.

If n is the number of variables and k is the number of clauses in ψ, then the instance of n-player Cutthroat will have $n + k + 2$ players and $2n + 5$ stars, organized as follows:

- For each variable x_i, with $1 \leq i \leq n$, we add two new stars. In the first one, the internal node is labeled $\{i\}$ and there is a leaf for each clause containing x_i; in the second one, the internal node is labeled $\{i\}$ and there is a leaf for each clause containing $\overline{x_i}$. These leaves are labeled $\{j\}$, with $n + 1 \leq j \leq n + k$.
- The last 5 stars are all S_1 star where the internal node is labeled $\{n+k+1\}$ and the leaf is labeled $\{n+k+2\}$

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except one star where the leaf is labeled $\{n+1, n+2, \ldots, n+k\}$.

Let us suppose that:

- The first coalition is formed by $\lfloor n/2 \rfloor + 1$ players corresponding to the dominoes labeled $2, 4, \ldots, 2\lfloor n/2 \rfloor$, and $n+k+2$,
- The second coalition is formed by the remaining players.

An example is shown in Fig. 2 where

$$\varphi \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4 (C_5 \wedge C_6 \wedge C_7)$$

and

$$\begin{aligned} C_5 &\equiv (\overline{x_1} \vee x_2 \vee \overline{x_3}) \\ C_6 &\equiv (x_1 \vee \overline{x_2} \vee \overline{x_4}) \\ C_7 &\equiv (x_1 \vee \overline{x_3} \vee x_4) \end{aligned}$$

The problem to determine the winning coalition is strictly connected to the problem of QBF, as shown in the following theorem.

Theorem 1: Let G be a general instance of n-player Cutthroat played on stars. Then, to establish whether or not a given coalition has a winning strategy is a $\mathcal{PSPACE}$-complete problem.

Proof: We show that it is possible to reduce every instance of QBF to a set of stars G representing an instance of n-player Cutthroat. Previously we have described how to construct the instance of n-player Cutthroat, therefore we just have to prove that QBF is satisfiable if and only if the second coalition has a winning strategy.

If QBF is satisfiable, then there exists an assignment of x_i such that ψ is true with $i \in \{1, 3, \ldots, 2\lceil n/2 \rceil - 1\}$. If x_i is true, then the i-th player removes the vertex labeled $\{i\}$ and, consequently, all the vertices corresponding to the clauses containing $\overline{x_i}$. If x_i is false, then the i-th player removes the vertex labeled $\{i\}$ and, consequently, all the vertices corresponding to the clauses containing x_i.

Every clause contains at least a true literal, therefore the i-th player with $i \in \{n+1, n+2, \ldots, n+k\}$ can always remove one vertex from the star corresponding to that literal. The $n+k+1$-th player removes the star with the leaf labeled $\{n+1, n+2, \ldots, n+k\}$. In this way, at the end of the game, the $n+k+1$-th player will be able to make the last move and therefore, the second coalition has a winning strategy.

Conversely, let us suppose that the second coalition has a winning strategy. We observe that the $n+k+1$-th player, during his/her first move, must be able to remove the star with the leaf labeled $\{n+1, n+2, \ldots, n+k\}$ in order to assure a winning strategy for the second coalition. As a consequence, the i-th player with $i \in \{n+1, n+2, \ldots, n+k\}$ must always be able to remove a vertex from the other stars, i.e., every clause has at least one true literal and QBF is satisfiable.

Therefore, to establish whether or not a coalition has a winning strategy in n-player Cutthroat played on stars is $\mathcal{PSPACE}$-hard.

To show that the problem is in $\mathcal{PSPACE}$ we present a polynomial-space recursive algorithm to determine which coalition has a winning strategy.

Let us introduce some useful notations:

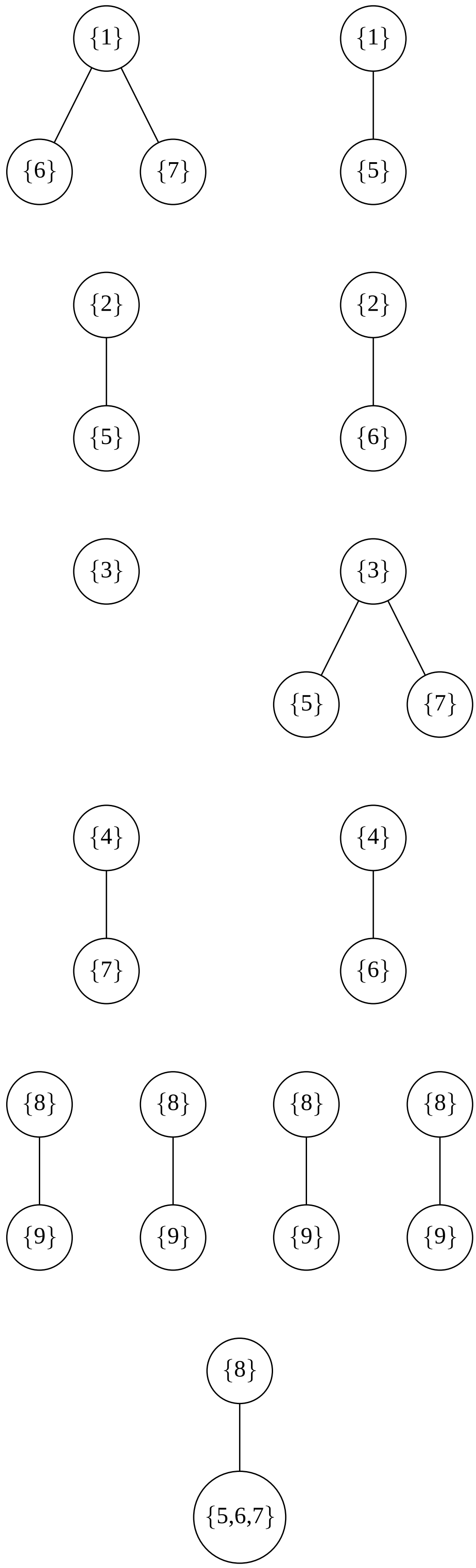

Fig. 2. The problem of QBF is reducible to the game of n-player Cutthroat played on stars.

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- $G = (V, E)$ is the graph representing an instance of n-player Cutthroat;
- p_i is the i-th player;
- C_0 is the set of current players belonging to the first coalition;
- C_1 is the set of current players belonging to the second coalition;
- $\texttt{coalition}(p_i)$ returns 0 if $p_i \in C_0$ and 1 if $p_i \in C_1$;
- $\texttt{label}(v)$ returns the label of the vertex v;
- $\texttt{next}(p_i)$ returns the player which has to play after p_i;
- $\texttt{remove}(G,v)$ returns the graph obtained after that the vertex v and any connected component, where all the vertices have the same label, have been removed from G.

Require: A graph $G = (V, E)$, the two initial coalitions C_0 and C_1 and the player p_i that has to move
Ensure: 0 if the first coalition wins and 1 if the second coalition wins

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Algorithm Check(G, C_0, C_1, p_i)
j ← coalition(p_i)
if ∄v ∈ V : label(v) = i then
  C_j ← C_j \ {p_i}
  if C_j = ∅ then
    return 1 − j
  else
    return Check(G, C_0, C_1, next(p_i))
  end if
else
  for all v ∈ V : label(v) = i do
    G' ← remove(G, v)
    if Check(G', C_0, C_1, next(p_i)) = j then
      return j
    end if
  end for
  return 1 − j
end if
```

Algorithm Check performs an exhaustive search until a winning strategy is found and its correctness can be easily proved by induction on the depth of the game tree.

Algorithm Check is clearly in $\mathcal{PSPACE}$ because the number of nested recursive calls is at most $|V|$ and therefore the total space complexity is $O(|V|^2)$. ■

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