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Monotonicity Preserving Interpolation using Rational Spline

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***Abstract*—The main spotlight of this work is to visualize the monotone data to envision of very smooth and pleasant monotonicity preserving curves by using piecewise rational cubic function. The piecewise rational cubic function has three shape parameters in each interval. We derive a simpler constrains on shape parameters which assurance to preservation of the monotonicity curve of monotonic data. The free shape parameters will provide the extra freedom to the user to interactively generate visually more pleasant curves as desired. The smoothness of the interpolation is C^1 continuity.**

***Index Terms*—Rational cubic function, monotone data, shape parameters, free shape parameters, Interpolation.**

I. Introduction

SMoothness and pleasant shape preserving curves is very important in Computer aided Geometric design (CAGD), Computer Graphics (CG), and Data Visualization (DV).In these fields, it is often needed to produce a monotonicity preserving interpolating curve corresponding to the given monotone data. The one very important feature from the interpolation methods is that make good judgment to study is its positivity and monotonicity which are important aspects of the shape. Many physical situations where entities are taken only monotone. In [14], monotonicity is applied in the specification of Digital to Analog Converters (DACs), Analog to Digital Converters (ADCs) and sensors. These devices are used to control system applications where non monotonicity is not acceptable. In cancer patients, Erythrocyte sedimentation rates (ESR) and Uric acid level in the patients who are suffering from gout are providing monotone data see[14]. Approximation of couples and quasi couples in statistics see [16], rate of dissemination of drug in blood see [16], empirical option of pricing models in finance see [16] are good examples of monotone data. Also in Engineering, the study of tensile strength of the material which give the monotone data because the tensile strength of a material can be defined as the maximum force that a material can withstand before breaking see for detail[12]. The forced applied usually is called stress and is studied alongside the stretch of the material referred as strain. So the data from these two entities is always monotone see for detail [12].

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The problem of monotonicity preserving has been considered by number of authors [1]-[19] and references therein. Hussain & Sarfraz [1] has developed monotonicity preserving interpolation by rational cubic function with four shape parameters which are arranged as two are free and the other two are automated shape parameters. The authors derived data dependent constraints on automated shape parameters which ensure the monotonicity and also provide very pleasant curves but one automated shape parameter is dependent on the other and hence the scheme is economically very expensive.

Fritch & Carlson [2] and Butland [3] have developed a piecewise cubic polynomials monotonicity preserving interpolation. With positive derivatives and increasing data in the cubic Hermite Polynomials they still violate the necessary monotonicity condition. These authors are made a modification in input derivatives in their interpolating schemes if they contravene the necessary monotonicity conditions. Schumaker [5] developed a very economical interpolation by piecewise quadratic polynomial but the author also for interpolation they put in an extra knot in each interval. The constructed interpolant preserve the monotonicity but have degree 2.Butt [18] & Higham [4] also used a piecewise cubic polynomials in which the derivatives are not modified but the authors inserted extra knots where the curve lost the monotonicity. Gregory & Delbourgo [6] has introduced a solution of monotonicity without inserting an extra knot and preserves monotonicity by rational quadratic function with quadratic denominator. Fahr and Kallay [7] used a monotone rational B-spline of degree one to preserve the shape of monotone data.

For the pithiness of monotonicity preserving interpolation the reader is referred to: Delbourgo & Gregory [8] has developed piecewise rational cubic interpolation for the preserving monotonicity with no freedom to user to refine the curve. Gregory & Sarfraz [9] introduced a rational cubic spline with one tension parameter in each subinterval. Sarfraz [10] & [11], Hussain & Hussain [12] and Sarfraz & Hussain [13], Sarfraz et al [17] have introduced rational cubic function with two shape parameters that provide the desired monotone curves but there is no freedom for the user to modify the curves hence is may be unsuitable for interactive design. Hussain et al [14] also developed a rational function for the preserving monotonicity with single shape parameter which is very economical and generates pleasant curve but have limited flexibility in particular curves modification. Sarfraz [15] introduced an C^2 interpolant using a general piecewise rational cubic (GPRC) with two shape parameters to preserves the monotonicity but there is also no freedom to the user as well as the scheme is expansive than the proposed scheme.

This work is particularly concerned with the monotone data using the cubic rational function with three shape parameters, they are arranged in such a manner that two shape parameters are provide extra independence to the user to modify the curve as desired and the other one is constrained to satisfy the monotonic properties of curves through monotone data. In section (I-C), we show how the single parameter produces monotonicity preserving curve through monotone data which are given in Tables I,II and III, also providing an interactive and pleasant curve as compared to [14] while the other two shape parameters are used to refine the curve as desired by the user. In the proposed scheme there is no insertion of extra knot and is also more economical as compared to the existing schemes.

The rest of this paper is organized as follows. we construct a rational cubic function in section (I-A) and the determination of the derivatives is given in section (I-B). The construction of the monotonicity preserving interpolation for monotone data in section (I-C) and some numerical examples are discussed in section (I-D). Finally the conclusion of the paper is discussed in section (II).

A. Rational cubic function

In this section, a C^1 rational cubic function [19] with three shape parameters with cubic denominator has been developed. Let $\{(x_i, f_i), i = 0, 1, 2, ...n\}$ be a given set of data points defined over the interval [a, b], where $a = x_0 < x_1 < .... < x_n = b$. The C^1 piecewise rational cubic function with three parameters are defined over each subinterval $I_i = [x_i, x_{i+1}]$, i =0, 1, 2,. . . ,n -1 as:

$$S(x) = S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \tag{1}$$

with

$$\begin{aligned} p_i(\theta) &= u_i f_i (1-\theta)^3 + (w_i f_i + u_i h_i d_i)\, \theta (1-\theta)^2 \\ &\quad + (w_i f_{i+1} - v_i h_i d_{i+1})\, \theta^2 (1-\theta) + v_i f_{i+1} \theta^3 \\ q_i(\theta) &= u_i (1-\theta)^3 + w_i \theta (1-\theta) + v_i \theta^3 \end{aligned}$$

where

$$h_i = x_{i+1} - x_i \ , \quad \Delta_i = \frac{f_{i+1} - f_i}{h_i} \tag{2}$$

$$\theta = \frac{(x - x_i)}{h_i} \ ,0 \le \theta \le 1 \tag{3}$$

The rational cubic function (1) has the following properties,

$$\begin{aligned} S(x_i) = f_i,\ S(x_{i+1}) = f_{i+1} \\ S'(x_i) = d_i,\ S'(x_{i+1}) = d_{i+1} \end{aligned} \tag{4}$$

Where $S'(x)$ denotes the derivative with respect to 'x' and d_i denotes the value of the derivative at the knots x_i. It is very easy to prove the existence and uniqueness of the interpolation for given data (x_i, f_i) with parameters u_i, v_i, w_i. It is interesting that when $u_i = 1, v_i = 1 \ and \ w_i = 3$ in each interval I_i, the piecewise rational cubic functions (1) reduce to the standard cubic Hermite interpolation shown in (5).

$$\begin{aligned} S_i(x) &= (1-\theta)^2 (1+2\theta) f_i + \theta^2 (3-2\theta) f_{i+1} \\ &\quad + \theta (1-\theta)^2 h_i d_i - \theta^2 (1-\theta) h_i d_{i+1} \end{aligned} \tag{5}$$

In this paper, u_i, v_i are free parameters which can be used freely for the modification of the curve and w_i be the automated shape parameter which preserves the monotonicity of monotone data.

B. Determination of Derivatives

Mostly the derivative values or parameters $\{d_i\}$ are not given then it has been derived by two means which one from the give data set $(x_i, f_i),\ i = 1, 2, 3, ..., n$ or by some other methods. In this paper, we are determined from the given data for the smoothness of interpolant (1). These methods are the approximation based on many mathematical theories.The following two method are borrowed from Hussain & Sarfraz [1]. The descriptions of such approximations are as follow:

1) Arithmetic Mean Method: This method is the three point difference approximation with

$$d_i = \begin{cases} 0 & if\, \Delta_{i-1} = 0 \ or \ \Delta_i = 0 \\ d_i^* \quad otherwise & ,i = 2, 3, ...n-1 \end{cases} \tag{6}$$

And the end conditions are given as,

$$d_1 = \begin{cases} 0 & if\, \Delta_1 = 0 \ or \ sgn(d_1^*) \neq sgn(\Delta_1) \\ d_1^* & otherwise \end{cases} \tag{7}$$

$$d_n = \begin{cases} 0 & if\, \Delta_{n-1} = 0 \ or \ sgn(d_n^*) \neq sgn(\Delta_{n-1}) \\ d_n^* & otherwise \end{cases} \tag{8}$$

Where

$$\begin{aligned} d_i^* &= (h_i \Delta_{i-1} + h_{i-1} \Delta_i)/(h_i + h_{i-1}) \\ d_1^* &= \Delta_1 + (\Delta_1 - \Delta_2) h_1/(h_1 + h_2) \\ d_n^* &= \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2}) h_{n-1}/(h_{n-1} + h_{n-2}) \end{aligned}$$

2) Geometric Mean Method: These are non-linear approximations which are defined as:

$$d_i = \begin{cases} 0, \quad if\, \Delta_{i-1} = 0 \, or\, \Delta_i = 0 \\ \Delta_{i-1}^{h_i/h_{i-1}+h_i} \Delta_i^{h_{i-1}/h_{i-1}+h_i}, \quad i = 2, 3, 4, ..., n-1 \end{cases} \tag{9}$$

The end conditions are given as:

$$d_1 = \begin{cases} 0, \ if\, \Delta_1 = 0 \ or \ \Delta_{3,1} = \frac{f_3 - f_1}{x_3 - x_1} = 0 \\ \Delta_1 \left\{ \frac{\Delta_1}{\Delta_{3,1}} \right\}^{h_1/h_2}, otherwise \end{cases} \tag{10}$$

$$d_n = \begin{cases} 0, \ if\, \Delta_{n-1} = 0 \ or \ \Delta_{n,n-2} = \frac{f_n - f_{n-2}}{x_n - x_{n-2}} = 0 \\ \Delta_{n-1} \left\{ \frac{\Delta_{n-1}}{\Delta_{n,n-2}} \right\}^{h_{n-1}/h_{n-2}}, otherwise \end{cases} \tag{11}$$

C. Monotonicity Preserving Interpolation

The rational cubic function in section (I-A) does not provide a guarantee to preserves monotonicity curve of the monotone data (see Fig.1, Fig.4 & Fig.7). It appears that cubic Hermite spline scheme does not provide the desired shape features thus some further treatment is required to attain a pleasant and interactive shape from the given monotone data. To proceed with this approach, some mathematical treatments are required which will be shown in the following steps.

Let $(x_i, f_i), i = 1, 2, 3, ..., n$ be a given monotone data set,

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where $x_1 < x_2 < x_3 < ... < x_n$. Thus for monotone increasing(use same approach for monotone decreasing data)

$$f_i \le f_{i+1}, i = 1, 2, 3, ..., n \quad (12)$$

Suppose $h_i = x_{i+1} - x_i$, $\Delta_i = (f_{i+1} - f_i)/h_i$ and $d_i, i = 1, 2, 3, ..., n$ be the derivative values at $x_i, i = 1, 2, 3, ..., n$. Moreover, for the monotonic interpolation $S_i(x)$, it is then necessary that the derivative should be as given below
For $\Delta_i = 0$,

$$d_i = d_{i+1} = 0 \quad (13)$$

And for $\Delta_i \neq 0$,

$$sgn(d_i) = sgn(d_{i+1}) = sgn(\Delta_i) \quad (14)$$

The main focus to preserve the monotonicity of rational function (1) is to assign suitable values to shape parameters u_i, v_i & w_i.
We are discussing two cases for monotonicity preserving interpolation.

1) Case.1: When we have $\Delta_i = 0$, then $d_i = 0$ .In this case $S(x)$ reduces to

$$S_i(x) = f_i \qquad \forall x \in [x_i, x_{i+1}] \quad (15)$$

So is a monotone.

2) Case.2: When we have $\Delta_i \neq 0$, and then $S_i(x)$ is monotonically increasing if and only if $S_i^{(1)}(x) \ge 0 \ \ \forall x \in [x_i, x_{i+1}]$ .So we calculate $S_i^{(1)}(x)$ which after simplification is given as

$$S_i^{(1)}(x) = \frac{\sum_{k=1}^{6} (1-\theta)^{6-k} \theta^k A_{k,i}}{\{q_i(\theta)\}^2} \quad (16)$$

Where

$$A_{1,i} = v_i^2 d_{i+1}$$
$$A_{2,i} = A_{1,i} + 2v_i(w_i\Delta_i - d_i u_i)$$
$$A_{3,i} = A_{2,i} - A_{1,i} + 3u_i v_i \Delta_i + w_i(w_i\Delta_i - u_i d_i - v_i d_{i+1})$$
$$A_{4,i} = A_{5,i} - A_{6,i} + 3u_i v_i \Delta_i + w_i(w_i\Delta_i - u_i d_i - v_i d_{i+1})$$
$$A_{5,i} = A_{6,i} + 2u_i(w_i\Delta_i - d_{i+1} v_i)$$
$$A_{6,i} = u_i^2 d_i$$

So $S_i^1(x) \ge 0 \quad \forall x \in [x_i, x_{i+1}], \ \ if$

$$A_{k,i} \ge 0\,, \qquad k = 1, 2, 3, ..., 6$$

Where the necessary conditions

$$d_i \ge 0\,,\ d_{i+1} \ge 0,\ u_i \ge 0\ \&\ v_i \ge 0 \quad (17)$$

must be hold.
It is obvious that both $A_{1,i}$ & $A_{6,i}$ are positive from (17).However,$A_{2,i} \ge 0 \ \ if$

$$w_i > \frac{u_i d_i}{\Delta_i} \quad (18)$$

Similarly, $A_{5,i} \ge 0, \ \ if$

$$w_i > \frac{v_i d_{i+1}}{\Delta_i} \quad (19)$$

Finally, both $A_{3,i}$ & $A_{4,i}$ are positive if

$$w_i > \frac{u_i d_i + v_i d_{i+1}}{\Delta_i} \quad (20)$$

So constraints on w_i are efficient & reasonably acceptable choice to use the condition given above in (18- 20) can be summarized as:

Theorem 1.1: The piecewise C^1 rational cubic interpolant $S(x)$, defined over the interval [a,b] in (1), is preserve the monotonicity if in each subinterval $I_i = [x_i, x_{i+1}]$, the following sufficient conditions are satisfied:
$w_i > \max\left\{ 0, \frac{u_i d_i}{\Delta_i}, \frac{v_i d_{i+1}}{\Delta_i}, \frac{u_i d_i + v_i d_{i+1}}{\Delta_i} \right\}$
The above result can be rearranged as:
$w_i = l_i + \max\left\{ 0, \frac{u_i d_i}{\Delta_i}, \frac{v_i d_{i+1}}{\Delta_i}, \frac{u_i d_i + v_i d_{i+1}}{\Delta_i} \right\}, l_i > 0$

D. Numerical Examples

1) : In this example, the monotone data is taken from [1] as shown in Table I. Fig.1 is the curve generated by using cubic Hermite spline scheme with this monotone data, but does not preserve the monotonicity. To remove this obscurity, we show the visualization of the same monotone data in Fig.2 with $u_i = v_i = 0.1$ & Fig.3 with $u_i = v_i = 3.0$ using developed rational cubic interpolation scheme in section (I-C) to preserve the shape of monotone data. Fig.3 is the improvement of Fig.2 which is visually more pleasant curve.

TABLE I
MONOTONE DATA

i	1	2	3	4	5
x_i	0	6	10	29.5	30
f_i	0	15	15	25	30

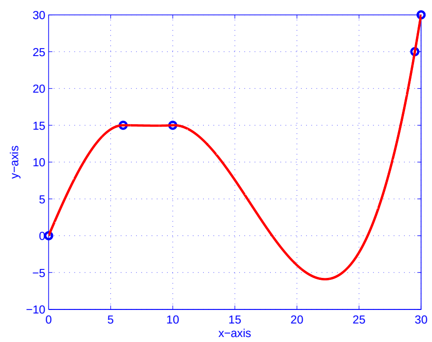

Fig. 1. Cubic Hermite Spline

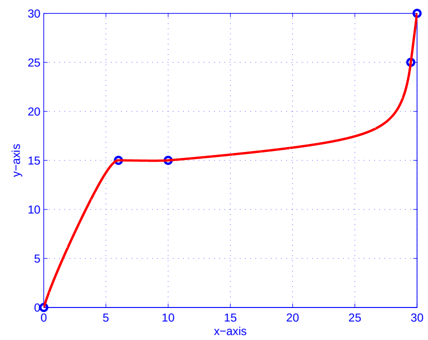

Fig. 2. Developed piecewise rational cubic function with $u_i = v_i = 0.1$

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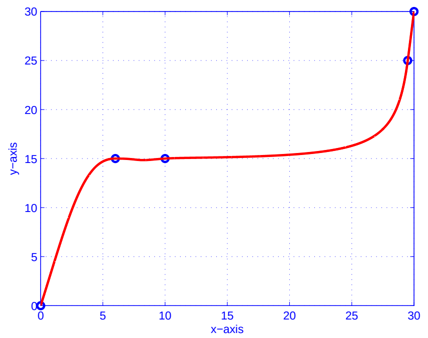

Fig. 3. Developed piecewise rational cubic function with $u_i = v_i = 3$

2) : The monotone data is taken from [1] as shown in Table II. Fig.4 is generated by cubic Hermite spline which loses the monotonicity. To remove this flaw, we show the visualization of the same monotone data in Fig.5 with $u_i = v_i = 0.1$ & Fig.6 with $u_i = v_i = 7.0$ using developed rational cubic interpolation which nicely preserve the shape of monotone data. Fig.6 is the improvement of Fig.5 which is also visually more pleasant curve.

TABLE II
MONOTONE DATA

i	1	2	3	4	5
x_i	5	6	11	12	15
f_i	10	11	15	50	85

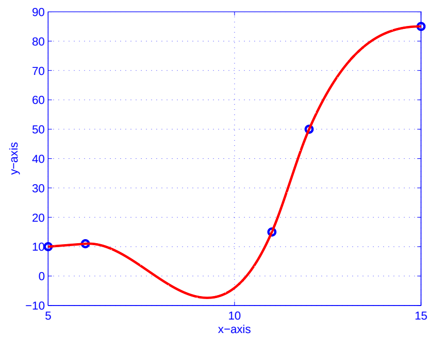

Fig. 4. Cubic Hermite Spline

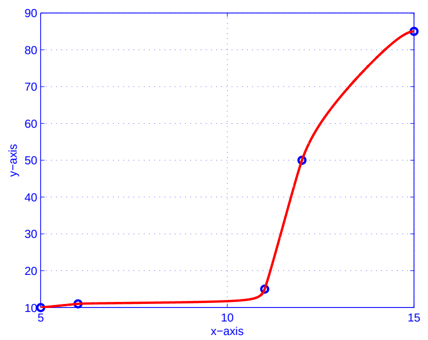

Fig. 5. Developed piecewise rational cubic function with $u_i = v_i = 0.1$

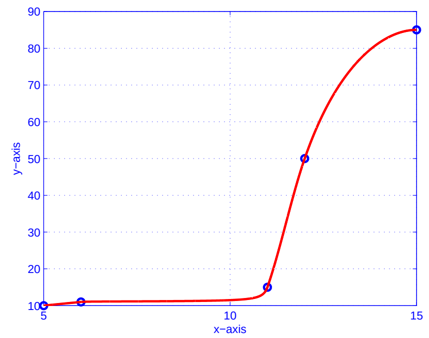

Fig. 6. Developed piecewise rational cubic function with $u_i = v_i = 7$

3) : In this example, the monotone data is taken from [14] as shown in Table III. Fig.7 is generated by cubic Hermite spline scheme which does not preserve monotonicity. Fig.8 with $u_i = v_i = 0.1$ & Fig.9 with $u_i = v_i = 3.0$ are generated by developed rational cubic interpolation given in section (I-C) that preserve the shape of monotone data. However, Fig.9 which is the improvement of Fig.8 is visually more pleasant curve.

TABLE III
MONOTONE DATA

i	1	2	3
x_i	1760	2650	2760
f_i	500	1360	2940

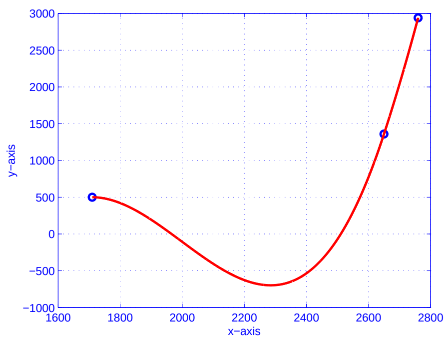

Fig. 7. Cubic Hermite Spline

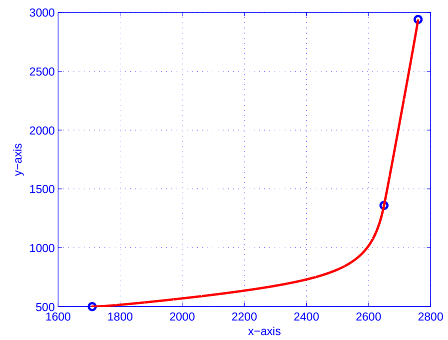

Fig. 8. Developed piecewise rational cubic function with $u_i = v_i = 0.1$

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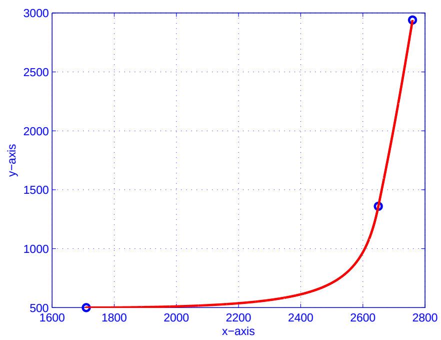

Fig. 9. Developed piecewise rational cubic function with $u_i = v_i = 3$

II. Conclusion & Suggestions

In this work the problem of shape preserving of monotonicity curves which is nicely controlled by piecewise ration cubic spline is proposed. Simple data dependent constraints are developed in the description of rational cubic function with three shape parameters with two are free shape parameters provided to the user to refine the shape as desired and other one is the automated shape parameter. The advantageous features on the existing schemes are: no extra knots are inserted to preserve the monotonicity of monotone data. The schemes [10], [11], [12], [13] and [17] are developed by using rational cubic function with two shape parameters that generates a nice monotonicity preserving curves but they have no degree of freedom. The method [14] is developed by using rational cubic function with single shape parameter to preserve the monotonicity but also has limited flexibility of the curves. Further, unlike [10], no derivative constraints are imposed. So, the our method applies on equally to data or data with given derivatives. The method developed in this paper is not only local and computationally economical but visually pleasant and interactive as well. Thus the method in section (I-C) is more flexible and may be more suitable for interactive CAD. Extension to monotone surfaces is under process.

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