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Delayed System Approach to the Stability of Networked Control Systems

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***Abstract*—This paper deals with the problem of stability analysis for networked control systems via the time-delayed system approach. The network-induced delays are modeled as two additive time-varying delays in the closed-loop system. To check the stability of such particular featured systems, an appropriate Lyapunov-Krasovskii functional is proposed and the Jensen inequality lemma is applied to the integral terms that are derived from the derivative of the Lyapunov-Krasovskii functional. Here, the cascaded structure of the delays in the system enables one to partition the domain of the integral terms into three parts, which produces a linear combination of positive functions weighted by inverses of convex parameters. This is handled efficiently by the authors' lower bounds lemma.**

Some numerical examples are given to demonstrate the effectiveness of the proposed method.

***Index Terms*—reciprocally convex combination; delay systems; stability; networked control systems.**

I. Introduction

It is well known that the presence of delay elements can bring about system instability and performance degradation, leading to design flaws and incorrect analysis conclusions [1]. Hence, a lot of attention has been paid to the time-delayed systems in recent years.

Among the relevant topics in this field, networked control systems have emerged as one of the most attractive issues in line with the rapidly growing network environments [2]–[5].

In the networked control systems, interpreting in the time-delayed system perspective, signals transmitted from one point to another may encounter two network-induced delays: $d_s(t)$ from the sampler (sensor) to the controller and $d_h(t)$ from the controller to the holder (actuator). This causes an introduction of delay elements in the closed-loop system. For example, when the ordinary plant is considered, $\dot{x}(t) = Ax(t) + Bu(t)$, the closed-loop system becomes

$$\dot{x}(t) = Ax(t) + BK(x - d_s(t) - d_h(t)), \tag{1}$$

i.e. systems with two additive time-varying delays.

As concerns about it, a conventional approach is to assemble the induced delays as a single one, $d(t) = d_s(t) + d_h(t)$, and to consider (1) as a single delayed system with $d(t)$ as the system delay [6], [7]. Recently, [3], [4] have also shown the possible reduction of conservatism in the analysis and synthesis problems by treating $d_s(t)$, $d_h(t)$ separately in constructing the Lyapunov-Krasovskii functional. However, to take into account the relationship between the two delays, they have to introduce slightly excessive free weighting matrices.

As a way of reducing the number of decision variables, this paper focuses on the Jensen inequality lemma [1]. It is well known that relaxations based on Jensen inequality lemma in delayed systems produce a special type of function combinations, a linear combination of positive functions weighted by inverses of convex parameters, *say a reciprocally convex combination*. Here, the cascaded structure of delays in the system enables one to partition the domain of the integral terms that are derived from the derivative of the Lyapunov-Krasovskii functional into three parts, which produces a reciprocally convex combination having three convex parameters as the weights. This can be handled efficiently by [8]'s lower bounds lemma that can be applied for all finite reciprocally convex combinations. It is notable that, to avoid the emergence of the reciprocally convex combination, [9] has to introduce a very conservative approximation on the difference between delays, $-\int_{t-d(t)}^{t-d_s(t)} \dot{x}^T(s)X\dot{x}(s)ds \leq -\frac{1}{\bar{d}_h}\int_{t-d(t)}^{t-d_s(t)} \dot{x}^T(s)dsX\int_{t-d(t)}^{t-d_s(t)} \dot{x}(s)ds$, $0 < d_h(t) \leq \bar{d}_h$, in the middle stage of the derivation.

The paper is organized as follows. Section 2 will explain the structure of the networked control systems and develop the corresponding stability criterion. Section 3 will show simple examples for verification of the criterion.

II. Main results

A. System description

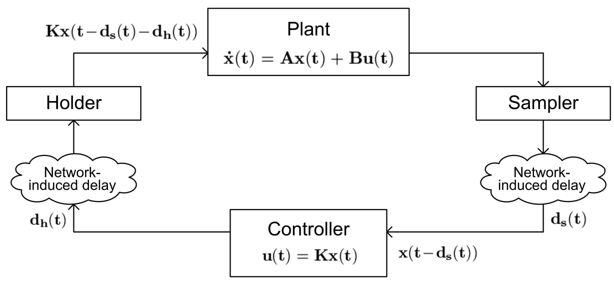

Fig. 1. Networked control systems

Let us consider the system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

which, with the introduction of the two network-induced delays: $d_s(t)$ from the sampler (sensor) to the controller and

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$d_h(t)$ from the controller to the holder (actuator), produces a special type of closed-loop system as

$$\dot{x}(t) = Ax(t) + BKx(t - d_s(t) - d_h(t)),$$

i.e. systems with two additive time-varying delays. To check the stability of such particular featured systems is the focus of the forthcoming section.

B. Stability analysis

Let us consider the following delayed system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t - d_1(t) - d_2(t)),\ t \ge 0, \\ x(t) &= \phi(t),\ \ -\bar{d}_1 - \bar{d}_2 \le t \le 0, \end{aligned} \tag{3}$$

where $0 \le d_1(t) \le \bar{d}_1$, $0 \le d_2(t) \le \bar{d}_2$, $\dot{d}_1(t) \le \bar{\tau}_1$, $\dot{d}_2(t) \le \bar{\tau}_2$ and $\phi(t) \in \mathcal{C}^1(\bar{d}_1 + \bar{d}_2)$, the set of continuously differentiable functions in the domain $[-2(\bar{d}_1 + \bar{d}_2), 0]$. Let us define $\bar{d} \triangleq \bar{d}_1 + \bar{d}_2$, $\bar{\tau} \triangleq \bar{\tau}_1 + \bar{\tau}_2$, $d(t) \triangleq d_1(t) + d_2(t)$, $\chi(t) \triangleq col\{x(t), x(t - d_1(t)), x(t - d(t)), x(t - \bar{d})\}$ and the corresponding block entry matrices as

$$\begin{aligned} &e_1 \triangleq [I\ 0\ 0\ 0]^T, e_2 \triangleq [0\ I\ 0\ 0]^T, e_3 \triangleq [0\ 0\ I\ 0]^T, \\ &e_4 \triangleq [0\ 0\ 0\ I]^T, e_5 \triangleq (Ae_1^T + A_d e_3^T)^T, \end{aligned} \tag{4}$$

so that the system can be written as $\dot{x}(t) = e_5^T \chi(t)$.

Consider the following Lyapunov-Krasovskii functional:

$$V(t) \triangleq V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{5}$$
$$V_1(t) = x^T(t)Px(t),\ P > 0, \tag{6}$$
$$V_2(t) = \int_{t-d_1(t)}^{t} x^T(\alpha)Q_1 x(\alpha)d\alpha,\ Q_1 > 0, \tag{7}$$
$$V_3(t) = \int_{t-d(t)}^{t} x^T(\alpha)Q_2 x(\alpha)d\alpha,\ Q_2 > 0, \tag{8}$$
$$V_4(t) = \int_{t-\bar{d}}^{t} x^T(\alpha)Q_3 x(\alpha)d\alpha,\ Q_3 > 0, \tag{9}$$
$$V_5(t) = \bar{d}\int_{-\bar{d}}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha)R\dot{x}(\alpha)d\alpha d\beta,\ R > 0. \tag{10}$$

Theorem 1: The delayed system (3) is asymptotically stable if there exist matrices P, Q_1, Q_2, Q_3, R, $S_{1,2}$, $S_{1,3}$ and $S_{2,3}$ such that the following conditions hold:

$$\Omega_1 + \Omega_2 < 0, \tag{11}$$
$$\begin{bmatrix} R & S_{1,2} \\ * & R \end{bmatrix} \ge 0, \begin{bmatrix} R & S_{1,3} \\ * & R \end{bmatrix} \ge 0, \begin{bmatrix} R & S_{2,3} \\ * & R \end{bmatrix} \ge 0, \tag{12}$$
$$P > 0,\ Q_1 > 0,\ Q_2 > 0,\ Q_3 > 0,\ R > 0, \tag{13}$$

where

$$\begin{aligned} \Omega_1 = {} & e_5 P e_1^T + e_1 P e_5^T + e_1(Q_1 + Q_2 + Q_3)e_1^T - e_4 Q_3 e_4^T \\ & -(1-\bar{\tau}_1)e_2 Q_1 e_2^T - (1-\bar{\tau})e_3 Q_2 e_3^T + \bar{d}^2 e_5 R e_5^T, \end{aligned} \tag{14}$$

$$\Omega_2 = -\begin{bmatrix} (e_1-e_2)^T \\ (e_2-e_3)^T \\ (e_3-e_4)^T \end{bmatrix}^T \begin{bmatrix} R & S_{1,2} & S_{1,3} \\ * & R & S_{2,3} \\ * & * & R \end{bmatrix} \begin{bmatrix} (e_1-e_2)^T \\ (e_2-e_3)^T \\ (e_3-e_4)^T \end{bmatrix}. \tag{15}$$

Proof: The time derivatives of $V_i(t)$ become

$$\dot{V}_1(t) = 2\dot{x}^T(t)Px(t) = 2\chi^T(t)e_5 P e_1^T \chi(t), \tag{16}$$
$$\dot{V}_2(t) = \chi^T(t)\{e_1 Q_1 e_1^T - (1 - \dot{d}_1(t))e_2 Q_1 e_2^T\}\chi(t), \tag{17}$$
$$\dot{V}_3(t) = \chi^T(t)\{e_1 Q_2 e_1^T - (1 - \dot{d}(t))e_3 Q_2 e_3^T\}\chi(t), \tag{18}$$
$$\dot{V}_4(t) = \chi^T(t)\{e_1 Q_3 e_1^T - e_4 Q_3 e_4^T\}\chi(t), \tag{19}$$
$$\dot{V}_5(t) = \bar{d}^2\chi^T(t)e_5 R e_5^T \chi(t) - \bar{d}\int_{t-\bar{d}}^{t} \dot{x}^T(\beta)R\dot{x}(\beta)d\beta, \tag{20}$$

so that $\dot{V}(t)$ can be upper-bounded by the following quantity:

$$\begin{aligned} \dot{V}(t) \le {} & \chi^T(t)\Omega_1\chi(t) \\ & -\bar{d}\int_{t-d_1(t)}^{t} \dot{x}^T(\beta)R\dot{x}(\beta)d\beta \\ & -\bar{d}\int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(\beta)R\dot{x}(\beta)d\beta \\ & -\bar{d}\int_{t-\bar{d}}^{t-d(t)} \dot{x}^T(\beta)R\dot{x}(\beta)d\beta & (21) \\ \le {} & \chi^T(t)\{\Omega_1 & (22) \\ & -\frac{1}{\alpha}(e_1-e_2)R(e_1-e_2)^T \\ & -\frac{1}{\beta}(e_2-e_3)R(e_2-e_3)^T \\ & -\frac{1}{\gamma}(e_3-e_4)R(e_3-e_4)^T\}\chi(t) & (23) \\ \le {} & \chi^T(t)(\Omega_1 + \Omega_2)\chi(t), & (24) \end{aligned}$$

where the inequality (22) comes from the Jensen inequality lemma [1], and that of (24) from [8]'s lower bounds lemma (see Appendix) as

$$\begin{aligned} & -\chi^T(t)\{\begin{bmatrix} \sqrt{\frac{\beta}{\alpha}}(e_1-e_2)^T \\ -\sqrt{\frac{\alpha}{\beta}}(e_2-e_3)^T \end{bmatrix}^T \begin{bmatrix} R & S_{1,2} \\ * & R \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\beta}{\alpha}}(e_1-e_2)^T \\ -\sqrt{\frac{\alpha}{\beta}}(e_2-e_3)^T \end{bmatrix} \\ & + \begin{bmatrix} \sqrt{\frac{\gamma}{\alpha}}(e_1-e_2)^T \\ -\sqrt{\frac{\alpha}{\gamma}}(e_3-e_4)^T \end{bmatrix}^T \begin{bmatrix} R & S_{1,3} \\ * & R \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\gamma}{\alpha}}(e_1-e_2)^T \\ -\sqrt{\frac{\alpha}{\gamma}}(e_3-e_4)^T \end{bmatrix} \\ & + \begin{bmatrix} \sqrt{\frac{\gamma}{\beta}}(e_2-e_3)^T \\ -\sqrt{\frac{\beta}{\gamma}}(e_3-e_4)^T \end{bmatrix}^T \begin{bmatrix} R & S_{2,3} \\ * & R \end{bmatrix} \begin{bmatrix} \sqrt{\frac{\gamma}{\beta}}(e_2-e_3)^T \\ -\sqrt{\frac{\beta}{\gamma}}(e_3-e_4)^T \end{bmatrix}\}\chi(t) \le 0, \end{aligned} \tag{25}$$

where

$$\alpha = \frac{d_1(t)}{\bar{d}}, \beta = \frac{d_2(t)}{\bar{d}}, \gamma = \frac{\bar{d} - d(t)}{\bar{d}}.$$

Note that when $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$, we have $\chi^T(t)(e_1 - e_2) = 0$ or $\chi^T(t)(e_2 - e_3) = 0$ or $\chi^T(t)(e_3 - e_4) = 0$, respectively. So the relation (24) still holds. This completes the proof. ∎

Remark 1: It is well known that relaxations based on Jensen inequality lemma in delayed systems produce a special type of function combinations, a linear combination of positive functions weighted by inverses of convex parameters. And, to the knowledge of the authors, all such particular featured combinations of functions in the literature have had only two convex parameters as the weights. However, here the cascaded structure of delays in the system (3) enables one to partition the domain of the integral term in (20) into three parts as (21), which produces the reciprocally convex combination (23) having three convex parameters (α, β, γ) as

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the weights. This is handled efficiently by [8]'s lower bounds lemma that can be applied for all finite reciprocally convex combinations.

It is notable that [10]'s relaxation method, which properly approximates the coefficient $\bar{d}$ in the integral terms (21) so as to obtain a condition that is linear in some parameters, has not addressed how to extend it for the case where integral terms are partitioned more than twice.

III. Examples

Example 1: Consider the system (3) taken from [3] with

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}. \tag{26}$$

The maximum upper bounds on the delays (*MUBDs*) under the assumption:

$$\dot{d}_1(t) \leq 0.1, \quad \dot{d}_2(t) \leq 0.8 \tag{27}$$

are listed in Tables I–II. Comparing with the recently developed results, we can see that the result in this paper is less conservative with relatively low decision variables.

TABLE I
MUBDs of d_2 for given $\bar{d}_1$ in Example 1

Method	$\bar{d}_1 = 1.0$	$\bar{d}_1 = 1.2$	$\bar{d}_1 = 1.5$	# of variables
Li et al.(2009) [7]	0.378	0.178	infeasible	32
Lam et al.(2007) [3]	0.415	0.340	0.248	59
Gao et al.(2008) [4]	0.512	0.406	0.283	85
Du et al.(2009) [9]	0.512	0.406	0.283	15
Peng & Tian(2008) [6]	0.665	0.465	0.165	18
Theorem 1	0.873	0.673	0.373	27

TABLE II
MUBDs of d_1 for given $\bar{d}_2$ in Example 1

Method	$\bar{d}_2 = 1.0$	$\bar{d}_2 = 1.2$	$\bar{d}_2 = 1.5$	# of variables
Lam et al.(2007) [3]	0.212	0.090	infeasible	59
Gao et al.(2008) [4]	0.378	0.178	infeasible	85
Li et al.(2009) [7]	0.378	0.178	infeasible	32
Du et al.(2009) [9]	0.378	0.178	infeasible	15
Peng & Tian(2008) [6]	0.665	0.465	0.165	18
Theorem 1	0.873	0.673	0.373	27

Example 2: Consider the system (3) taken from [9] with

$$A = \begin{bmatrix} -1.7073 & 0.6856 \\ 0.2279 & -0.6368 \end{bmatrix}, A_d = \begin{bmatrix} -2.5026 & 1.0540 \\ -0.1856 & -1.5715 \end{bmatrix}. \tag{28}$$

The derivatives of the delays are assumed to be

$$\dot{d}_1(t) \leq 0.3, \quad \dot{d}_2(t) \leq 0.8. \tag{29}$$

By Theorem 1, the improvement of this paper is shown in Tables III–IV.

TABLE III
MUBDs of d_2 for given $\bar{d}_1$ in Example 2

Method	$\bar{d}_1 = 0.1$	$\bar{d}_1 = 0.2$	$\bar{d}_1 = 0.3$	# of variables
Lam et al.(2007) [3]	0.412	0.290	0.225	59
Li et al.(2009) [7]	0.484	0.384	0.284	32
Gao et al.(2008) [4]	0.484	0.385	0.293	85
Du et al.(2009) [9]	0.484	0.385	0.293	15
Peng & Tian(2008) [6]	0.577	0.477	0.377	18
Theorem 1	0.684	0.584	0.484	27

TABLE IV
MUBDs of d_1 for given $\bar{d}_2$ in Example 2

Method	$\bar{d}_2 = 0.1$	$\bar{d}_2 = 0.2$	$\bar{d}_2 = 0.3$	# of variables
Li et al.(2009) [7]	0.484	0.384	0.284	32
Lam et al.(2007) [3]	0.547	0.343	0.185	59
Peng & Tian(2008) [6]	0.577	0.477	0.377	18
Gao et al.(2008) [4]	0.585	0.419	0.292	85
Du et al.(2009) [9]	0.585	0.419	0.292	15
Theorem 1	0.684	0.584	0.484	27

IV. Conclusions

This paper proposed an efficient stability criterion for networked control systems via the time-delayed system approach. The network-induced delays were modeled as two additive time-varying delays in the closed-loop system. To check the stability of such particular featured systems, an appropriate Lyapunov-Krasovskii functional was constructed and the Jensen inequality lemma was applied to the integral terms that were derived from the derivative of the Lyapunov-Krasovskii functional. Here, owing to the cascaded structure of delays in the system, the domain of the integral terms could be partitioned into three parts, which produced a linear combination of positive functions weighted by inverses of convex parameters. This was handled efficiently by Park et al. (2010)'s lower bounds lemma.

Examples showed the resulting criterion outperforms all existing ones with relatively low decision variables.

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