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Optimal Control for Linear Partial Differential Algebraic Equations Using Simulink

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***Abstract*—In this paper, optimal control for linear partial differential algebraic equations (PDAE) with quadratic performance is obtained using Simulink. By using the method of lines, the PDAE is transformed into a differential algebraic equations (DAE). Hence, the optimal control of PDAE can be found out by finding the optimal control of the corresponding DAE. The goal is to provide optimal control with reduced calculus effort by the Simulink solutions of the matrix Riccati differential equation (MRDE). Accuracy of the Simulink solution to this problem is qualitatively better. The advantage of the proposed approach is that, once the Simulink model is constructed, it allows to evaluate the solution at any desired number of points spending negligible computing time and memory. The corresponding solution curves can be obtained from the Simulink model without writing any codes. An illustrative numerical example is presented for the proposed method.**

***Index Terms*—Differential algebraic equation, Matrix Riccati differential equation, Optimal control, Partial differential algebraic equation and Simulink.**

I. INTRODUCTION

THE theory of PDAE is comparatively a recent topic. As expected, PDAE can also be characterized by an index and many efforts have been made to define indices for PDAE. The first concept deals with linear PDAE with constant coefficients. The linear PDAE of the form

$$\left.\begin{aligned} Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x),\\ (t,x)\in J\times\Omega \end{aligned}\right\}, \qquad (1)$$

where $J=(0,t_e)$, $\Omega=(-l,l)$, $t_e>0$, $l>0$, $f:[0,t_e]\times[-l,l]\rightarrow R^n$, at least one of the matrices $A,B\in R^{n\times n}$ is singular and $C\in R^{n\times n}$. The two special cases $A=0$ or $B=0$ lead to ordinary differential equations (ODE) or differential algebraic equations (DAE). The DAE is studied to find the optimal control of PDAE using Simulink.

The boundary conditions (BC) for the components u_j of u for all $j\in\mathcal{M}_{BC}\subseteq\{1,2,...,n\}$ and for simplicity, assume Dirichlet BC

$\mathrm{R}_{BC}u_j(t,\pm l)=0, \quad j\in\mathcal{M}_{BC}.$

The initial conditions (IC) are given as follows:

$\mathrm{u}_i(0,x)=g_i(x), for x\in[-l,l], i\in\mathcal{M}_{IC}\subseteq\{1,2,...,n\}.$

Any components g_i of g for all i can be chosen. The compatibility conditions between the IC and BC are given as

$\mathrm{R}_Bg(x)=R_Bu(0,x)=0.$

There are numerous applications in other scientific areas. Examples of PDAE can be found in the field of Navier-Stokes equations [18], in chemical engineering [12], in magneto-hydrodynamics [4] and in the theory of elastic multibody systems [16]. The PDAE (1) can be transformed into DAE using the method of lines (MOL). The DAE is also called MOL-DAE and Singular system.

Consider linear singular systems represented by:

$$E\dot{x}(t)=Ax(t)+Bu(t), \quad x(0)=x_0, \qquad (2)$$

where the matrix E is singular, $x(t)\in R^n$ is a generalized state space vector, $u(t)\in R^m$ is a control variable. $A\in R^{n\times n}$ and $B\in R^{n\times m}$ are known as coefficient matrices associated with $x(t)$ and $u(t)$ respectively, x_0 is given initial state vector and $m\leq n$.

Many practical processes can be modelled as descriptor systems such as constrained control problems, electrical circuits, certain population growth models and singular perturbations. In the past years, stability and control problems of descriptor systems have been extensively studied due to the fact that the descriptor system describes physical systems in a better manner than the state-space systems. Compared to state-space systems, the descriptor system has a more complicated yet richer structure. Furthermore, the study of the dynamic performance of descriptor systems is much more difficult than that of state-space systems since descriptor systems usually have three types of modes, namely, finite dynamic modes, impulsive modes and non-dynamic modes [5], while the latter two do not appear in the state-space systems.

Singular systems contain a mixture of algebraic and differential equations. In that sense, the algebraic equations represent the constraints to the solution of the differential part. These systems are also known as degenerate, descriptor or semi-state and generalized state-space systems. The system arises naturally as a linear approximation of system models or linear system models in many applications such as electrical networks, aircraft dynamics, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, robotics, biology, etc see [2], [3], [9].

As the theory of optimal control of linear systems with quadratic performance criteria is well developed, the results are most complete and close to use in many practical designing problems. The theory of the quadratic cost control

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problem has been treated as a more interesting problem and the optimal feedback with minimum cost control has been characterized by the solution of a Riccati equation. Da Prato and Ichikawa [6] showed that the optimal feedback control and the minimum cost are characterized by the solution of a Riccati equation. Solving the Matrix Riccati Differential Equation (MRDE) is a central issue in optimal control theory. The needs for solving such equations often arise in analysis and synthesis such as linear quadratic optimal control systems, robust control systems with H_2 and H_∞ - control [19] performance criteria, stochastic filtering and control systems, model reduction, differential games etc. One of the most intensely studied nonlinear matrix equations arising in Mathematics and Engineering is the Riccati equation. This equation, in one form or another, has an important role in optimal control problems, multivariable and large scale systems, scattering theory, estimation, detection, transportation and radiative transfer [7]. The solution of this equation is difficult to obtain from two points of view. One is nonlinear and the other is in matrix form. Most general methods to solve MRDE with a terminal boundary condition are obtained by transforming MRDE into an equivalent linear differential Hamiltonian system [8]. By using this approach, the solution of MRDE is obtained by partitioning the transition matrix of the associated Hamiltonian system [17]. Another class of methods is based on transforming MRDE into a linear matrix differential equation and then solving MRDE analytically or computationally [10], [14], [15]. However, the method in [13] is restricted for cases when certain coefficients of MRDE are non-singular. In [8], an analytic procedure of solving the MRDE of the linear quadratic control problem for homing missile systems is presented. The solution $K(t)$ of MRDE is obtained by using $K(t) = \frac{p(t)}{f(t)}$, where f(t) and p(t) are solutions of certain first order ordinary linear differential equations. However, the given technique is restricted to single input.

Simulink is a MATLAB add-on package that many professional engineers use to model dynamical processes in control systems. Simulink allows to create a block diagram representation of a system and run simulations very easily. Simulink is really translating a block diagram into a system of ordinary differential equations. Simulink is the tool of choice for control system design, digital signal processing (DSP) design, communication system design and other simulation applications [1]. This paper focuses upon the implementation of Simulink approach for solving MRDE in order to get the optimal control of the PDAE.

This paper is organized as follows. In section 2, the statement of the problem is given. In section 3, solution of the MRDE is presented. In section 4, numerical example is discussed. The final conclusion section demonstrates the efficiency of the method.

II. STATEMENT OF THE PROBLEM

Consider the PDAE (1) with BC and IC conditions. Let us discretize the PDAE on an equidistant grid [11]

$\Omega_h = \left\{ x_k : s_k = -l + kh, k = 1, ..., N, h = \frac{2l}{N+1} \right\}$

for a given integer $N \in N^+$. Replacing $u_{xx}(t, x_k)$ by

$u_{xx}(t, x_k) \approx \frac{1}{h^2}(u_{k+1}(t) - 2u_k(t) + u_{k-1}(t)), k = 1, ..., N.$

The following semi-discretized equation ($u_k(t) \approx u(t, x_k)$) is obtained:

$$\left.\begin{aligned} &Au_k'(t) + \frac{1}{h^2}B(u_{k+1}(t) \\ &-2u_k(t) + u_{k-1}(t)) + CU_k(t) = f_k(t) \end{aligned}\right\}$$

or in matrix representation using Kronecker product

$$\left.\begin{aligned} &(I_N \otimes A)U'(t) + \Big(\frac{1}{h^2}P \otimes B \\ &+I_N \otimes C\Big)U(t) = F(t) - r(t) \end{aligned}\right\}, \tag{3}$$

where the component $u_k(t)$ of the vector $U(t) \in R^{nN}$ is the continuous time approximation to $u(t, x_k)$ and the nN dimensional vectors $r(t)$, $F(t)$ are given by

$$\left.\begin{aligned} &r(t) = \Big(\frac{1}{h^2}I_N \otimes B\Big)(u^T(t, -l), 0, ..., 0, u^T(t, l))^T, \\ &F(t) = (f_1^T, ..., f_N^T(t))^T, f_k(t) = f(t, x_k) \end{aligned}\right\},$$

where I_N is the $(N \times N)$ unit matrix and the matrix P is defined by

$$P= \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{bmatrix} \in R^{N \times N}$$

Equation (3) is completed by the consistent initial vector

$U(0)= \left(\tilde{g}^T(x_1), ..., \tilde{g}^T(x_N)\right)^T \in R^{nN},$

where the difference $\tilde{g} - g$ goes (component wise) to zero for $h \to 0$. If A is singular, then equation (3) is a DAE for fixed h. Hence, the DAE is obtained by discretizing PDAE.

Consider the linear singular time-invariant system (2) for (1). In order to minimize both state and control signals of the feedback control system, a quadratic performance index is usually minimized:

$$\left.\begin{aligned} &J = \frac{1}{2}x^T(t_f)E^TSEx(t_f) \\ &+\frac{1}{2}\int_0^{t_f}[x^T(t)Qx(t) + u^T(t)Ru(t)]dt \end{aligned}\right\},$$

where the superscript T denotes the transpose operator, S and $Q \in R^{n \times n}$ are symmetric and positive definite (or semidefinite) weighting matrices for $x(t)$ and $R \in R^{m \times m}$ is a symmetric & positive definite weighting matrix for $u(t)$. It will be assumed that $|sE - A| \neq 0$ for some s. This assumption guarantees that any input u(t) will generate one and only one state trajectory x(t).

If all state variables are measurable, then a linear state feedback control law

$$u(t) = -R^{-1}B^T\lambda(t) \tag{4}$$

can be obtained to the system described by equation (2) where

$$\lambda(t) = K(t)Ex(t), \tag{5}$$

the symmetric matrix $K(t) \in R^{n \times n}$ is the solution of the MRDE and it satisfies the terminal conditions (TCs) $K(t_f) = E^TSE$.

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It is well known in the control literature that to minimize J is equivalent to minimize the Hamiltonian equation

$$H(x(t), u(t), \lambda(t)) = \tfrac{1}{2}x^T(t)Qx(t) + \tfrac{1}{2}u^T(t)Ru(t) + \lambda^T(t)[Ax(t) + Bu(t)].$$

The necessary conditions for optimality are

$$\frac{\partial H}{\partial u}(x, u, \lambda, t) = 0$$

implies that

$$Ru(t) + B^T\lambda(t) = 0$$

and

$$\frac{\partial H}{\partial x} = E^T\dot{\lambda}(t)$$

$$\Rightarrow E^T\dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) \qquad (7)$$

$$\frac{\partial H}{\partial \lambda} = E\dot{x}(t)$$

$$\Rightarrow E\dot{x}(t) = Ax(t) + Bu(t)$$

Using (4), we have

$$E\dot{x}(t) = Ax(t) - BR^{-1}B^T\lambda(t). \qquad (8)$$

Equations (7) and (8) can be written in a matrix form as follows :

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix},$$

where $x(0) = x_0$ and $E^T\lambda(t_f) = E^TSEx(t_f)$.

Assuming that $|R| \neq 0$, from (5) we have

$$\dot{\lambda}(t) = \dot{K}(t)Ex(t) + K(t)E\dot{x}(t)$$

and

$$E^T\dot{\lambda}(t) = E^T\dot{K}(t)Ex(t) + E^TK(t)E\dot{x}(t). \qquad (9)$$

Using the equations (5)-(8) in (9), we obtain

$$\left.\begin{array}{l} [E^T\dot{K}(t)E + E^TK(t)A + A^TK(t)E \\ +Q - E^TK(t)BR^{-1}B^TK(t)E]x(t) = 0 \end{array}\right\}. \qquad (10)$$

Since equation (10) holds for all non-zero x(t), the term pre-multiplying x(t) must be zero. Therefore, we obtain the following MRDE for the linear singular system (2)

$$\left.\begin{array}{l} E^T\dot{K}(t)E + E^TK(t)A + A^TK(t)E \\ +Q - E^TK(t)BR^{-1}B^TK(t)E = 0 \end{array}\right\}. \qquad (11)$$

In the following section, the MRDE (11) is solved for $K(t)$ in order to get the optimal solution.

III. SIMULINK SOLUTION OF MRDE

While solving the MRDE (11), the following system of nonlinear differential equation has occurred.

$$\left.\begin{array}{l} \dot{k}_{ij}(t) = \phi_{ij}(k_{ij}(t)), \\ (k_{ij})(t_f) = A_{ij} \quad (i, j = 1, 2, ..., n) \end{array}\right\}. \qquad (12)$$

Simulink is an interactive tool for modelling, simulating and analyzing dynamic systems. It enables engineers to build graphical block diagrams, evaluate system performance and refine their designs. Simulink integrates seamlessly with MATLAB and is tightly integrated with state flow for modelling event driven behavior. Simulink is built on top of MATLAB. A Simulink model for the given problem can be constructed using building blocks from the Simulink library. The solution curves can be obtained from the model without writing any codes.

A Simulink model is constructed for the following system of two differential equations as shown in the Figure 1.

$$\begin{aligned} x'(t) &= -x(t) + 1, \quad x(0) = -1 \\ y'(t) &= -y(t) + 1, \quad y(0) = 1. \end{aligned}$$

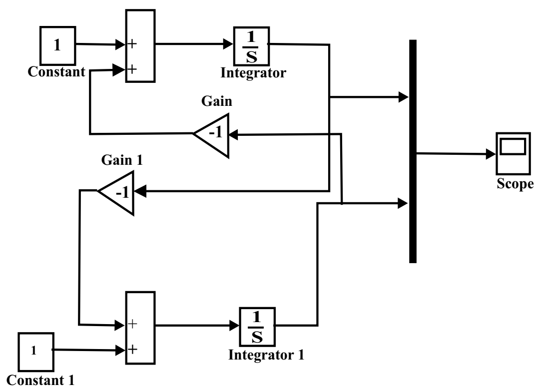

Fig. 1. Simulink Model

As soon as the model is constructed for (12), the simulink parameters can be changed according to the problem. The solution of the system of differential equation can be obtained in the display block by running the model.

A. Procedure for Simulink Solution

Step 1. Select the required number of blocks from the Simulink Library.
Step 2. Connect the appropriate blocks.
Step 3. Make the required changes in the simulation parameters.
Step 4. Run the Simulink model to obtain the solution.

IV. NUMERICAL EXAMPLE

Consider the optimal control problem after discretizing the PDAE into DAE/Singular system:

$$\text{Minimize} \quad \left.\begin{array}{l} J = \frac{1}{2}x^T(t_f)E^TSEx(t_f) \\ +\frac{1}{2}\int_0^{t_f}[x^T(t)Qx(t) + u^T(t)Ru(t)]dt \end{array}\right\}$$

subject to the singular system

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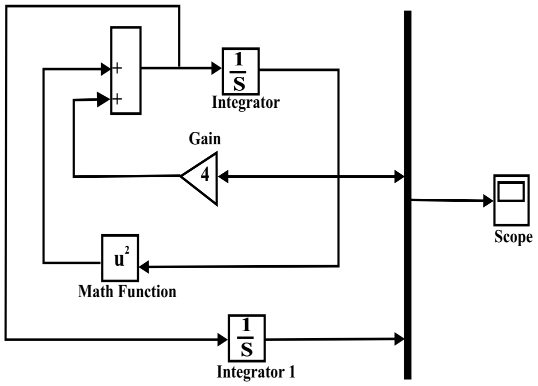

Fig. 2. Simulink Model

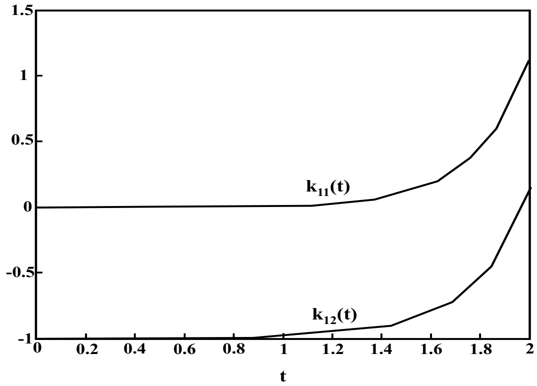

Fig. 3. Simulink Curve for MRDE

TABLE I
SIMULINK SOLUTION OF MRDE

t	k_{11}	k_{12}
0.0	0.0003	-0.9997
0.2	0.0007	-0.9993
0.4	0.0015	-0.9985
0.6	0.0033	-0.9967
0.8	0.0074	-0.9926
1.0	0.0164	-0.9836
1.2	0.0368	-0.9632
1.4	0.0828	-0.9172
1.6	0.1891	-0.8109
1.8	0.4467	-0.5533
2.0	1.1517	0.1517

$$E\ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

where

$$S=\begin{bmatrix} 1.1517 & 0.1517 \\ 0.1517 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A=\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$R=1, \quad Q=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The numerical implementation could be adapted by taking $t_f = 2$ for solving the related MRDE of the above linear singular system with the matrix A . The appropriate matrices are substituted in MRDE. The MRDE is transformed into differentia algebraic equations (DAE) in k_{11} and k_{12}. The DAE can be changed into a system of differential equations by differentiating the algebraic equation. In this problem, the value of k_{22} of the symmetric matrix $K(t)$ is free and let $k_{22} = 0$. Then the optimal control of the system can be found out by the solution of MRDE.

A. Solution Obtained Using Simulink

The Simulink model is constructed for MRDE. The Simulink model is shown in Figure 2. The numerical solution of MRDE is calculated by Simulink and displayed in Table 1. The numerical solution curve of MRDE by Simulink is given in Figure 3.

V. CONCLUSION

The optimal control of PDAE can be obtained by finding the optimal control of the corresponding DAE. The optimal control of DAE can be found out by solving the relative MRDE. The numerical results of the MRDE in Table 1 indicate that the Simulink solutions are much more efficient and accurate. The long calculus time of the MRDE is avoided by using Simulink. The solution curves can be obtained from the model without writing any codes. The efficient optimal solution is done with PC, CPU 2.0 GHz in MATLAB. In future, Simulink approach can be used to solve linear and nonlinear stochastic partial differential algebraic equations.

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