An Extended Model for the Uncapacitated Single Allocation Hub Covering Problem in a Fuzzy Environment

A. Eydi and A. Mirakhorli

Abstract—**This paper addresses the uncapacitated single allocation hub covering location problem with fuzzy travel time and cover radius. The main objective is to find the minimum number of hubs, considering the maximum allowable travel time. Computational results show that due to uncertainty, number of hubs and the allocation of non-hub nodes to the located hub nodes are changing. CAB data set are used to show the performance and validity of the proposed model.**

Index Terms— **fuzzy linear programming, hub covering problem, hub location**

I. Introduction

Hub location problem is one of the new topics in location problems because it is widely used in many transportation and telecommunication networks. Hubs are facilities that serve as transshipment and switching point to consolidate flows at certain locations for transportation and telecommunication systems [1],[2]. Hub and spokes network is used when it is not possible or it is so costly to transport the products directly between nodes (figure 1).

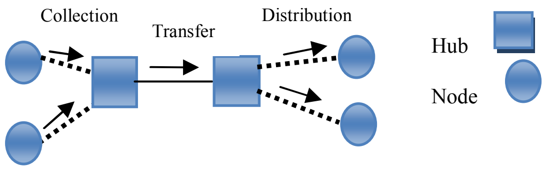

Fig. 1. Configuration of hub and spoke network

The location and number of hubs are important issues in hub and spoke network design problems. Hub location problems can be categorized into four main categories as follows: the P-hub median problem, the hub location problem with fixed costs, the p-hub center problem, and the hub covering problem [3]. In hub covering location problem (HCLP), the objective is to find the best location of hubs and assignment of non-hub nodes to hubs, considering the predetermined covering constrians, in order to minimize the total number of hubs or total operating cost associated with each hub. The first published formulation for HCLP was presented in [4]. Based on Campbell's opinion [4] ,an origin-destination pair (i,j) is covered in the following conditions:1) the total distance (cost or time) of a path through hub k and m does not exceed the predefined value, 2) the distance (cost or time) of each link in the path from node i to j via hub k and m does not exceed the specified value, and 3) the distance (cost or time) of each of links between the hubs and non-hub nodes in a path doesnot exceed the predefined value.

Kara and Tansel [5] presented linear model for hub covering problem and compared the computational time of their model with the previous approaches. Tan and Kara [6] tested the work of Kara and Tansel [5], on the Turkish network Wagner [7] proposed the pre-processing methods to fix the value of some decision variables, and to reduce the size of problem by removing redundant constraints. His model achieves better computational performance than the model presented in [5]. Ernest *et al.* [8] presented a new formulation for single and multiple allocation hub covering problem using the concept of cover radius. Calick *et al.* [9] presented a model for the hub covering problem over incomplete network and proposed a heuristic approach based on tabu search to solve the model.

The most application for the hub covering problem within transportation systems is delivering goods to customer at a predetermind time. Previous studies on hub location problems consider certainty conditions and assume that transshipment times are fixed, whereas delays arising from uncertainties such as unpredictable events, traffic congestion and bad weather during transshipment and delivering goods to customers are possible. These uncertainties also affect the cover radius, and the cover radius has the great impact on the structure of the network. Designing a network with desirable reliability level which garanties the performance and survivability of network from any disrubtion and malfunction is the most importand issue in the network design problems. In order to take into account the real world uncertainties, this paper formulizes the hub covering location problem under fuzzy environment considering fuzzy travel time and cover radius.

Rest of the paper is organized as follows. We will describe the fuzzy linear programming and propose fuzzy hub covering formulation in the second section. The third section is dedicated to the computational analysis. The performance of the model is tested with the optimization solver LINGO 8 on the well-known CAB data set. The last section is devoted to concluding remarks.

Alireze Eydi, Assistant Professor in the Faculty of Engineering, University of Kurdistan, Sanandaj, Iran (corresponding author to provide email: Alireza.eydi@uok.ac.ir)
Abbas Mirakhorli, M.Sc. student of Industrial Engineering, University of Kurdistan, Sanandaj, Iran (email: mirakhorli.a@acm.org)

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

II. Fuzzy Hub Covering Problem

A. Fuzzy Linear Programming (FLP)

Based on the empirical surveys, Linear Programming (LP) is the most applicable methods for solving real world problems. Bellman and Zadeh [10] proposed fuzzy linear programming as an extension of the classical linear programming. In fuzzy linear programming, for expression the imprecise parameters, the objective function and constraints are presented by fuzzy sets.

Based on Zimmermann approach [11], FLP model can be stated as:

$$\widetilde{max}(\text{or } \widetilde{min})\ f(x) = CX \qquad (1)$$

s.t.

$$AX \ \tilde{\lesssim}\ b \qquad (2)$$

$$X \geq 0 \qquad (3)$$

Here X is a decision vector, C is an objective coefficient vector, A is a constraint coefficient matrix and b is a constant vector. "$\widetilde{max}$", "$\tilde{\lesssim}$" denote the fuzzy version of *max*, <. The fuzzy objective function and the fuzzy constraints are defined by their corresponding membership functions [12].

B. Mathematical Formulation

A mathematical model for the fuzzy hub covering problem is given in this section. First, underlying assumptions are listed below:

- In this paper we study the hub covering problem.
- We model the single allocation case of the problem. For single allocation problems, every node is assigned to exactly one such hub.
- We also study this problem over a complete hub network, that is, every hub pair in the hub network is interconnected with a hub link.
- The economy of scale is modeled by a discount factor $\propto \in [0,1]$ on each of the hub to hub links [13].
- In this research, it is assumed that each hub is uncapacitated, that is, the amount of flow passing through the hub is infinite.
- Each arc and transportation path has infinite capacity.
- Because of uncertainties, the travel time between any origin-destination is modeled by fuzzy sets.

Here, we propose a formulation for the hub covering problem under fuzzy environment. We assume that there is a given node set N with n nodes and a potential hub set $H \subseteq N$ with h nodes. The mathematical model locates hubs from the potential hub set, constructs the hub network, and allocates the remaining nodes in set N to these hubs, such that the travel time between any origin–destination pairs to be less than a given time bound β, where β is the cover radius. The objective of our model is to minimize the total number of establishing hubs.

All other parameters of the model are as follows The parameter ∝ is the discount factor for hub to hub connections, and $\widetilde{t_{ij}}$ is the travel time between nodes i and j.

The decision variables of the model are presented as follows:

$$X_{ik} = \begin{cases} 1 & \text{if node } i \text{ is assigned to hub } k \\ 0 & \text{otherwise} \end{cases}$$

$$X_{kk} = \begin{cases} 1 & \text{if node } k \text{ is selected as hub} \\ 0 & \text{otherwise} \end{cases}$$

An integer programming formulation of the fuzzy hub covering problem can be written as follows:

$$\text{Min} \sum_k x_{kk} \qquad (4)$$

$$(\widetilde{t_{ir}} + \propto \widetilde{t_{rk}})\ X_{ir} + \widetilde{t_{jk}}\ X_{jk} \leq \tilde{\beta} \qquad \forall\ i, j \in N,\ k, r \in H \qquad (5)$$

$$\sum_{k \in H} X_{ik} = 1 \qquad \forall\ i \in N \qquad (6)$$

$$X_{ik} \leq X_{kk} \qquad \forall\ i \in N,\ k \in H \qquad (7)$$

$$X_{ik} \in \{0, 1\} \qquad \forall\ i \in N,\ k \in H \qquad (8)$$

The mathematical model consists of the objective function (4) and the constraints (5)–(8). Equation (5) is linear constraint (coverage constraint) and states that total travel time from node i to node j through hub k and m cannot be more than β (figure 2). Constraints (6) and (8) ensure that every node is allocated to exactly one hub. Constraint (7) states that a node cannot be allocated to another node unless that node is a hub.

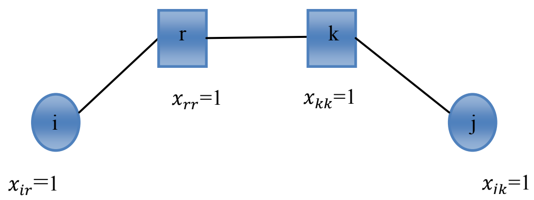

Fig. 2.Schema i llustration of (5)

In (5), the coefficients of decision variables and right hand side are represented by fuzzy numbers. Hence, the mathematical model is in form of FLP, based on fuzzy numbers.

Definition 1: $\tilde{C} \in$ f(R) is called a fuzzy number, where R represents the set of whole real numbers, if $\tilde{C}$ is convex and normal. In other words $x_0 \in$R exists, such that $\mu_{\tilde{C}}(x_0)$=1. The membership function of $\tilde{C}$ is $\mu_{\tilde{C}}(x_0)$. There are different types of fuzzy numbers like L-R-type and T-type [12].

Definition 2: Denote f(R) as a set of all fuzzy numbers on R. Here suppose $A \epsilon$f(R), if membership function $\tilde{A}$ is shown by figure 3, then $\tilde{A}$ =(A^l, A^C, A^R) is called a triangular fuzzy number. Where A^l, A^C, A^R are real numbers.

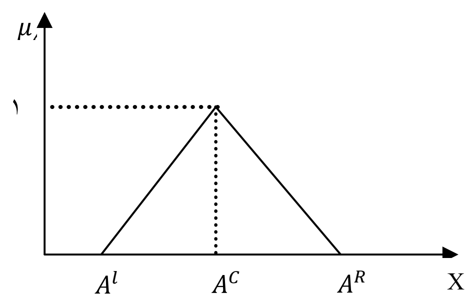

Fig. 3. Configuration of triangular fuzzy number [12]

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

This paper uses a triangular fuzzy number to represent the uncertainty of parameters such as travel time and cover radius. Because of uncertain parameters in (5), this constraint should be converted to its crisp equivalent. In this paper, we propose a method which is based on possibility theory to convert the fuzzy constraint to its crisp equvalent [14].

Definition 3: Based on Dubois and prade [15], if $\tilde{a}, \tilde{b}$ are two fuzzy numbers with membership functions $\mu_{\tilde{a}}$, $\mu_{\tilde{b}}$, then we can state the following equation:

$$\text{Pos}\ \{\ \tilde{a} \leq \tilde{b}\ \} = \text{Sup}\ \{\min\ (\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)) |\ x, y \epsilon\ R, x \leq y\} \quad (9)$$

Therefore, based on the expressed method, i.e., chance constrained programming in the fuzzy environment; equation (5) can be written as follows:

$$\text{Pos}\ \{\ (\widetilde{t_{ir}} + \propto \widetilde{t_{rk}})\ X_{ir} + \widetilde{t_{jk}}\ X_{jk} \leq \tilde{\beta}\ \} \geq \Phi \quad (10)$$

In (10), Φ is predetermined confidence level ($\Phi\ \epsilon[0,100]$). Constraint (10) states that the possibility of approximate total travel time is less than approximate cover radius is at least Φ .

Theorem 1: Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ are triangular fuzzy numbers, then the following result can be drawn :

I) $\text{Pos}\ (\tilde{a} \leq \tilde{b}) = 1 \Leftrightarrow a_2 \leq b_2$
II) $\text{Pos}\ (\tilde{a} \leq \tilde{b}) \geq \Phi \Leftrightarrow a_2 \leq b_3 - \Phi(b_3 - b_2)$

Proof: Equation (I) is alwayes true while $a_2 \leq b_2$ because there is alwayes a point like a_2 in ã which is smaller than or equal to a point like b_2 in $\tilde{b}$ as presented in figure 4.

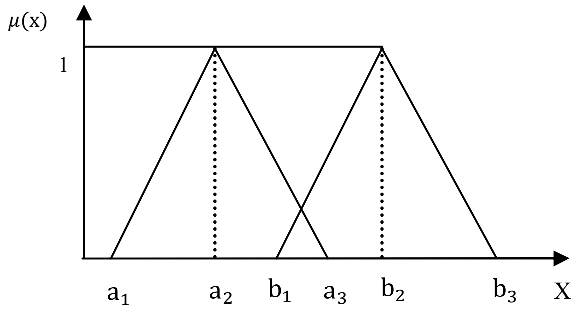

Fig. 4. Comparison of two traingular fuzzy numbers $\tilde{a}, b$: $a_2 \leq b_2$

when $a_2 > b_2$, equation (I) is not true any more. Thus, if we want to be sure that $\text{Pos}\ (\tilde{a} < \tilde{b}) \geq \Phi$, the following result can be obtained respecting figure 5.

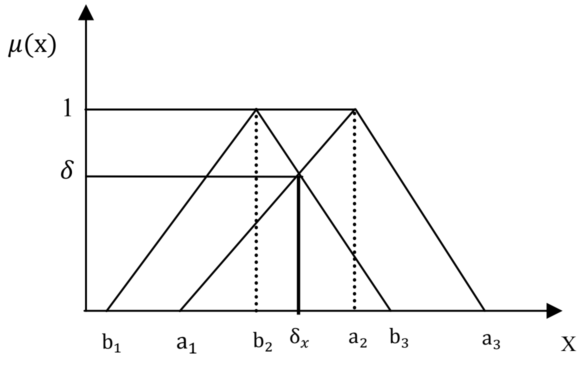

Fig. 5. Comparison of two traingular fuzzy numbers $\tilde{a}, \tilde{b}$: $a_2 > b_2$

Regarding triangles $\delta_x\ b_3 \delta$ and $b_2\ b_3\ 1$:

$$\frac{\delta}{1} = \frac{b_3 - \delta_x}{b_3 - b_2} \rightarrow \delta_x \leq b_3 - \delta(b_3 - b_2).$$

Therefore, (II) is proven.

Let $\tilde{\beta} = (\beta_1, \beta_2, \beta_3)$ is a triangular fuzzy number for representing cover radius and $\tilde{t}_{ir} = (t_{ir_1}, t_{ir_2}, t_{ir_3})$, $\tilde{t}_{rk} = (t_{rk_1}, t_{rk_2}, t_{rk_3})$, $\tilde{t}_{jk} = (t_{jk_1}, t_{jk_2}, t_{jk_3})$ are triangular fuzzy number for representing travel times. Based on theorem 1, (10) can be converted to crisp equivalent as follows:

$$(t_{ir_2} + \propto t_{rk_2})\ X_{ir} + t_{jk_2}\ X_{jk} \leq \beta_3 - \Phi(\beta_3 - \beta_2) \quad (11)$$

Therefore, the mathematical model for the fuzzy hub covering problem can be converted to its crisp equivalent as follows:

$$\text{Min} \textstyle\sum_k x_{kk} \quad (4)$$

$$(t_{ir_2} + \propto t_{rk_2})\ X_{ir} + t_{jk_2}\ X_{jk} \leq \beta_3 - \Phi(\beta_3 - \beta_2) \quad (11)$$

$$\textstyle\sum_{k \epsilon H} X_{ik} = 1 \qquad \forall\ i \epsilon N \quad (6)$$

$$X_{ik} \leq X_{kk} \qquad \forall\ i \epsilon N, k \epsilon H \quad (7)$$

$$X_{ik}\ \epsilon\ \{0, 1\} \qquad \forall\ i \epsilon N, k \epsilon H \quad (8)$$

In the next section, results of computational experiments on the well-known data set are presented.

III. MODEL EXPERIMENTS

In this section, we present the results of computational experiments to evaluate the performance of the proposed model. The tests are based on the CAB data set [16]. The CAB data set from the Civil Aeronautics Board Survey of 1970 contains the passenger flows and distances between 25 cities in the USA. The CAB data set introduced by O'Kelly [13]. The experimental study on the CAB data set are grouped into four problem sizes, *n*= {10, 15, 20, 25} with different values α = {0.2,0.4,0.6,0.8,1} reported in the literature.

In this reseach, we set instance of *n*=25, α=0.4 and consider the effects of uncertainties on the hub covering problem. The right and left speards of the fuzzy travel times are set 1.2 and 0.8 fold of their crisp values. For parameter β, this paper uses the radius presented in [5] as follows: this paper assumes the cover radius as triangular fuzzy number $\tilde{\beta}$=(β1,β2,β3). For example for (*n*, α) =(25,0.4), the cover radius is 2401,2099,1881,1597. Finally, we can take different values of Φ between 0 and 100%, but in this paper, the confidence level of 75%, 95% is considered.

We solved our model by using LINGO 8 on a personal computer Pentium 4 with a 2.4 GHZ processor and 1 GB of RAM. Computational results are shown in figures 6, 7 and 8.

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

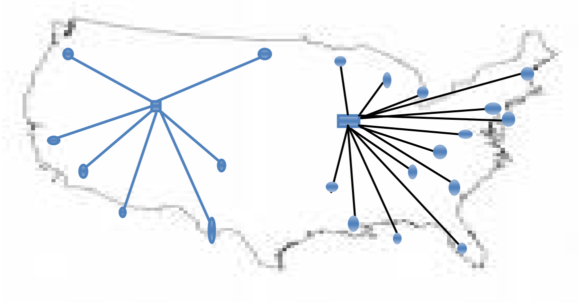

Fig. 6. Configuration of hub allocations in certain case

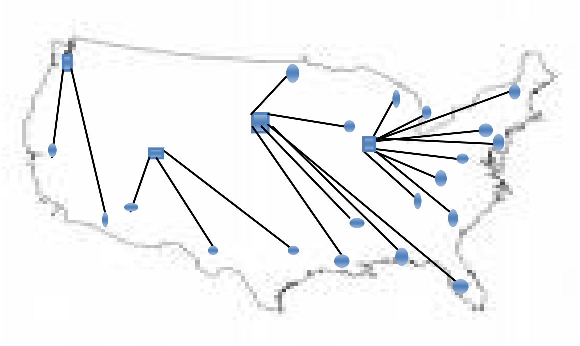

Fig.7. Configuration of hub allocations in fuzzy case for Φ=75%

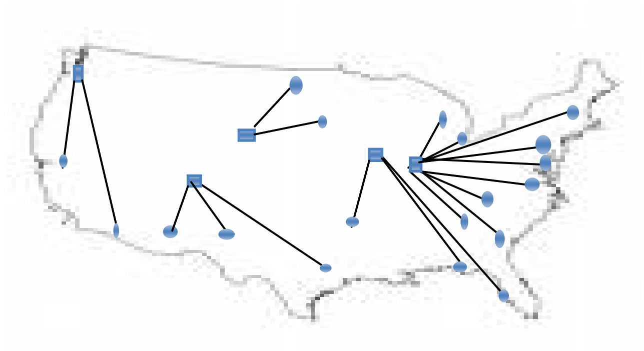

Fig. 8. Configuration of hub allocations in fuzzy case for Φ=95%

The computational results indicate that the number of hubs will increase when the problem is under fuzzy environment. It is rational to increase the number of hubs when cover radius is imprecise in order to meet customer demand as soon as possible and with least possible delay. Figure 6 shows the result of experiment for n= 25 and ∝ =0.4 with precise cover radius 2400, there are two hubs as presented in figure 6. But if the cover radius is considered imprecise, with confidence level of 75%, the number of hubs is increased to 4 as presented in figure 7. By increasing the confidence level to 95%, the numbers of hubs are increased to 5, as presente in figure 8.

IV. Conclusion

This paper studies the uncapacitated single allocation hub covering problem under fuzzy environment. In order to take account the uncertainty of travel times, we presented a fuzzy linear programming formulation. The computational results show that the number of hubs are increased by considering the problem with fuzzy parameters. When travel time and cover radius are imprecise, the number of hubs should be increased to satisfy customer demand as soon as possible. Also, we tested our model on the CAB data set.

For further research, one can apply the proposed model in a cargo delivery system. Developing heuristic algorithms for solving the problem on larger sized networks is another issue for further research.

Acknowledgment

The authors are grateful to Dr. H. Bevrani for carefully proof reading an earlier draft of this paper.

References

[1] M.E. O'Kelly and H.J. Miller," The hub network design problem: A review and synthesis", *Journal of Transport Geography*, vol. 2, no. 1, pp. 31-40, 1994.
[2] J.F. Campbell, A.T. Ernst and M. Krishnamoorthy, "Hub location problems", Z. Drezner and H.W. Hamacher, Eds., *Facility Location: Applications and Theory*, New York: Springer-Verlag, 2002, pp. 373-408.
[3] S. Alumur and B.Y. Kara, "Network hub location problems: The state of the art", *European Journal of Operational Research*, vol. 190, no. 1, pp. 1-21, 2008.
[4] J.F. Campbell, "Integer programming formulations of discrete hub location problems", *European Journal of Operational Research*, vol.72, no.2, pp. 387–405, 1994.
[5] B.Y. Kara and B.C. Tansel, "The single-assignment hub covering problem: models and linearization", *Journal of the Operational Research Society*, vol. 54, pp.59-64, 2003.
[6] P.Z. Tan and B.Y. Kara, "A hub covering model for cargo delivery systems", *Networks*, vol.49, no.1, pp. 28-39, 2007.
[7] B. Wagner, "Model formulations for hub covering problems". *Journal of the Operational Research Society*, vol.59, no.7, pp.932-938, 2008.
[8] A.T. Ernst, H. Jiang and M. Krishnamoorthy, "Reformulations and computational results for uncapacitated single and multiple allocation hub covering problems", unpublished report, CSIRO Mathematical and Information Sciences, Australia, 2005.
[9] H. Calik, S.A. Alumur, B.Y. Kara and O.E. Karasan, "A tabu-search based heuristic for the hub covering problem over incomplete hub networks", *Computers & Operations Research*, vol.36, no. 12, pp.3088-3096, 2009
[10] R.E. Bellman and L.A.Zadeh, "Decision making in a fuzzy environment", *Management Science*, vol .17, no. 4, pp. 141-164, 1970
[11] H.J. Zimmermann, *Fuzzy set theory and its applications*, 2nd ed., USA: Kluwer Academic,1991.
[12] B.-Y. Cao, *Optimal models and methods with fuzzy quantities*, Germany: Springer-Verleg, 2010, pp.139-191.
[13] M.E. O'Kelly, "A quadratic integer program for the location of interacting hub facilities", *European Journal of Operational Research*, vol. 32, no. 3, pp. 393–404, 1987.
[14] B. Liu and K. Iwamuva, "Chance constrained programming with fuzzy parameter", *Fuzzy Sets and Syatems*, vol. 94, no. 2, pp. 227-237, 1998.
[15] D. Dubois, and H. Prade, *Possibility theory*, USA: Plenum, 1988.
[16] J.E. Beasley, 2004, *OR-library*, Available: http://www.brunel.ac.uk/debpts/ma/research/jeb/info.html.