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Simple Policies for Joint Replenishment Can Perform Badly

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***Abstract*—We consider the stochastic joint replenishment problem in which several items must be ordered in the face of stochastic demand. Previous authors proposed multiple heuristic policies for this economically-important problem. We show that several such policies are not good approximations to an optimal policy, since as some items grow more expensive than others, the cost rate of the heuristic policy can grow arbitrarily larger than that of an optimal policy. These policies include the well-known RT policy, the $P(s, S)$ policy, the $Q(s, S)$ policy and the recently-proposed (Q, S, T) policy. To compensate for this problem, we propose a $Q_{\mathcal{I}}(s, S)$ policy, which is a generalization of the $Q(s, S)$ policy, and in which items are ordered if an expensive item is demanded or if demand for other items reaches Q. Our numerical results demonstrate that $Q_{\mathcal{I}}(s, S)$ policies do indeed overcome the weakness of the other heuristics, and can cost less than the $Q(s, S)$ heuristic even when the ratio of the cost of expensive items to other items is only a factor of three.**

***Index Terms*—inventory: approximations / heuristics / multi-item / policies; joint replenishment.**

I. Introduction

Most real-world supply chains face the stochastic joint replenishment problem (SJRP), where several items must be ordered to trade-off the costs of ordering, holding inventory and backlogging, in the face of stochastic demand. The existence and structure of optimal policies for the SJRP is well-established [1] and algorithms for computing optimal policies have been proposed [2]. However, these algorithms are not practical for more than 4 or 5 items as their cost grows exponentially with the number of items, yet in reality, inventories often consist of over 100 types of item. Therefore heuristic policies are used in practice.

To avoid extra costs from using a poor policy, it is important to understand how well such heuristic policies might perform. Yet such policies have only been evaluated on test problems with a limited range of manually-specified parameters [3] and compared with lower bounds on the cost rate of an optimal policy. With such an evaluation it is not possible to guarantee that such policies will perform well in all instance of the SJRP. Indeed, this paper shows that the cost rates of several heuristic policies can be arbitrarily larger than the cost rate of an optimal policy.

First we formally state the problem (Section 2). After describing existing heuristics (Section 3), we prove that even for two-item settings with independent Poisson demand, the ratio of the cost rate of an optimal policy from the given class of heuristics to the cost rate of an optimal policy can tend to infinity, as the parameters of the problem are varied (Theorem 1, Section 4). The policies in question include the well-known RT policy, the QS policy and the recently-proposed (Q, S, T) policy. We also show that the cost rate of $P(s, S)$ and $Q(s, S)$ policies relative to an optimal policy tends to infinity as the number of items is increased. These observations motivate us to suggest a way of partitioning the items to reduce the cost in pathalogical cases, resulting in a new policy which we call the $Q_{\mathcal{I}}(s, S)$ policy (Section 5). The $Q_{\mathcal{I}}(s, S)$ policy generalizes the $Q(s, S)$ policy, so it is always possible to find a $Q_{\mathcal{I}}(s, S)$ policy that performs as well as a $Q(s, S)$ policy. Finally, we demonstrate the advantage of the $Q_{\mathcal{I}}(s, S)$ policy over $Q(s, S)$ policies numerically (Section 5) and show that they succeed in overcoming the weaknesses of the other heuristics.

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II. The Stochastic Joint Replenishment Problem (SJRP)

A *stochastic joint replenishment problem (SJRP)* is given by a tuple $\langle n, d, k, k_i, h_i, p_i\rangle$. The number of item types is $n \in \mathbb{Z}_+$. The total amount of item i demanded up to time $t \in \mathbb{R}_+$ is the increasing random process $d_i(t)$. Major ordering cost $k \in \mathbb{R}_+$ is incurred per order in addition to a minor order setup cost of k_i for each item i that is ordered. The penalty cost rate for item i is $p_i(x_i)$ when the inventory position of item i is x_i, and the holding cost rate is $h_i(x_i)$, where $p_i : \mathbb{R} \to \mathbb{R}_+$ and $h_i : \mathbb{R} \to \mathbb{R}_+$.

A *policy* π *for a SJRP* is given by a tuple $\langle S_\pi, T_\pi\rangle$. The n-vector $S_\pi(x) \in \mathbb{R}^n$ is called the *order-up-to point* and has the interpretation that just after each order, the inventory position of item i is $S_{\pi,i}(x)$ if the inventory position just before the order is x. The policy orders at a random time T_π which may depend on the time t since the last order and on the realization of the demand d up to time T_π. Time interval $[0, T_\pi)$ is called a *cycle*. The *cost rate* v_π of policy π is the mean total cost per cycle divided by the mean time per cycle

$$v_\pi := \frac{\mathbb{E}K_\pi + \mathbb{E}H_\pi + \mathbb{E}P_\pi}{\mathbb{E}T_\pi} \tag{1}$$

$$K_\pi := k + \sum_{i=1}^{n} \mathbf{1}_{S_{\pi,i}(x(T_\pi)) \neq x_i(T_\pi)} k_i \tag{2}$$

$$H_\pi := \int_0^{T_\pi} \sum_{i=1}^{n} h_i(S_{\pi,i} - d_i(t))dt \tag{3}$$

$$P_\pi := \int_0^{T_\pi} \sum_{i=1}^{n} p_i(S_{\pi,i} - d_i(t))dt \tag{4}$$

where K_π, H_π, P_π are the mean ordering, holding and penalty cost accumulated in a cycle and the indicator function $\mathbf{1}_A$ for proposition A is one if A is true and zero otherwise.

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The *objective* of the SJRP is to minimize cost rate v_π with respect to the policy π.

III. Existing Heuristics

For the sake of internal consistency (regarding the symbols s, T and P), our nomenclature differs from some of the existing literature, where our PS policy is called the RT policy (yet the literature also refers to a $P(s, S)$ policy), our (Q, S, P) policy is called the (Q, S, T) policy (yet the literature also refers to a $Q(s, S)$ policy) and the can-order policy is usually denoted by (s, c, S) where s plays the role of our m and c plays the role of our s.

Existing policies can be defined in terms of an order-up-to point $S \in \mathbb{R}^n$. If $y_i(t)$ is the inventory position that would be obtained in the absence of an order at time t, then a policy is said to order an item i to S if it orders amount $S_i - y_i(t)$ of that item. Existing policies specify the subset to order $I_s(t)$ with a parameter $s \in \mathbb{R}^n$ as the set of items i whose inventory positions in the absence of an order $y_i(t)$ would otherwise be less than or equal to s_i, that is $I_s(t) := \{i \mid y_i(t) \leq s_i, 1 \leq i \leq n\}$.

If the last order was at time τ, then existing policies can be defined as follows.

Definition **1.** *Let $Q \in \mathbb{Z}_+, P \in \mathbb{R}_+$ and $m, s, S \in \mathbb{R}^n$ be any parameters.*

1) *PS policies [4] order all items to S when $t - \tau \geq P$.*
2) *$P(s, S)$ policies [3], [5] order items $I_s(t)$ to S when $t - \tau \geq P$.*
3) *QS policies [4], [6] order all items to S when $N_d(t) - N_d(\tau) \geq Q$.*
4) *$Q(s, S)$ policies [5], [7] order items $I_s(t)$ to S when $N_d(t) - N_d(\tau) \geq Q$.*
5) *(Q, S, P) policies [8] order all items to S when $N_d(t) - N_d(\tau) \geq Q$ or $t - \tau \geq P$.*
6) *Can-order policies [9], [10] order items $I_s(t)$ to S when $y_i(t) \leq m_i$ for some $1 \leq i \leq n$.*

IV. Example of Poor Performance

Our counterexample shows that (Q, S, P) policies and hence QS and PS policies are not constant-factor approximations, even for two items, zero minor setup costs and independent Poisson demand. Intuitively, if one item has large holding and penalty cost rates, then P or Q should be small so that demands for this item do not result in large item costs. However, if a second item is demanded frequently and ordering is costly, then a small P or Q implies a high ordering cost rate. No choice of P and Q provides a near-optimal tradeoff between these effects.

We use the following definitions. The demand processes for items 1 and 2 are $X(t)$ and $Y(t)$ which are independent Poisson processes with rates λ_X, λ_Y and with $X(0) = Y(0) = 0$. The cumulative demand process is $Z(t) := X(t) + Y(t)$. By the well-known properties of Poisson processes, $Z(t)$ is also a Poisson process with rate $\lambda_Z := \lambda_X + \lambda_Y$. The ordering time of a (Q, S, P) policy with $P \in \mathbb{R}_+, Q \in \mathbb{Z}_+$ is

$$T := \min\{P, \inf\{t \mid Z(t) \geq Q\}\}. \tag{5}$$

The following Lemma says that either the expected cycle time of a (Q, S, P) policy is short, or a significant proportion of the cycle time is spent having non-zero demand for item 1.

Lemma **1.** *For any $X(t), Y(t), P, Q$ as defined above*

$$\text{either} \quad \mathbb{E}T < \frac{2}{\lambda_Z}\mathbb{P}(Z(T) > 0) \tag{6}$$

$$\text{or} \quad \frac{\mathbb{E}\int_0^T \mathbf{1}_{X(t)>0} dt}{\mathbb{E}T} \geq \frac{\lambda_X}{2\lambda_Z}. \tag{7}$$

Proof: We consider two cases that cover all possibilities:

$$\text{either } \mathbb{E}T < 2\int_0^P \mathbb{P}(Z(t) = 0)dt$$
$$\text{or} \quad \mathbb{E}T \geq 2\int_0^P \mathbb{P}(Z(t) = 0)dt. \tag{8}$$

In the first case, the fact that $Z(t)$ is Poisson with rate λ_Z gives

$$\mathbb{E}T < 2\int_0^P \mathbb{P}(Z(t) = 0)dt \tag{9}$$
$$= \frac{2}{\lambda_Z}\left(1 - e^{-\lambda_Z P}\right) \tag{10}$$
$$= \frac{2}{\lambda_Z}(1 - \mathbb{P}(Z(P) = 0)) \tag{11}$$
$$= \frac{2}{\lambda_Z}(1 - \mathbb{P}(Z(T) = 0)) \tag{12}$$
$$= \frac{2}{\lambda_Z}\mathbb{P}(Z(T) > 0) \tag{13}$$

where the penultimate line follows since $Z(P) = 0$ implies that $T = P$ by definition of T and the fact that $Z(t)$ is non-decreasing, and similarly $Z(T) = 0$ also implies that $T = P$.

In the second case, we first use the definition of T to obtain

$$\mathbb{E}T = \mathbb{E}\int_0^P \mathbf{1}_{Z(t)<Q} dt \tag{14}$$
$$= \sum_{q=0}^{Q-1} \int_0^P \mathbb{P}(Z(t) = q)dt. \tag{15}$$

Now using the fact that $X(t)$ given $Z(t) = q$ has a binomial

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distribution with parameters λ_X/λ_Z and q for any t we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \mathbf{1}_{X(t)>0}dt \\
&= \mathbb{E}\int_0^P \mathbf{1}_{X(t)>0,Z(t)<Q}dt && (16)\\
&= \sum_{q=0}^{Q-1}\int_0^P \mathbb{P}(X(t)>0, Z(t)=q)dt && (17)\\
&\geq \sum_{q=1}^{Q-1}\int_0^P \mathbb{P}(Z(t)=q)\mathbb{P}(X(t)>0 \mid Z(t)=q)dt \\
&\geq \sum_{q=1}^{Q-1}\int_0^P \mathbb{P}(Z(t)=q)\mathbb{P}(X(t)>0 \mid Z(t)=1)dt \\
&= \sum_{q=1}^{Q-1}\int_0^P \mathbb{P}(Z(t)=q)\frac{\lambda_X}{\lambda_Z}dt && (18)\\
&= \frac{\lambda_X}{\lambda_Z}\left(\mathbb{E}T - \int_0^P \mathbb{P}(Z(t)=0)dt\right) && (19)
\end{aligned}$$

where the last line follows from (14).

Dividing (19) by $\mathbb{E}T$ and using the second case of (8) gives

$$\begin{aligned}
&\mathbb{E}\int_0^T \mathbf{1}_{X(t)>0}dt/\mathbb{E}T \\
&\geq \frac{\lambda_X}{\lambda_Z}\left(1 - \frac{\int_0^P \mathbb{P}(Z(t)=0)dt}{\mathbb{E}T}\right) && (20)\\
&\geq \frac{\lambda_X}{\lambda_Z}\left(1 - \frac{\frac{1}{2}\mathbb{E}T}{\mathbb{E}T}\right) = \frac{\lambda_X}{2\lambda_Z}. && (21)
\end{aligned}$$

Together (13) and (21) complete the proof. ■

The following Lemma says that the expected cycle time for which item 1 has demand $d \in \mathbb{Z}_+$ is less than or equal to the expected cycle time for which item 1 has zero demand.

LEMMA **2.** *For any $X(t), Y(t), P, Q$ as defined above and for any $d \in \mathbb{Z}+$,*

$$\mathbb{E}\int_0^T \mathbf{1}_{X(t)=d}dt \leq \mathbb{E}\int_0^T \mathbf{1}_{X(t)=0}dt. \tag{22}$$

Proof: Let $\tau_d := \int_0^\infty \mathbf{1}_{X(t)=d}dt$ be the time during which $X(t) = d$ and let $t_d := \sum_{d'=0}^{d-1}\tau_{d'}$ be the time until $X(t) \geq d$. The sequence $(\tau_d)_{d=0}^\infty$ consists of independent identically distributed exponential random variables, by definition of the Poisson process.

The following inequalities are a consequence of the facts that t_d and τ_d are independent of Y, that $\mathbf{1}_{Y(t)<Q-d,t\leq P}$ is decreasing in t and in d, and that τ_d and τ_0 are identically distributed:

$$\begin{aligned}
&\mathbb{E}\int_0^T \mathbf{1}_{X(t)=d}dt \\
&= \mathbb{E}_X\int_{t_d}^{t_d+\tau_d} \mathbb{E}_Y\mathbf{1}_{Y(t)<Q-d,t\leq P}dt && (23)\\
&\leq \mathbb{E}_X\int_0^{\tau_d} \mathbb{E}_Y\mathbf{1}_{Y(t)<Q-d,t\leq P}dt && (24)\\
&\leq \mathbb{E}_X\int_0^{\tau_d} \mathbb{E}_Y\mathbf{1}_{Y(t)<Q,t\leq P}dt && (25)\\
&= \mathbb{E}_X\int_0^{\tau_0} \mathbb{E}_Y\mathbf{1}_{Y(t)<Q,t\leq P}dt && (26)\\
&= \mathbb{E}\int_0^T \mathbf{1}_{X(t)=0}dt. && (27)
\end{aligned}$$

This completes the proof. ■

THEOREM **1.** *Suppose that $X(t), Y(t), P, Q$ are as defined above and that $k, \theta_1, \theta_2, \lambda_X, \lambda_Y > 0$ are any parameters. Let the item cost rate as a function of inventory position $x := (x_1, x_2)$ be*

$$c(x) := \theta_1|x_1| + \theta_2|x_2|. \tag{28}$$

Then for any order-up-to point $S \in \mathbb{R}^2$, the cost rate $\bar{v}$ of a (Q, S, P) policy using these parameters satisfies

$$\bar{v} \geq \min\{2k\lambda_Z, \theta_1\lambda_X/\lambda_Z\}/4 \tag{29}$$

whereas the cost rate $\bar{v}_*$ of an optimal policy satisfies

$$\bar{v}_* \leq \lambda_X k + \theta_2(\lambda_Y/\lambda_X). \tag{30}$$

Thus if we set $\lambda_X = \theta_2 = 1, k = \lambda_Y = \alpha, \theta_1 = \alpha^3$ for any $\alpha \geq 0$, then

$$\frac{\bar{v}}{\bar{v}_*} \geq \frac{1}{8}\frac{\alpha^2}{1+\alpha}. \tag{31}$$

Proof: First we place a lower bound on the cost rate $\bar{v}$ of the (Q, S, P) policy. Since $X(t) \in \mathbb{Z}_+$, for any $S_1 \in \mathbb{R}$ there exists some $d \in \mathbb{Z}_+$ for which

$$|S_1 - X(t)| \geq \mathbf{1}_{X(t)\neq d}/2 \text{ for all } t. \tag{32}$$

Thus, we may bound $\bar{v}$ by dropping costs associated with item 2, as

$$\begin{aligned}
&\bar{v}\mathbb{E}T \\
&\geq k\mathbb{P}(Z(T)>0) + \theta_1\mathbb{E}\int_0^T |S_1 - X(t)|dt && (33)\\
&\geq k\mathbb{P}(Z(T)>0) + (\theta_1/2)\mathbb{E}\int_0^T \mathbf{1}_{X(t)\neq d}dt && (34)\\
&\geq k\mathbb{P}(Z(T)>0) + (\theta_1/2)\mathbb{E}\int_0^T \mathbf{1}_{X(t)>0}dt && (35)
\end{aligned}$$

by Lemma 2. So by Lemma 1

$$\bar{v} \geq \min\left\{\frac{k}{2/\lambda_Z}, (\theta_1/2)\frac{\lambda_X}{2\lambda_Z}\right\}. \tag{36}$$

This proves (29).

Secondly, we observe that the cost rate $\bar{v}_*$ of an optimal policy is no more than the cost rate $\bar{v}_1$ of a policy that uses order-up-to-point $S = 0$ and that orders as soon as there is a demand for item 1. This policy orders at time $T_1 := \inf\{t \mid X(t) > 0\}$. Since T_1 is exponentially distributed,

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Algorithm 1: Find a $Q_{\mathcal{I}}(s,S)$ policy
Input: An SJRP

let $\mathcal{C}$ be the set of items i having $\theta_i > (k+k_i)\lambda_i$
sort these items so that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{|\mathcal{C}|}$
return a policy corresponding to
$\min_{0\leq j\leq|\mathcal{C}|}\{\bar{v}_{\mathcal{I}} \mid \mathcal{I}=\{1,2,\ldots,j\}\}$
where
$\bar{v}_{\mathcal{I}} := \sum_{i\in\mathcal{I}} c_i(x_i^*) + \min_Q \left(\frac{k}{\mathbb{E}T^{Q,\mathcal{I}}} + \sum_{i\in\mathcal{I}^c} g_i^{Q,\mathcal{I}}\right)$

$\mathbb{E}T_1 = 1/\lambda_X, \mathbb{E}T_1^2 = 2/\lambda_X^2$. Thus this policy has a cost rate

$$\bar{v}_1 = \left(k + \mathbb{E}\int_0^{T_1} \theta_2|Y(t)|dt\right)/\mathbb{E}T_1 \tag{37}$$
$$= \left(k + \theta_2\mathbb{E}\int_0^{T_1} \lambda_Y t\ dt\right)/\mathbb{E}T_1 \tag{38}$$
$$= \lambda_X\left(k + \theta_2\lambda_Y\mathbb{E}T_1^2/2\right) \tag{39}$$
$$= \lambda_X k + \theta_2\frac{\lambda_Y}{\lambda_X} \geq \bar{v}_*. \tag{40}$$

This proves (30).

Finally, taking the ratio of (29) and (30) for $\lambda_X = \theta_2 = 1, k = \lambda_Y = \alpha, \theta_1 = \alpha^3$ yields

$$\frac{\bar{v}}{\bar{v}_*} \geq \frac{1}{4}\frac{\min\{2k\lambda_Z, \lambda_X\theta_1/\lambda_Z\}}{\lambda_X k + \theta_2(\lambda_Y/\lambda_X)} \tag{41}$$
$$= \frac{1}{4}\frac{\min\{2\alpha(1+\alpha), \alpha^3/(1+\alpha)\}}{\alpha+\alpha} \tag{42}$$
$$= \frac{1}{8}\frac{\alpha^2}{1+\alpha}. \tag{43}$$

This completes the proof. ■

Theorem 1 does not immediately show that $Q(s,S)$ and $P(s,S)$ policies can perform badly relative to an optimal policy. However, the same type of argumentation can be made in this case. For instance, in the case of a $Q(s,S)$ policy, if there is a single expensive item with cost rate parameter θ_1 and $n-1$ identical inexpensive items, then either the policy sets $Q=1$ or it will sometimes pay the expensive rate θ_1. As θ_1 is increased, the cost rate of such a policy with $Q>1$ increases without bound relative to that of an optimal policy. On the other hand, any policy with $Q=1$ makes an order as soon as there is some item i whose inventory position reaches s_i. Thus, such a policy makes orders for only one item at a time. Any policy which orders only one item at a time fails to benefit from joint replenishment opportunities, so as the number of items n increases, the cost rate of such a policy increases without bound relative to that of an optimal policy.

In the same setting, a $P(s,S)$ policy either sets P to a very small value or it will sometimes pay the expensive rate θ_1 with finite probability. As θ_1 increases, the cost rate of a policy that pays θ_1 with finite probability increases without bound relative to that of an optimal policy. On the other hand, a policy with a small value for P will only rarely benefit from joint replenishment opportunities, and will mostly order only one item at a time. So, as the number of items n increases, the cost rate of such a policy increases without bound relative to that of an optimal policy.

V. Partitioning Items

The example of the previous section immediately suggests that an improvement of the $Q(s,S)$ policy in which we split the items into two groups: $\mathcal{I}$, which is the set of items with expensive holding and penalty costs; and $\mathcal{I}^c$, which is the complement of $\mathcal{I}$. We then set two monitoring variables $Q_{\mathcal{I}}$ and Q, where $Q_{\mathcal{I}}$ monitors the demand from $\mathcal{I}$ and Q monitors the demand from all items. We make orders for all items i having inventory positions $x_i \leq s_i$ whenever the demand for items from $\mathcal{I}$ exceeds $Q_{\mathcal{I}}$ or whenever demand for items from all items exceeds Q.

If the monitoring variable for $\mathcal{I}$ is set to one, $Q_{\mathcal{I}}=1$, then it turns out that we can compute a corresponding policy for essentially the same cost as computing a $Q(s,S)$ policy. We call such a policy a $Q_{\mathcal{I}}(s,S)$ policy.

We do not know if the $Q_{\mathcal{I}}(s,S)$ policy in general is a constant-factor approximation for the SJRP. However it is possible to imagine some (unnatural but still quasi-convex) cost rate functions $c_i(x)$ for which such a policy with $Q_{\mathcal{I}} = 1$ would not work well. For instance, consider the cost rate function

$$c_i(x) = K\max\{|x|-a, 0\} \tag{44}$$

where $K, a \in \mathbb{R}_+$ are large constants. To approximate an optimal policy for a SJRP in which all items had this cost rate function, one would require that all such items were placed in $\mathcal{I}$, otherwise large penalties would sometimes be incurred. On the other hand, such a policy (with $Q_{\mathcal{I}}=1$) would order every time an item in $\mathcal{I}$ is demanded, whereas an optimal policy would not need to order so frequently if a is large.

A. Computing a $Q_{\mathcal{I}}(s,S)$ Policy

In this section, we present an algorithm for computing a $Q_{\mathcal{I}}(s,S)$ policy (see Algorithm 1). The first step is to select a candidate set $\mathcal{C}$ of expensive items, so we must define the notion of expensive. Theorem 1 suggests that we identify items with a large value of $\min\{h_i,p_i\}/(k\lambda_i)$ where h_i and p_i are the holding and penalty costs for the item. However, we require a definition that takes leadtime and minor order setup costs into account. We meet this requirement as follows. Let $x_i^* \in \mathbb{Z}$ be an inventory position for which item i has its lowest cost rate

$$c_i(x_i^*) = \min_{x\in\mathbb{Z}} c_i(x). \tag{45}$$

We define a parameter θ_i as the minimum change in cost rates around inventory position x_i^*, that is

$$\theta_i := \min\{c_i(x_i^*+1), c_i(x_i^*-1)\} - c_i(x_i^*). \tag{46}$$

The candidate set of items $\mathcal{C}$ then consists of those items i having $\theta_i > (k+k_i)\lambda_i$.

Our algorithm then loops over subsets $\mathcal{I}$ of $\mathcal{C}$ in order of decreasing θ_i, at each stage computing the cost rate $\bar{v}_{\mathcal{I}}$ of the corresponding $Q_{\mathcal{I}}(s,S)$ policy. For any given Q, the total cost rate is the sum of: the cost rate for the items in $\mathcal{I}$, which is $\sum_{i\in\mathcal{I}} c_i(x_i^*)$, since such items stay at inventory position x_i^*; plus the cost rate associated with the major order setup costs, when orders are made at an average time interval of $\mathbb{E}T^{\mathcal{I},Q}$; plus the sum of the cost rates $g_i^{\mathcal{I},Q}$ of the remaining items. We explain the computation of $\mathbb{E}T^{\mathcal{I},Q}$ and $g_i^{\mathcal{I},Q}$ below.

For each choice of $\mathcal{I}$, we must search over the parameter Q. No guarantees about this search have been given, even

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Algorithm 2: Minimum cost rate for item $i \in \mathcal{I}^c$
Input: Parameter $Q \in \mathbb{N}$, set $\mathcal{I}$ and an SJRP

```
Step 1
μ := Σ_{j=1}^{n} λ_j
p := λ_i/μ
r := Σ_{j∈I} λ_j/μ
q := 1 − p − r
for y = Q − 1 to 0
  P(y) := r(1 − r)^y
end for
P(Q) := 1 − Σ_{y=0}^{Q−1} P(Y(T) = y)
for x = 0 to Q
  a(x) := Σ_{y=x}^{Q} P(y) (y choose x) (p/(p+q))^x (q/(p+q))^{y−x}
end for
Step 2 (Assume v(x, y) = 0 if otherwise undefined)
for y = Q − 1 to 0
  for x = x_min to x_max
    v(x, y) := c(x)/μ + pv(x − 1, y + 1)
      +qv(x, y + 1)
  end for
end for
Step 3 (Assume w(x, y) = 0 if otherwise undefined)
for d = 1 to Q
  for x = x_min to x_max
    S_w := Σ_{j=1}^{d−1} a(j)w(x − j, d − j)
    w(x, d) := (v(x, 0) + S_w) /(1 − a(0))
  end for
end for
Step 4
repeat step 2 replacing c(x) by 1
  and v(x, y) by v_t(x, y)
repeat step 3 replacing w(x, d) by t(x, d)
  and v(x, y) by v_t(x, y)
return cost rate g_i^{Q,I} := min_{S_i,s_i} (k_i+w(S_i,S_i−s_i))/t(S_i,S_i−s_i)
  and time between orders ET^{Q,I} := v_t(0, 0)
```

for computing a $Q(s,S)$ policy [7] as the cost rate is not a quasi-convex function of Q. To do this search, we start with a guess for an interval of Q-values that might contain the best value. Then we investigate the endpoints and midpoint of this interval. If the endpoints give better results, we extend the interval, otherwise we narrow it.

B. Computing the Cost Rate of an Individual Item

We now explain how to compute the mean time between orders $\mathbb{E}T^{\mathcal{I},Q}$ and the cost rate $g_i^{\mathcal{I},Q}$ or an individual item i (see Algorithm 2).

Before describing the algorithm, let us set up some notation. Let $X(t)$ be the demand for item $i \in \mathcal{I}^c$ (the complement of $\mathcal{I}$) since the last order of item i. Let $Y(t), Z(t)$ be the cumulative demand for items in $\mathcal{I}^c$ and $\mathcal{I}$ since the last order. Then, for a given Q, the first order is made at time

$$T := \inf\{t \mid Y(t) \geq Q \text{ or } Z(t) > 0\}. \tag{47}$$

Similarly, if $S_i - s_i = d$, then the first order for item i is made at time

$$\begin{aligned} T_i(d) := \inf\{t \mid X(t) \geq d \text{ and} \\ (Y(t) \geq Q \text{ or } Z(t) > 0)\}. \end{aligned} \tag{48}$$

Note that $X(t)$ can still increase after an order is made at time T, since that order might not involve item i.

First (Step 1) we find the probability mass function $a(x) := \mathbb{P}(X(T) = x)$ in terms of the total demand rate μ and the probabilities p, q, r that a demand is from item i, from an item in $\mathcal{I}^c\backslash\{i\}$ or from an item in $\mathcal{I}$ respectively. This follows from the fact that

$$Y(T) \sim \min\{Q, \text{Geometric}(r)\}$$

and that

$$X(T) \mid Y(T) \sim \text{Binomial}(p/(p+q), Y(T)).$$

Then (Step 2) we find the expected holding and penalty cost $v(x,y)$ associated with item i over time interval $[0,T)$, given that the inventory position of item i at $t = 0$ is x and that $X(0) = 0, Y(0) = y, Z(0) = 0$. We use a recurrence for $v(x,y)$ which follows from the facts that the mean time between demands is $1/\mu$ and that a demand for item i decreases $x - X(t)$ and increases $Y(t)$, whereas a demand for item $j \in \mathcal{I}^c\backslash\{i\}$ only increases $Y(t)$.

Then (Step 3), we find the expected holding and penalty cost $w(x,d)$ associated with item i over time interval $[0, T_i(d))$, given that $X(0) = Y(0) = Z(0) = 0$. By definition of ordering time $T_i(d)$ we have $w(x,d) = 0$ for $d \leq 0$. Otherwise, cost $w(x,d)$ is the cost until the next order $v(x,0)$ plus the cost incurred from the inventory position at the next order $w(x - X(T), d - X(T))$. This gives the following recurrence for $0 < d \leq Q$

$$\begin{aligned} w(x,d) &= v(x,0) + \mathbb{E}_{X(T)} w(x - X(T), d - X(T)) \\ &= v(x,0) + a(0)w(x,d) \\ &\quad + \sum_{j=1}^{d-1} a(j)w(x-j, d-j) \end{aligned} \tag{49}$$

which is easily solved for $w(x,d)$.

Finally (Step 4), we find the expected time $t(x,d)$ between orders of item i. This satisfies the same recurrence as $w(x,d)$ except with $v(x,0)$ replaced by the expected time between orders $\mathbb{E}T := v_t(0,0)$, which can be found by imagining that item i has a constant cost rate of one. The cost rate of item i for any given ordering parameters S_i, s_i is simply the cost between orders of i, including the minor order setup cost k_i, divided by the time between orders of i

$$\frac{k_i + w(S_i, S_i - s_i)}{t(S_i, S_i - s_i)}. \tag{50}$$

VI. Results

To illustrate the performance of the $Q_{\mathcal{I}}(s,S)$ policy we compare it with the $Q(s,S)$ policy, a policy which controls items in $\mathcal{I}$ and $\mathcal{I}^c$ with independent $Q(s,S)$ policies and the Atkins-Iyogun-Viswanathan lower bound [11] on a simple 4-item problem (see Table 1). In this problem we fixed leadtimes $L = 0.1$, fixed penalties $\pi = 0$ and minor order setup costs $k_i = 0.5$ for each item. The holding and proportional penalty costs are $h_i = p_i = 1$ for items $i \in \mathcal{I} := \{1,2\}$, while the demand rates are $\lambda_i = 1$ for items $i \in \mathcal{I}^c = \{3,4\}$. The other parameters are varied.

The table clearly demonstrates the benefits of $Q_{\mathcal{I}}(s,S)$ policies over $Q(s,S)$ policies and to a lesser extent the benefits of $Q_{\mathcal{I}}(s,S)$ policies over policies that treat expensive items independently. Indeed, the cost of the $Q(s,S)$ policy is always between 9% and 56% more than that of the $Q_{\mathcal{I}}(s,S)$ policy. While Theorem 1 would suggest that such benefits only become noticeable when some items are

TABLE I
COST RATES OF $Q_{\mathcal{I}}(s,S)$ AND $Q(s,S)$ POLICIES FOR 4-ITEM PROBLEM.

k	$h_{3:4}$ $= p_{3:4}$	$\lambda_{1:2}$	$Q_{\mathcal{I}}(s,S)$	$Q(s,S)$	Indep-endent	Lower Bound
1	3	1	5.09	5.55	5.93	3.80
1	8	2	7.06	8.91	7.84	5.64
2	27	2	13.01	18.88	14.36	10.44
3	64	3	23.33	36.32	25.22	19.39
3	125	4	36.35	56.56	38.31	32.08

far more expensive than other items, the first row shows that the benefits are apparent even when items 3 and 4 are only three times more expensive than the other items. Such a ratio of three in holding cost rates could clearly arise in practice.

VII. CONCLUSION

We showed that commonly-used policies for the stochastic joint replenishment problem do not provide a good approximation to an optimal policy. Indeed the cost of such policies can grow without bound relative to the cost of an optimal policy if some items are significantly more expensive than others.

To overcome this weakness, we proposed a $Q_{\mathcal{I}}(s,S)$ policy in which items are ordered if an expensive item is demanded or if demand for other items reaches Q. Our results demonstrate that this policy does indeed prevent this weakness and even provides benefits when the ratio of the cost of expensive items to other items is only a factor of three.

For future work, it would be of interest to consider extensions of the $Q_{\mathcal{I}}(s,S)$ policy which may be able to guarantee a constant-factor approximation to the optimal policy for the stochastic joint replenishment problem.

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