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Robust-based Random Fuzzy Mean-Variance Model Using a Fuzzy Reasoning Method

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Abstract—**This paper considers a robust-based random fuzzy mean-variance portfolio selection problem using a fuzzy reasoning method, particularly a single input type fuzzy reasoning method. Capital Asset Pricing Model is introduced as a future return of each security, and the market portfolio is assumed to be a random fuzzy variable whose mean is derived from a fuzzy reasoning method. Furthermore, under interval inputs of fuzzy reasoning method, a robust programming approach is introduced in order to minimize the worst case of the total variance. The proposed model is equivalently transformed into the deterministic nonlinear programming problem, and so the solution steps to obtain the exact optimal portfolio are developed.**

Index Terms—**Portfolio selection problem, Random fuzzy programming, Fuzzy reasoning method, Robust programming.**

I. Introduction

The decision of optimal asset allocation among various securities is called portfolio selection problem, and it is one of the most important research themes in investment and financial research fields since the mean-variance model was proposed by Markowitz [19]. Then, after this outstanding research, numerous researchers have contributed to the development of modern portfolio theory (cf. Elton and Gruber [1], Luenberger [18]), and many researchers have proposed several types of portfolio models extending Markowitz model; mean-absolute deviation model (Konno [12], Konno, et al. [13]), safety-first model [1], Value at Risk and conditional Value at Risk model (Rockafellar and Uryasev [22]), etc.. As a result, nowadays it is common practice to extend these classical economic models of financial investment to various types of portfolio models because investors correspond to present complex markets. In practice, many researchers have been trying different mathematical approaches to develop the theory of portfolio model. Particularly, Capital Asset Pricing Model (CAPM), which is a single factor model proposed by Sharpe [23], Lintner [15], and Mossin [21], has been one of the most useful tools in the investment fields and also used in the performance measure of future returns for portfolios and the asset pricing theory.

In such previous researches, expected future return and variance of each asset are assumed to be known. Then, in previous many studies in the sense of mathematical programming for the investment, future returns are assumed to be continuous random variables according to normal distributions. However, investors receive effective or ineffective information from the real markets and economic analysts, and ambiguous factors usually exist in it. Furthermore, investors often have the subjective prediction for future markets which are not derived from the statistical analysis of historical data, but their long-term experiences of investment. Then, even if investors hold a lot of information from the real market, it is difficult that the present or future random distribution of each asset is strictly set. Consequently, we need to consider not only random conditions but also ambiguous and subjective conditions for portfolio selection problems.

As recent studies in mathematical programming, some researchers have proposed various types of portfolio models under randomness and fuzziness. These portfolio models with probabilities and possibilities are included in stochastic programming problems and fuzzy programming problems, respectively, and there are some basic studies using stochastic programming approaches, goal programming approaches, and fuzzy programming approaches to deal with ambiguous factors as fuzzy sets (Inuiguchi and Ramik [8], Leon, et al. [14], Tanaka and Guo [25], Tanaka et al. [26], Vercher et al. [27], Watada [28]). Furthermore, some researchers have proposed mathematical programming problems with both randomness and fuzziness as fuzzy random variables (for instance, Katagiri et al. [10, 11]). In the studies [10, 11], fuzzy random variables were related with the ambiguity of the realization of a random variable and dealt with a fuzzy number that the center value occurs according to a random variable. On the other hand, future returns may be dealt with random variables derived from the statistical analysis, whose parameters are assumed to be fuzzy numbers due to the decision maker's subjectivity, i.e., random fuzzy variables which Liu [16] defined. There are a few studies of random fuzzy programming problem (Hasuike et al. [3, 4], Huang [7], Katagiri et al. [9]). Most recently, Hasuike et al. [4] proposed several portfolio selection models including random fuzzy variables and developed the analytical solution method.

However, in [4], each membership function of fuzzy mean values of future returns was set by the investor, and the mathematical detail of setting the membership function. Of course, it is also important to determine the fuzzy mean values of future returns with the investor's long-term experiences and economical analysts' effective information.

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Therefore, in order to involve the necessary information into mean values of future returns mathematically, we introduce a fuzzy inference or reasoning method based on fuzzy if-then rules. The fuzzy reasoning method is the most important approach to extract and decide effective rules under fuzziness mathematically. Since outstanding studies of Mamdani [20] and Takagi and Sugeno [24], many researchers have extended these previous approaches, and proposed new fuzzy reasoning methods. Particularly, we focus on a single input type fuzzy reasoning method proposed by Hayashi et al. [5, 6]. This method sets up rule modules to each input item, and the final inference result is obtained by the weighted average of the degrees of the antecedent part and consequent part of each rule module. Nevertheless this approach is one of the simplest mathematical approaches in fuzzy reasoning methods, the final inference result is similar to the other standard approaches. Therefore, in this paper, we proposed a random fuzzy mean-variance model introducing CAPM-based future returns and Hayashi's single input type fuzzy reasoning method for the mean value of market portfolio of CAPM.

The proposed random fuzzy mean-variance model is not formulated as a well-defined problem due to fuzziness, we need to set some certain optimization criterion so as to transform into well-defined problems. In this paper, assuming the interval values as a special case of fuzzy numbers and introducing the concept of robust programming, we transform the main problem into a robust programming problem. Recently, the robust optimization problem becomes a more active area of research, and there are some studies of robust portfolio selection problems determining optimal investment strategy using the robust approach (For example, Goldfarb and Iyengar [2], Lobo [17]). In robust programming, we obtain the exact optimal portfolio.

This paper is organized in the following way. In Section 2, we introduce mathematical concepts of random fuzzy variables, Capital Asset Pricing Model, and a single input type fuzzy reasoning method. In Section 3, we propose a random fuzzy portfolio selection problem with mean values derived from the fuzzy reasoning method. Performing the deterministic equivalent transformations, we obtain a fractional programming problem with one variable. Finally, in Section 4, we conclude this paper.

II. Mathematical Definition and Notation

In many existing studies of portfolio selection problems, future returns are assumed to be random variables or fuzzy numbers. However, since there are few studies of them treated as the CAPM with random fuzzy variables and fuzzy reasoning method, simultaneously. Therefore, in this section, we explain definitions and mathematical formulations of random fuzzy variable, CAPM, and single input type fuzzy reasoning method proposed by Hayashi et al. [5, 6].

A. *Random Fuzzy Variables*

First of all, we introduce a random fuzzy variables defined by Liu [16] as follows.

Definition 1 (Liu [16])

A random fuzzy variable is a function ξ from a collection of random variables R to $[0,1]$. An n-dimensional random fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ is an n-tuple of random fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$.

That is, a random fuzzy variable is a fuzzy set defined on a universal set of random variables. Furthermore, the following random fuzzy arithmetic definition is introduced.

Definition 2 (Liu [16])

Let $\xi_1, \xi_2, \ldots, \xi_n$ be random fuzzy variables, and $f: R^n \to R$ be a continuous function. Then, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a random fuzzy variable on the product possibility space $(\Theta, P(\Theta), \mathrm{Pos})$, defined as

$$\xi(\theta_1, \theta_2, \ldots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), \ldots, \xi_n(\theta_n))$$

for all $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

From these definitions, the following theorem is derived.

Theorem 1(Liu[16])

Let ξ_i be random fuzzy variables with membership functions μ_i, $i = 1, 2, \ldots, n$, respectively, and $f: R^n \to R$ be a continuous function. Then, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a random fuzzy variable whose membership function is

$$\mu(\eta) = \sup_{\eta_i \in R_i, 1 \le i \le n} \left\{ \min_{1 \le i \le n} \mu_i(\eta_i) \middle| \eta = f(\eta_1, \eta_2, \ldots, \eta_n) \right\}$$

for all $\eta \in R$, where

$$R = \left\{ f(\eta_1, \eta_2, \ldots, \eta_n) \middle| \eta_i \in R_i,\ i = 1, 2, \ldots, n \right\}.$$

B. *Capital Asset Pricing Model*

In portfolio models, Capital Asset Pricing Model (CAPM) proposed by Sharpe [23], Lintner [15], and Mossin [21] has been used in many practical investment cases by not only researchers but also practical investors. The main advantage of CAPM is to deal with the relation between returns of each asset and market portfolio such as NASDAQ and TOPIX as the the following simple linear formulation;

$$r_j = d_j^1 + d_j^2 r_m$$

where r_m is the return of market portfolio. Then, d_j^1 and d_j^2 are inherent values derived from historical data in investment fields. However, market portfolio r_m is not entirely equal to NASDAQ and TOPIX, and so it is almost impossible to observe r_m exactly in the investment field. Furthermore, in the case that the decision maker predicts the future return using CAPM, it is obvious that market portfolio r_m also occurs according to a random distribution with the investor's subjectivity. Therefore, in these situations, we propose a random fuzzy CAPM model. In this model we assume that $\bar{\tilde{r}}_m$ is a random fuzzy variable, and the "dash above" and "wave above", i.e., "$-$" and "~", denote randomness and fuzziness of the coefficients, respectively. In this paper, $\bar{\tilde{r}}_m$ occurs according to a random distribution with fuzzy mean value $\tilde{r}_m$ and constant variance σ_m^2. To simplify, we assume that each fuzzy expected return $\tilde{r}_m$ is an interval values $\tilde{r}_m = \left[r_m^L, r_m^U\right]$ derived from a fuzzy reasoning method in the next subsection.

C. Single Input Type Fuzzy Reasoning Method

Many researchers have proposed various fuzzy inference and reasoning methods based on or extending Mamdani [20] or Takagi and Sugeno's [24] outstanding studies. In this paper, as a mathematically simple approach, we introduce a single input type fuzzy reasoning method proposed by Hayashi et al. [5, 6]. In this method, we consider the following *m* rule modules:

Rule-*i*: $\left\{\zeta_i = A_s^i \rightarrow r_{mi} = r_s^i\right\}_{s=1}^{S_i}, \ \left(i = 1, 2, ..., m\right)$

where ζ_i and r_{mi} are the *i*th input and consequent data, respectively. Then, r_s^i is the real value of output for the consequent part. A_s^i is the fuzzy set of the *s*th rule of the Rules-*i*, and S_i is the total number of membership function of A_s^i. The degree of the antecedent part in the *s*th rule of Rules-*i* is obtained as $h_s^i = A_s^i\left(\zeta_i^0\right)$. In Hayashi's single input type fuzzy reasoning method, the inference result r^0 is calculated as follows:

$$r^0 = \frac{\sum_{s=1}^{S_1} h_s^1 r_s^1 + \cdots + \sum_{s=1}^{S_m} h_s^m r_s^m}{\sum_{s=1}^{S_1} h_s^1 + \cdots + \sum_{s=1}^{S_m} h_s^m} = \frac{\sum_{i=1}^{m}\sum_{s=1}^{S_1} h_s^i r_s^i}{\sum_{i=1}^{m}\sum_{s=1}^{S_1} h_s^i} \quad (1)$$

Particularly, if membership functions of all fuzzy sets A_s^i are triangle fuzzy numbers, formula (1) is a linear fractional function on input column vector $\boldsymbol{\zeta}^0$. In this paper, using this fuzzy reasoning method, we obtain the mean value of market portfolio r_m. We assume that input column vector $\boldsymbol{\zeta}^0$ means important financial and social factors to decide the mean value of market portfolio. However, it is difficult to set input column vector $\boldsymbol{\zeta}^0$ as constant values $\left[\underline{\zeta}_i^0, \overline{\zeta}_i^0\right]$. Therefore, we set each input ζ_i^0 as an interval value, and in order to obtain the maximum and minimum values of r_m under interval values of $\left[\underline{\zeta}_i^0, \overline{\zeta}_i^0\right]$, we introduce the following mathematical programming:

$$\begin{aligned} &\text{Maximize}\left(\text{Minimize}\right) \ \frac{\sum_{i=1}^{m}\sum_{s=1}^{S_1} h_s^i r_s^i}{\sum_{i=1}^{m}\sum_{s=1}^{S_1} h_s^i} \\ &\text{subject to } \ \underline{\zeta}_i^0 \le \zeta_i^0 \le \overline{\zeta}_i^0, \ \left(i = 1, 2, ..., m\right) \end{aligned} \quad (2)$$

Each problem is a fractional linear programming problem under triangle fuzzy numbers A_s^i, and so we obtain the optimal solutions. Let r_m^U and r_m^L be the optimal solution maximizing and minimizing the object, respectively.

III. Formulation of Portfolio Selection Problem with Random Fuzzy Returns

The previous studies on random and fuzzy portfolio selection problems often have considered standard mean-variance model or safety first models introducing probability or fuzzy chance constraints based on modern portfolio theories (e.g. Hasuike et al. [4]). However, there is no study to the random fuzzy mean variance model using the fuzzy reasoning method to obtain the interval mean value of market portfolio. Therefore, in this paper, we extend the previous random fuzzy mean-variance model to a robust programming-based model using the fuzzy reasoning method.

First, we deal with the following most simple portfolio selection problem involving the random fuzzy variable based on the standard asset allocation problem to maximize total future returns.

$$\begin{aligned} &\text{Maximize } \ \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \\ &\text{subject to } \ \sum_{j=1}^{n} x_j = 1, \ x_j \ge 0, \ j = 1, 2, \cdots, n \end{aligned} \quad (3)$$

where the notation of parameters used in this paper is as follows:

$\bar{\tilde{r}}_j$: Future return of the j th financial asset assumed to be a random fuzzy variable, whose fuzzy expected value is $\tilde{m}_j$ and variance-covariance matrix is $\mathbf{V}$, respectively. Then, we denote randomness and fuzziness of the coefficients by the "dash above" and "wave above", i.e., "$-$" and "~", respectively.

r_G : Target total return

n : Total number of securities

x_j : Budgeting allocation to the *j*th security

In [4], we consider several models and solution approaches based on standard safety-first models of portfolio selection problems. However, in order to solve the previous models analytically, we must assume that each return occurs according to the normal distributions in the sense of randomness. This assumption is a little restricted. Therefore, in this paper, we do not assume certain random distributions for future returns. Alternatively, we introduce the following portfolio model minimizing the worst total variance, i.e., maximizing the total variance, as a robust portfolio model:

$$\begin{aligned} &\underset{x}{\text{Minimize}} \quad \max_{r_m \in [r_m^L, r_m^U]} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right\} \\ &\text{subject to } E\left(\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \right) \geq r_G, \\ &\qquad \sum_{j=1}^{n} x_j = 1 \end{aligned} \tag{4}$$

By assuming this robust programming problem, the investor may be able to avoid the latent risk including the worst case of future return. In order to solve this problem, we fix the value of interval mean value of random fuzzy market portfolio $\tilde{\bar{r}}_m$ as $\bar{r}_m(\omega)$, i.e., each future return is also only a random variable as $\bar{r}_j(\omega) = d_j^1 + d_j^2 \bar{r}_m(\omega)$. Therefore, problem (4) is transformed into the following standard mean-variance portfolio model:

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ &\text{subject to } E\left(\sum_{j=1}^{n} \bar{r}_j(\omega) x_j \right) \geq r_G, \\ &\qquad \sum_{j=1}^{n} x_j = 1 \\ &\text{where } \bar{r}_j(\omega) = d_j^1 + d_j^2 r_m(\omega) \end{aligned} \tag{5}$$

This problem is a convex quadratic programming problem due to positive definite matrix, and so we obtain the exact optimal portfolio by using the following steps in nonlinear programming.

First we introduce the Lagrange function for problem (5) as follows:

$$L = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j + \lambda \left(r_G - \sum_{j=1}^{n} \bar{r}_j(\omega) x_j \right) + \xi \left(1 - \sum_{j=1}^{n} x_j \right) \tag{6}$$

where λ and ξ are Lagrange multiples. Then, by using Karush-Kuhn-Tucker (KKT) condition, we obtain the following equation on each variable x_j:

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = \sum_{i=1}^{n} \sigma_{ij} x_i - \lambda \bar{r}_j(\omega) - \xi = 0, \ (j = 1, 2, ..., n) \\ \sum_{j=1}^{n} \bar{r}_j(\omega) x_j = r_G \\ \sum_{j=1}^{n} x_j = 1 \end{cases} \tag{7}$$

In order to solve equations derived from KKT condition, we set the vector notation, and obtain the solution of $\boldsymbol{x}$ as follows:

$$\begin{aligned} &\mathbf{V}\boldsymbol{x} - \lambda \boldsymbol{r}(\omega) - \xi \boldsymbol{I} = \mathbf{0} \\ &\Leftrightarrow \mathbf{V}\boldsymbol{x} - \mathbf{A} \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = 0, \\ &\qquad \left(\mathbf{A} = \begin{pmatrix} \bar{r}_1(\omega) & \cdots & \bar{r}_n(\omega) \\ 1 & \cdots & 1 \end{pmatrix} \right) \\ &\Leftrightarrow \boldsymbol{x} = \mathbf{V}^{-1} \mathbf{A}^t \begin{pmatrix} \lambda \\ \xi \end{pmatrix} \end{aligned} \tag{8}$$

By substituting this solution into second and third equations in KKT condition (7), we obtain the following optimal values of Lagrange multiples:

$$\begin{aligned} &\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = \hat{\boldsymbol{r}}_G, \ \left(\hat{\boldsymbol{r}}_G = \begin{pmatrix} r_G \\ 1 \end{pmatrix} \right) \\ &\Leftrightarrow \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \end{aligned} \tag{9}$$

Consequently, we obtain the optimal portfolio $\boldsymbol{x}^*$ and the objective value $\left(\boldsymbol{x}^*\right)^t \mathbf{V}\boldsymbol{x}^*$ as follows:

$$\begin{aligned} &\boldsymbol{x}^* = \mathbf{V}^{-1}\mathbf{A}^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \\ &\left(\boldsymbol{x}^*\right)^t \mathbf{V}\boldsymbol{x}^* = \left(\hat{\boldsymbol{r}}_G^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \mathbf{A}\mathbf{V}^{-1} \right) \mathbf{V} \left(\mathbf{V}^{-1}\mathbf{A}^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \right) \\ &\qquad = \hat{\boldsymbol{r}}_G^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \\ &\qquad = \hat{\boldsymbol{r}}_G^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \end{aligned} \tag{10}$$

From $\boldsymbol{x}^*$ and the optimal objective value, we secondly consider a robust programming-based portfolio model, i.e. the worst case of the total variance:

$$\begin{aligned} &\underset{r_m \in [r_m^L, r_m^U]}{\text{Maximize}} \ \hat{\boldsymbol{r}}_G^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G \\ &\text{subject to } r_m \in \left[r_m^L, r_m^U \right] \end{aligned} \tag{11}$$

In objective function $\hat{\boldsymbol{r}}_G^t \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} \hat{\boldsymbol{r}}_G$, inverse matrix $\left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1}$ is calculated as the following form:

$$\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t = \begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_i r_j & \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_j \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_j & \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} \end{pmatrix}$$

$$\begin{cases} \left(\mathbf{A}\mathbf{V}^{-1}\mathbf{A}^t \right)^{-1} = \dfrac{1}{\Delta(\boldsymbol{r})} \begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} & -\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_j \\ -\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_j & \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_i r_j \end{pmatrix} \\ \Delta(\boldsymbol{r}) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_i r_j \right) \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} \right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \hat{\sigma}_{ij} r_j \right)^2 \end{cases} \tag{12}$$

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Therefore, objective function $\hat{\boldsymbol{r}}_G^t\left(\mathbf{AV}^{-1}\mathbf{A}^t\right)^{-1}\hat{\boldsymbol{r}}_G$ is also calculated as follows:

$$\hat{\boldsymbol{r}}_G^t\left(\mathbf{AV}^{-1}\mathbf{A}^t\right)^{-1}\hat{\boldsymbol{r}}_G = \frac{1}{\Delta(\boldsymbol{r})}\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_ir_j - 2r_G\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_j + r_G^2\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\right) \tag{13}$$

Consequently, problem (11) is equivalently transformed into the following problem:

$$\begin{aligned}&\text{Maximize } \frac{1}{\Delta(\boldsymbol{r})}\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_ir_j - 2r_G\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_j + r_G^2\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\right)\\ &\text{subject to } r_j = d_j^1 + d_j^2 r_m,\ r_m \in \left[r_m^L, r_m^U\right]\\ &\left(\text{where } \Delta(\boldsymbol{r}) = \left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_ir_j\right)\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\right) - \left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_j\right)^2\right)\end{aligned} \tag{14}$$

In this problem, we substitute $\overline{r}_j(\omega) = d_j^1 + d_j^2 r_m(\omega)$ derived from CAPM, and the numerator and denominator of objective function are calculated as follows:

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_ir_j - 2r_G\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}r_j + r_G^2\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\\ &= \sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}d_i^2d_j^2r_m^2\\ &\quad + 2r_G\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\left(d_i^1d_j^2 + d_j^1d_i^2 - 2r_Gd_j^2\right)r_m\\ &\quad + (\text{constant value})\\ &= p_0 + p_1r_m + p_2r_m^2\\ &\Delta(\boldsymbol{r}) = \left(\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\right)\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}d_i^2d_j^2\right)\right)r_m^2\\ &\quad + \left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\left(d_i^1d_j^2 + d_j^1d_i^2\right)\right)\left(\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ij}\right)r_m\\ &\quad - \left(\sum_{k=1}^n\sum_{i=1}^n\sum_{j=1}^n\hat{\sigma}_{ki}\hat{\sigma}_{kj}d_i^2d_j^2\right)^2 r_m^2\\ &\quad - \left(\sum_{k=1}^n\sum_{i=1}^n\sum_{j=1}^n 2\hat{\sigma}_{ki}\hat{\sigma}_{ij}d_i^1d_j^2\right)r_m\\ &\quad + (\text{constant value})\\ &= q_0 + q_1r_m + q_2r_m^2\end{aligned} \tag{15}$$

Therefore, the main problem is represented as the following mathematical programming problem with variable r_m :

$$\begin{aligned}&\text{Maximize } \frac{p_0 + p_1r_m + p_2r_m^2}{q_0 + q_1r_m + q_2r_m^2}\\ &\text{subject to } r_m \in \left[r_m^L, r_m^U\right]\end{aligned} \tag{16}$$

This problem is a nonlinear fractional programming problem, and so it is generally difficult to obtain the optimal solution. However, this problem has only one variable r_m , and so we can obtain the optimal solution by using standard nonlinear programming approaches or illustrating the objective function directly.

IV. Conclusion

In this paper, we have proposed a robust-based mean-variance portfolio selection problem with random fuzzy CAPM using a single input type fuzzy reasoning method. In order to deal with the market portfolio of CAPM as a random interval variable, and to perform the deterministic equivalent transformations, the proposed model has been nonlinear programming problem with only one variable. Therefore, we have obtained the exact optimal portfolio using standard nonlinear programming approaches.

As future studies, we need to develop the solution algorithm in cases of general fuzzy numbers including interval values. Furthermore, we also need to consider random fuzzy portfolio models derived from not only a single input type fuzzy reasoning method but also more general fuzzy reasoning methods.

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