An Approach to Solve Bilevel Quadratic-linear Programming Problems

Sanjeet Singh

Abstract— **In this paper, we develop a method to solve bilevel quadratic-linear programming problems in which the upper level objective function is quadratic and lower level objective function is linear. In this method, we replace the lower level problem by its duality gap equaling zero. The resulting bilevel quadratic linear programming problem can be transformed into a traditional single objective programming problem, which can be solved using a series of finite number of standard convex quadratic programming problems.**

Index Terms—**bilevel programming, linear programming, optimization, quadratic programming**

I. Introduction

In a free market economy economic decisions are made by individual economic agents (the producers and consumers), while policy makers only influence the behavior of these economic agents. Therefore, there are two types of decision makers (i) the policy maker and (ii) economic agents representing consumers and producers. Each decision maker has his own objective function and constraints. Thus, this type of system can be represented by a bilevel programming problem. The Bilevel programming problem is a class of optimization problems where two criteria are two be optimized with respect to the two levels of hierarchy. In these problems, there are decision makers at two levels i.e. upper level and lower level. Many real life problems, such as structural design [9], transportation problems [15], traffic signal optimization [11], network flows [10] and so on can be formulated into the hierarchy problems. In research literature several methods have been proposed [2, 5, 8, 16, 18] to solve bilevel programming problems such as interior point, transformation approach, intelligent computation, extreme point search, descent and heuristics. However, bilevel programming problems are intrinsically hard because these problems are neither convex nor continuous. Even the simplest case of linear bilevel programming problem (BLPP), where both upper and lower level objectives, and all constraints are linear, was shown to be NP-hard [2, 5, 14]. Therefore, bilevel programming problems are very complex because of their non-continuity and non-convexity nature, especially the nonlinear bilevel programming problems. Therefore, most researchers in the field have been focused to only special class of these problems or have kept their study limited to obtaining the local optimal solutions. The quadratic programming problem is a special type of nonlinear programming problems, which involves maximization/minimization of a quadratic function subject to linear constraints. Research literature is full of variety of applications of quadratic programming problems such as portfolio optimization [6], structural analysis [1], optimal control [3], classification [12].

In this paper, we develop an approach to solve a special class of nonlinear bilevel programming problem-bilevel quadratic-linear programming (BLQLP) which can be written as follows:

$$\begin{aligned} \min \quad & \mathrm{F}(x_1,x_2) = (x_1,x_2)^{\mathrm{T}} Q (x_1,x_2), \qquad \text{where } x_2 \text{ solves} \\ & \min_{x_2} \quad \mathrm{f}(x_1,x_2) = \mathrm{p}^{\mathrm{T}} x_1 + q^T x_2 \\ & \quad \text{subject to} \quad A_1 x_1 + A_2 x_2 \le b \\ & \qquad\qquad\qquad x_1, x_2 \ge 0 \end{aligned} \tag{1}$$

where $F(x_1,x_2)$ and $f(x_1,x_2)$ are the upper level's objective function and lower level's objective function of BLQLP, respectively; $x_1 \in \Re^{n_1}$ and $x_2 \in \Re^{n_2}$ are the decision variables controlled by the upper level and lower level decision maker, respectively; p is n_1-vector; q is n_2-vectors; A_1 is an $m \times n_1$ matrix; A_2 is an $m \times n_2$ matrix; b is an m-vector and Q is an $(n_1+n_2)\times(n_1+n_2)$ symmetric matrix with

$$Q = \begin{pmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{pmatrix}$$

where Q_1, Q_2 and Q_3 are matrices of conformal dimensions.

We assume that the polyhedron $S = \{(x_1, x_2) : A_1 x_1 + A_2 x_2 \le b, x_1, x_2 \ge 0\}$ is a non-empty and compact. In addition, it is also assumed that Q_1 is positive definite matrix. Let S_1 be the projection of S onto $\Re^{n_1}$.

II. Notations and Preliminaries

For some fixed $x_1 \ge 0$, $p^T x_1$ is constant. Therefore, $p^T x_1$ can be taken equal to zero to ignore this term without loss of generality while solving the lower level problem. Thus, the lower level linear programming problem can be simplified to the following problem:

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$$\begin{aligned}&\min_{x_2}\ \bar{f}(x_1,x_2)=q^T x_2\\&\text{subject to}\ \ A_2x_2\le b-A_1x_1\\&\qquad\qquad x_2\ge 0\end{aligned}\tag{2}$$

Next we give the following notations and definitions of BLQLP:

The feasible region of BLQLP is

$$S=\{(x_1,x_2)\,|\,A_1x_1+A_2x_2\le b,\ x_1,x_2\ge 0\}.$$

For some fixed $x_1\ge 0$, the feasible region of the lower level problem in BLQLP is

$$S(x_1)=\{x_2\ge 0\,|\,A_2x_2\le b-A_1x_1\}.$$

Projection of S onto the upper level decision space

$S(X_1)=\{x_1\ge 0\,|\,\text{there exist } x_2\ge 0, \text{such that } (x_1,x_2)\in S\}.$

For some fixed $x_1\in S(X_1),$ the rational reaction set of the lower level programming problem is

$$Y(x_1)=\{x_2\,|\,x_2\in\arg\min f(x_1,x_2),x_2\in S(x_1)\}.$$

The inducible region of BLQLP is

$$IR=\{(x_1,x_2)\,|\,(x_1,x_2)\in S,x_2\in S(x_1)\}.$$

Here, S is assumed to be nonempty and compact and $S(x_1)$ is assumed to be bounded and nonempty.

Definition 1 A point (x_1,x_2) is a feasible solution for BLQLP if $(x_1,x_2)\in IR.$

III. PROBLEM FORMULATION AND THEORETICAL DEVELOPMENTS

According to duality theorem, we can get the dual problem of the problem (2)

$$\begin{aligned}&\max\ (b-A_1x_1)^T y\\&\text{subject to}\ A_2^T y\ge q\\&\qquad y\ge 0\end{aligned}\tag{3}$$

Thus, we have the following theorem:

Theorem 1 (x_1^*,x_2^*) is the optimal solution to the problem (1) if and only if there exists y^* such that (x_1^*,x_2^*,y^*) is the solution to the following programming problem:

$$\begin{aligned}&\min\ \mathrm{F}(x_1,x_2)=(x_1,x_2)^{\mathrm{T}}Q(x_1,x_2)\\&\text{subject to}\ A_1x_1+A_2x_2\le b\\&\qquad A_2^T y\ge q\\&\qquad \mathrm{q}^{\mathrm{T}}x_2-(b-A_1x_1)^T y=0\\&\qquad x_1,x_2,y\ge 0.\end{aligned}\tag{4}$$

Proof. According to the duality theorem of the linear programming, it is obvious that there exists (x_1^*,x_2^*,y^*) such that $\mathrm{q}^{\mathrm{T}}x_2^*-(b-A_1x_2^*)^T y^*=0$, $A_1x_1^*+A_2x_2^*\le b$, $A_2^T y^*\ge q$ if and only x_2^* solve the problem (2) for the fixed $x_1^*\in S(X_1).$

From the above Theorem 1, it is clear that the original bilevel quadratic linear programming problem (1) can be transformed into the usual single objective programming problem (4). In the programming problem (4), all the constraints except $\mathrm{q}^{\mathrm{T}}x_2-(b-A_1x_1)^T y=0$ are linear. We can now solve a series of standard convex quadratic programming problems to solve (4) by relaxing this linear constraint.

$Y=\{y\,|\,A_2^T y\ge q,y\ge 0\}$ denote the feasible region of the linear programming problem (3). Then the following results are drawn from the theory of the linear programming:

Remark 1. The feasible region of the linear programming has at least one vertex and at most finite vertices if it is non empty [17].

Remark 2. If there exists an optimal solution to the linear programming, it must be at least one of the vertex of the feasible region [17].

From the above remark 1-2, there are finite vertices in the feasible region Y and we denote the set of vertices of Y by Y^{E} and the vertices of S by S^{E}.

Remark 3. There exists $(x_1^*,x_2^*,y^*)\in S^E\times Y^E$, which is the optimal solution to the problem (4), and

$$(S\times Y)^E=S^E\times Y^E.$$

Thus, we can obtain all vertices of Y, denoted by $Y^E=\{y^1,y^2,\cdots,y^r\},$ by using the linear programming [17]. Hence, we can transform the problem (4) into a series of quadratic programming problem with linear constraints formulated as follows:

$$\begin{aligned}
&(QP(y^i)) \quad \min \mathrm{F}(x_1, x_2) = (x_1, x_2)^{\mathrm{T}} Q(x_1, x_2) \\
&\text{subject to } A_1 x_1 + A_2 x_2 \le b \\
&\qquad A_2^T y \ge q \qquad\qquad (5) \\
&\qquad \mathrm{q}^{\mathrm{T}} x_2 - (b - A_1 x_1)^T y^i = 0 \\
&\qquad x_1, x_2 \ge 0.
\end{aligned}$$

Problem (5) can be solved easily using any standard method of solving the convex quadratic programming problems [4, 7, 13, 19].

Because S is compact and nonempty, there exist optimal solutions or none feasible solution to the problem $QP(y^i)$ for $i \in \{1,2,\cdots,r\}$. Let $\Psi \subseteq \{1,2,\cdots.r\}$, there exist optimal solution to the problem $QP(y^i)$ when $i \in \Psi$. There must be i such that the problem $QP(y^i)$ has an optimal solution, hence $\Psi \ne \phi$. For $j \in \Psi$, let (x_1^j, x_2^j) be the optimal solution to the problem $QP(y^j)$ and

$F(x_1^k, x_2^k) = \max\{F(x_1^j, x_2^j) \mid j \in \Psi\}$. Hence, we have the following conclusion.

Remark 4. (x_1^k, x_2^k) is an optimal solution to the problem (1).

Therefore, we can obtain the solution of BLQLP by solving a series of convex quadratic programming problems with linear constraints. Quadratic programming problem (5) can be solved using the Karush-Kuhn-Tucker (KKT) conditions or by any other method.

Next, we give the steps of our proposed algorithm to solve the problem BLQLP as follows:

Step 1 Generate all vertices $Y^E = \{y^1, y^2, \cdots, y^r\}$ using the linear programming.

Step 2 Solve the problem $QP(y^i)$ for $\{i = 1, 2, \cdots, r\}$ using simplex method proposed in [19]. If there is no feasible solution, let (0, 0) denote the optimal solution and $F_k = +\infty$ denote the optimal value, otherwise let (x_1^k, x_2^k) denote the optimal solution and $F_k = (x_1^k, x_2^k)$ denote the optimal value.

Step 3 Choose $F^* = \max\{F_k, k = 1, 2, \cdots, r\}$ then F^* will be the optimal value and corresponding (x_1^k, x_2^k) be the optimal solution (x_1^*, x_2^*) of BLQLP.

Step 4 If $F^* = +\infty$, then there is no feasible solution to the problem BLQLP; otherwise (x_1^*, x_2^*) is the solution to the problem BLQLP, and F^* is the optimal value of the upper level objective function in BLQLP.

IV. CONCLUSION

In this paper, we have developed an approach to solve bilevel quadratic-linear programming problem in which the upper level objective function is quadratic and lower level objective function is linear. In this method, we replace the lower level problem by its duality equaling zero. The resulting bilevel quadratic linear programming problem can be transformed into a traditional single objective programming problem, which can be solved using a series of finite number of standard convex quadratic programming problem. Method has been formalized in the form of step wise algorithm. Case of modification of the proposed method for the bilevel quadratic linear fractional programming problem will be discussed in the author's future research.

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