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Second-Order Duality for Nondifferentiable Minimax Fractional Programming Involving (F, ρ)-Convexity

S. K. Gupta and Debasis Dangar

***Abstract*—We focus our study to formulate two different types of second-order dual models for a nondifferentiable minimax fractional programming problem and derive duality theorems under (F, ρ)-convexity. Several results including many recent works are obtained as special cases.**

***Index Terms*—minimax fractional programming, nondifferentiable programming, second-order duality, (F, ρ)-convexity.**

I. Introduction

STUDY with nonlinear programming problems is an interesting research topic for recent researchers in the field of mathematical programming. One of the application of nonlinear programming is to be maximized or minimized ratio of two functions. Ratio optimization is commonly called fractional programming. Minimax optimization problems has been of considerable interest in the past few years due to some important applications in the design of electronic circuits, game theory, economics, best approximation theory, engineering design, portfolio selection problems etc. For the theory, algorithms and applications of some minimax problems, the reader is referred to [1]. In the last two decades, several authors have shown their interest in developing optimality conditions and various duality results for minimax fractional programming problems dealing with differentiable case in [2-9] and nondifferentiable case in [10-16].

Liu and Wu [3, 5], considered the following fractional minimax problem

$$\text{(P)} \quad \text{Minimize} \quad \psi(x) = \sup_{y\in Y} \frac{f(x,y)}{h(x,y)}$$
$$\text{subject to} \quad g(x) \leq 0,$$

where Y is a compact subset of R^m, $f(.,.) : R^n \times R^m \to R$, $h(.,.) : R^n \times R^m \to R$ are C^1 mapping on $R^n \times R^m$, $g(.) : R^n \to R^p$ is C^1 mapping on R^n, $f(x,y) \geq 0$ and $h(x,y) > 0$.

They established sufficient condition for (P) and derived duality theorems for three different dual models under (F, ρ)-convexity/invexity assumptions. Lai and Lee [11] constructed two types of dual models for nondifferentiable case among the models considered in [3, 5] and proved duality relations involving pseudo/quasi-convex functions. A Mond-Weir type dual of (P) for nondifferentiable case considered in Ahmad and Husain [14] and further appropriate duality results are proved involving generalized convex functions. Jayswal [16] proposed three types of dual models for (P), in which one of the model constructed in [14] and proved duality theorems under α-unvexity.

In recent few years researchers are working with various second-order dual models for (P). Husain et al. [6] constructed two types of second-order dual models for (P) and obtained duality relations using the η-bonvexity assumptions. These models were further generalized by Hu et al. [7] by introducing an additional vector r and proved duality results for η-bonvex functions. Recently, Sharma and Gulati [8] and Ahmad [9] established duality relations for two types of second-order dual models of (P) under α-type I univex/generalized convex functions. In this paper, we formulate two different types of second-order dual models of (P) for nondifferentiable case and further derive duality results for involving the functions to be (F, ρ)-convex.

II. Notations and definitions

Consider the following nondifferentiable minimax fractional programming problem:

$$\text{(P1)} \quad \text{Minimize} \quad \psi(x) = \sup_{y\in Y} \frac{f(x,y) + (x^T Bx)^{1/2}}{h(x,y) - (x^T Dx)^{1/2}}$$
$$\text{subject to} \quad g(x) \leq 0,$$

where Y is a compact subset of R^l, $f(.,.) : R^n \times R^l \to R$, $h(.,.) : R^n \times R^l \to R$ are twice continuously differentiable on $R^n \times R^l$ and $g(.) : R^n \to R^m$ is twice continuously differentiable on R^n, B and D are $n \times n$ positive semidefinite matrix, $f(x,y) + (x^T Bx)^{1/2} \geq 0$ and $h(x,y) - (x^T Dx)^{1/2} > 0$ for each $(x,y) \in J \times Y$, where $J = \{x \in R^n : g(x) \leq 0\}$.

For each $(x,y) \in R^n \times R^l$, we define

$$J(x) = \{j \in M = \{1,2,...,m\} : g_j(x) = 0\},$$

$$Y(x) = \left\{ y \in Y : \frac{f(x,y) + (x^T Bx)^{1/2}}{h(x,y) - (x^T Dx)^{1/2}} = \sup_{z\in Y} \frac{f(x,z) + (x^T Bx)^{1/2}}{h(x,z) - (x^T Dx)^{1/2}} \right\},$$

$$K(x) = \Big\{ (s,t,\widetilde{y}) \in N \times R^s_+ \times R^{ls} : 1 \leq s \leq n+1,\ t = (t_1, t_2, ..., t_s) \in R^s_+, \sum_{i=1}^{s} t_i = 1,\ \widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_s),\ \widetilde{y}_i \in$$

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$$Y(x),\ i=1,2,...,s\Big\}.$$

Let R^n be the n-dimensional Euclidean space and $X \subseteq R^n$. Now, we need the following definitions in sequel:

Definition 2.1 ([9,15]) A functional $F : X \times X \times R^n \mapsto R$ is said to be sublinear with respect to the third variable if for all $(x,z) \in X \times X$,
(i) $F(x,z;a_1+a_2) \leqq F(x,z;a_1) + F(x,z;a_2)$, for all $a_1, a_2 \in R^n$,
(ii) $F(x,z;\alpha a) = \alpha F(x,z;a)$, for all $\alpha \in R_+$ and for all $a \in R^n$.

The definition of F-convexity in nonlinear programming was first introduced by Hanson and Mond [17]. Another definition of convexity called ρ-covexity was given by Vial [18]. Motivated by these concepts, Preda [19] first considered (F,ρ)-convexity for multiobjective programs. Zhang and Mond [20] extended the class of (F,ρ)-convex functions to second order (F,ρ)-convex functions and proved duality results for Mangasarian type and Mond-Weir type multiobjective dual problems.

Definition 2.2 A twice differentiable function ψ_i over X is said to be second-order (F,ρ_i)-convex at u on X, if for all $x \in X$, there exists vector $r \in R^n$, a real valued function $d_i(.,.) : X \times X \to R$ and a real number ρ_i such that

$$\psi_i(x) - \psi_i(z) + \frac{1}{2} r^T \nabla_{xx}\psi_i(z) r \geqq F(x,z;\nabla_x \psi_i(z) + \nabla_{xx}\psi_i(z) r) + \rho_i d_i^2(x,z).$$

Lemma 2.1 *(Generalized Schwartz inequality) Let B be a positive semidefinite matrix of order n. Then, for all $x,\ w \in R^n$,*

$$x^T B w \leq (x^T B x)^{1/2} (w^T B w)^{1/2}.$$

The equality holds if $Bx = \lambda B w$ for some $\lambda \geq 0$.

Following Theorem 2.1 ([10], Theorem 3.1) will be required to prove strong duality theorems:

Theorem 2.1 *(Necessary condition) If x^* is an optimal solution of problem (P1) satisfying $x^{*T}Bx^* > 0$, $x^{*T}Dx^* > 0$, and $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent, then there exist $(s,t^*,\widetilde{y}) \in K(x^*)$, $\lambda^* \in R_+$, $w, v \in R^n$ and $\mu^* \in R_+^m$ such that*

$$\sum_{i=1}^{s} t_i^* \{\nabla f(x^*,\widetilde{y}_i) + Bw - \lambda^*(\nabla h(x^*,\widetilde{y}_i) - Dv)\} + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0, \tag{1}$$

$$f(x^*,\widetilde{y}_i) + (x^{*T}Bx^*)^{1/2} - \lambda^*(h(x^*,\widetilde{y}_i) - (x^{*T}Dx^*)^{1/2}) = 0,\quad i=1,2,...,s, \tag{2}$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \tag{3}$$

$$t_i^* \geq 0,\ (i=1,2,...,s),\qquad \sum_{i=1}^{s} t_i^* = 1, \tag{4}$$

$$w^T B w \leq 1,\ v^T D v \leq 1,\ (x^{*T}Bx^*)^{1/2} = x^{*T}Bw,\quad (x^{*T}Dx^*)^{1/2} = x^{*T}Dv. \tag{5}$$

In the above theorem, both matrices B and D are positive semidefinite at the solution x^*. If either $x^{*T}Bx^*$ or $x^{*T}Dx^*$ is zero, then the functions involved in the objective function of problem (P1) are not differentiable. To derive necessary conditions under this situation, for $(s,t^*,\widetilde{y}) \in K(x^*)$, we define

$$Z_{\widetilde{y}}(x^*) = \{z \in R^n : z^T \nabla g_j(x^*) \leq 0, j \in J(x^*),$$
$$\text{with any one of the next conditions } (i)\text{-}(iii) \text{ holds}\}.$$

(i) $x^{*T}Bx^* > 0,\ x^{*T}Dx^* = 0$

$$\Rightarrow z^T\left(\sum_{i=1}^{s} t_i^* \left\{\nabla f(x^*,\widetilde{y}_i) + \frac{Bx^*}{(x^{*T}Bx^*)^{1/2}} - \lambda^* \nabla h(x^*,\widetilde{y}_i)\right\}\right) + (z^T(\lambda^{*2}D)z)^{1/2} < 0,$$

(ii) $x^{*T}Bx^* = 0,\ x^{*T}Dx^* > 0$

$$\Rightarrow z^T\left(\sum_{i=1}^{s} t_i^* \left\{\nabla f(x^*,\widetilde{y}_i) - \lambda^*\left(\nabla h(x^*,\widetilde{y}_i) - \frac{Dx^*}{(x^{*T}Dx^*)^{1/2}}\right)\right\}\right) + (z^T B z)^{1/2} < 0,$$

(iii) $x^{*T}Bx^* = 0,\ x^{*T}Dx^* = 0$

$$\Rightarrow z^T\left(\sum_{i=1}^{s} t_i^* \{\nabla f(x^*,\widetilde{y}_i) - \lambda^* \nabla h(x^*,\widetilde{y}_i)\}\right) + (z^T(\lambda^{*2}D)z)^{1/2} + (z^T B z)^{1/2} < 0.$$

If in addition, we insert the condition $Z_{\widetilde{y}}(x^*) = \phi$, then the result of Theorem 2.1 still holds.

III. DUALITY MODEL I

In this section, we consider the following dual problem to (P1):

$$\text{(DM1)} \qquad \max_{(s,t,\widetilde{y}) \in K(z)} \ \sup_{(z,\mu,w,v,p) \in H_1(s,t,\widetilde{y})} F(z),$$

where $F(z) = \sup_{y \in Y} \dfrac{f(z,y) + (z^T B z)^{1/2}}{h(z,y) - (z^T D z)^{1/2}}$ and $H_1(s,t,\widetilde{y})$ denotes the set of all

$(z,\mu,w,v,p) \in R^n \times R_+^m \times R^n \times R^n \times R^n$ satisfying

$$\sum_{i=1}^{s} t_i\{(\nabla f(z,\widetilde{y}_i) + Bw)(h(z,\widetilde{y}_i) - (z^T D z)^{1/2}) - (\nabla h(z,\widetilde{y}_i) - Dv)(f(z,\widetilde{y}_i) + (z^T B z)^{1/2})\} + \sum_{i=1}^{s} t_i\{(h(z,\widetilde{y}_i) - (z^T D z)^{1/2}) \nabla^2 f(z,\widetilde{y}_i) - (f(z,\widetilde{y}_i) + (z^T B z)^{1/2}) \nabla^2 h(z,\widetilde{y}_i)\} p + \sum_{j=1}^{m} \mu_j \nabla g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0. \tag{6}$$

$$\sum_{j=1}^{m}\mu_j g_j(z)-\frac{1}{2}p^T\Bigg(\sum_{i=1}^{s}t_i\{(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})$$
$$\nabla^2 f(z,\widetilde{y}_i)-(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})\nabla^2 h(z,\widetilde{y}_i)\}+$$
$$\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)\Bigg)p\geq 0 \quad (7)$$

$$w^TBw\leq 1,\ v^TDv\leq 1,\ (z^TBz)^{1/2}=z^TBw,$$
$$(z^TDz)^{1/2}=z^TDv. \quad (8)$$

If the set $H_1(s,t,\widetilde{y})=\phi$, we define the supremum of $F(z)$ over $H_1(s,t,\widetilde{y})$ equal to $-\infty$. Let

$$\phi_1(.)=\sum_{i=1}^{s}t_i[(h(z,\widetilde{y}_i)-z^TDv)(f(.,\widetilde{y}_i)+(.)^TBw)-$$
$$(f(z,\widetilde{y}_i)+z^TBw)(h(.,\widetilde{y}_i)-(.)^TDv)]$$

Theorem 3.1 (Weak Duality) *Let x and $(z,\mu,w,v,s,t,\widetilde{y},p)$ are feasible solutions of (P1) and (DM1) respectively. Assume that $f(.,\widetilde{y}_i)+(.)^TBw$ and $-h(.,\widetilde{y}_i)+(.)^TDv$ are second-order (F,ρ_i) and (F,ρ_i')-convex, $i=1,2,...,s$, respectively at z. Also let $g_j(.)$ be second-order (F,γ_j)-convex at z, $j=1,2,...,m$ and*

$$\sum_{i=1}^{s}t_i\Bigg\{(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\rho_i d_i^2(x,z)+(f(z,\widetilde{y}_i)+$$
$$(z^TBz)^{1/2})\rho_i'(d_i'(x,z))^2\Bigg\}+\sum_{j=1}^{m}\mu_j\gamma_j c_j^2(x,z)\geq 0. \quad (9)$$

Then

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}\geq F(z).$$

Proof Suppose to the contrary

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}< F(z). \quad (10)$$

Since $\widetilde{y}_i\in Y(z)$, $i=1,2,...,s$, we have

$$F(z)=\frac{f(z,\widetilde{y}_i)+(z^TBz)^{1/2}}{h(z,\widetilde{y}_i)-(z^TDz)^{1/2}}. \quad (11)$$

From (10) and (11), we have

$$\frac{f(x,\widetilde{y}_i)+(x^TBx)^{1/2}}{h(x,\widetilde{y}_i)-(x^TDx)^{1/2}}\leq\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}$$
$$<\frac{f(z,\widetilde{y}_i)+(z^TBz)^{1/2}}{h(z,\widetilde{y}_i)-(z^TDz)^{1/2}}.$$

$$\Rightarrow[(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})(f(x,\widetilde{y}_i)+(x^TBx)^{1/2})-$$
$$(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})(h(x,\widetilde{y}_i)-(x^TDx)^{1/2})]<0$$

As $t_i\geq 0$, $i=1,2,...,s$, $t\neq 0$ and $\widetilde{y}_i\in Y(z)$, from above, we have

$$\sum_{i=1}^{s}t_i[(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})(f(x,\widetilde{y}_i)+(x^TBx)^{1/2})-$$
$$(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})(h(x,\widetilde{y}_i)-(x^TDx)^{1/2})]<0 \quad (12)$$

This together with Lemma 2.1 and (8) give

$$\phi_1(x)=\sum_{i=1}^{s}t_i[(h(z,\widetilde{y}_i)-z^TDv)(f(x,\widetilde{y}_i)+x^TBw)-$$
$$(f(z,\widetilde{y}_i)+z^TBw)(h(x,\widetilde{y}_i)-x^TDv)]$$
$$\leq\sum_{i=1}^{s}t_i[(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})(f(x,\widetilde{y}_i)+(x^TBx)^{1/2})-$$
$$(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})(h(x,\widetilde{y}_i)-(x^TDx)^{1/2})]$$
$$<0=\phi_1(z)$$

Hence,

$$\phi_1(x)<\phi_1(z) \quad (13)$$

Since $f(.,\widetilde{y}_i)+(.)^TBw$ and $-h(.,\widetilde{y}_i)+(.)^TDv$ are second-order (F,ρ_i) and (F,ρ_i')-convex, $i=1,2,...,s$, respectively at z and $g_j(.)$ is second-order (F,γ_j)-convex, $j=1,2,...,m$ at z. Therefore, we have

$$f(x,\widetilde{y}_i)+x^TBw-(f(z,\widetilde{y}_i)+z^TBw)+\frac{1}{2}p^T\nabla^2 f(z,\widetilde{y}_i)p$$
$$\geq F\Bigg(x,z;\{\nabla f(z,\widetilde{y}_i)+Bw+\nabla^2 f(z,\widetilde{y}_i)p\}\Bigg)+$$
$$\rho_i d_i^2(x,z), \quad (14)$$

$$-h(x,\widetilde{y}_i)+x^TDv+h(z,\widetilde{y}_i)-z^TDv-\frac{1}{2}p^T\nabla^2 h(z,\widetilde{y}_i)p$$
$$\geq F\Bigg(x,z;\{-\nabla h(z,\widetilde{y}_i)+Dv-\nabla^2 h(z,\widetilde{y}_i)p\}\Bigg)+$$
$$\rho_i'(d_i'(x,z))^2 \quad (15)$$

$$-g_j(z)+\frac{1}{2}p^T\nabla^2 g_j(z)p$$
$$\geq F(x,z;\{\nabla g_j(z)+\nabla^2 g_j(z)p\})+\gamma_j c_j^2(x,z) \quad (16)$$

Multiplying (14) by $t_i[h(z,\widetilde{y}_i)-(z^TDz)^{1/2}]$ and (15) by $t_i[f(z,\widetilde{y}_i)+(z^TBz)^{1/2}]$, $i=1,2,...,s$ and then summing up these inequalities and using (8) and sublinearity of F, we get

$$\phi_1(x)-\phi_1(z)+\Bigg\{\frac{1}{2}p^T\sum_{i=1}^{s}t_i\{(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\nabla^2 f(z,\widetilde{y}_i)$$
$$-(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})\nabla^2 h(z,\widetilde{y}_i)\}\Bigg\}p$$
$$\geq F\Bigg(x,z;\sum_{i=1}^{s}t_i\{(\nabla f(z,\widetilde{y}_i)+Bw)(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})-$$
$$(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})(\nabla h(z,\widetilde{y}_i)-Dv)+((h(z,\widetilde{y}_i)-$$
$$(z^TDz)^{1/2})\nabla^2 f(z,\widetilde{y}_i)-(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})$$
$$\nabla^2 h(z,\widetilde{y}_i))p\}\Bigg)+\sum_{i=1}^{s}t_i\{(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\rho_i d_i^2(x,z)$$
$$+(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})\rho_i'(d_i'(x,z))^2\}. \quad (17)$$

As $\mu_j\geq 0$, $j=1,2,...,m$, from (16) and sublinearity of F, we have

$$-\sum_{j=1}^{m}\mu_j g_j(z)+\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\geq F\Bigg(x,z;\sum_{j=1}^{m}\mu_j($$

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$$\nabla g_j(z)+\nabla^2 g_j(z)p)\Big)+\sum_{j=1}^{m}\mu_j\gamma_j c_j^2(x,z). \quad (18)$$

Now, adding (17) and (18), using (6)-(8), (19) and sublinearity of F, we get

$$\phi_1(x)-\phi_1(z)\geq F\Bigg(x,z;\sum_{i=1}^{s}t_i\{(\nabla f(z,\widetilde{y}_i)+Bw)(h(z,\widetilde{y}_i)$$
$$-(z^TDz)^{1/2})-(\nabla h(z,\widetilde{y}_i)-Dv)(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})\}+$$
$$\sum_{i=1}^{s}t_i\{(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\nabla^2 f(z,\widetilde{y}_i)-(f(z,\widetilde{y}_i)+$$
$$(z^TBz)^{1/2})\nabla^2 h(z,\widetilde{y}_i)\}p+\sum_{j=1}^{m}\mu_j\nabla g_j(z)+$$
$$\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg)=0$$

which contradicts (13). Thus the theorem proved. □

Theorem 3.2 (Strong Duality) *Let x^* be an optimal solution for (P1) and let $\nabla g_j(x^*)$, $j\in J(x^*)$ be linearly independent. Then, there exist $(s^*,t^*,\widetilde{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,w^*,v^*,p^*=0)\in H_1(s^*,t^*,\widetilde{y}^*)$ such that $(x^*,\mu^*,\lambda^*,w^*,v^*,s^*,t^*,\widetilde{y}^*,p^*=0)$ is feasible solution of (DM1) and the two objectives have same values. If, in addition, the assumption of weak duality hold for all feasible solutions $(x,\mu,\lambda,w,v,s,t,\widetilde{y},p)$ of (DM1), then $(x^*,\mu^*,\lambda^*,w^*,v^*,s^*,t^*,\widetilde{y}^*,p^*=0)$ is an optimal solution of (DM1).*

Proof. Since x^* is an optimal solution of (P1) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, then by Theorem 2.1 , there exist $(s^*,t^*,\widetilde{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,w^*,v^*,p^*=0)\in H_1(s^*,t^*,\widetilde{y}^*)$ such that $(x^*,\mu^*,\lambda^*,w^*,v^*,s^*,t^*,\widetilde{y}^*,p^*=0)$ is feasible solution of (DM1) and the two objectives have same values. Optimality of $(x^*,\mu^*,\lambda^*,w^*,v^*,s^*,t^*,\widetilde{y}^*,p^*=0)$ for (DM1), thus follows from weak duality Theorem 3.1. □

Theorem 3.3 (Strict Converse Duality) *Let x^* be optimal solution to (P1) and $(z^*,\mu^*,\lambda^*,w^*,v^*,s,t^*,\widetilde{y}^*,p^*)$ be optimal solution to (DM1). Assume that $f(.,\widetilde{y}_i^*)+(.)^TBw^*$ and $-h(.,\widetilde{y}_i^*)+(.)^TDv^*$ are second-order (F,ρ_i) and (F,ρ_i')-convex, $i=1,2,...,s$ at z^*, respectively. Also let $g_j(.)$ be second-order (F,γ_j)-convex at z^*, $j=1,2,...,m$ and let*

$$\sum_{i=1}^{s}t_i^*\{(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})\rho_i d_i^2(x^*,z^*)+$$
$$(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})\rho_i'(d_i'(x^*,z^*))^2\}+$$
$$\sum_{j=1}^{m}\mu_j^*\gamma_j c_j^2(x^*,z^*)\geq 0 \quad (19)$$

holds and let $\{\nabla g_j(x^*),\ j\in J(x^*)\}$, are linearly independent. Then $z^*=x^*$.

Proof We shall assume that $z^*\neq x^*$ and reach a contradiction. Since x^* and $(z^*,\mu^*,\lambda^*,w^*,v^*,s,t^*,\widetilde{y}^*,p^*)$ be optimal solutions to (P1) and (DM1), respectively and $\{\nabla g_j(x^*),\ j\in J(x^*)\}$, are linearly independent, by strong duality theorem, we get

$$\sup_{\widetilde{y}^*\in Y}\frac{f(x^*,\widetilde{y}^*)+(x^{*T}Bx^*)^{1/2}}{h(x^*,\widetilde{y}^*)-(x^{*T}Dx^*)^{1/2}}=F(z^*).$$

Therefore, for $\widetilde{y}_i^*\in Y(z^*)$, we have

$$\frac{f(x^*,\widetilde{y}_i^*)+(x^{*T}Bx^*)^{1/2}}{h(x^*,\widetilde{y}_i^*)-(x^{*T}Dx^*)^{1/2}}\leq\sup_{\widetilde{y}^*\in Y}\frac{f(x^*,\widetilde{y}^*)+(x^{*T}Bx^*)^{1/2}}{h(x^*,\widetilde{y}^*)-(x^{*T}Dx^*)^{1/2}}$$
$$=\frac{f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2}}{h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2}}.$$

which from $t_i^*\geq 0$, $i=1,2,...,s$ and $t^*\neq 0$ follows that

$$\sum_{i=1}^{s}t_i^*[(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})(f(x^*,\widetilde{y}_i^*)+(x^{*T}Bx^*)^{1/2})-$$
$$(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})(h(x^*,\widetilde{y}_i^*)-(x^{*T}Dx^*)^{1/2})]<0.$$

This together with Lemma 2.1 and (8) give

$$\phi_1(x^*)=\sum_{i=1}^{s}t_i^*[(h(z^*,\widetilde{y}_i^*)-z^{*T}Dv^*)(f(x^*,\widetilde{y}_i^*)+x^{*T}Bw^*)-$$
$$(f(z^*,\widetilde{y}_i^*)+z^{*T}Bw^*)(h(x^*,\widetilde{y}_i^*)-x^{*T}Dv^*)]$$
$$\leq\sum_{i=1}^{s}t_i^*[(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})(f(x^*,\widetilde{y}_i^*)+(x^{*T}Bx^*)^{1/2})$$
$$-(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})(h(x^*,\widetilde{y}_i^*)-(x^{*T}Dx^*)^{1/2})]$$
$$<0=\phi_1(z^*)$$

Therefore

$$\phi_1(x^*)<\phi_1(z^*). \quad (20)$$

Since, $f(.,\widetilde{y}_i^*)+(.)^TBw^*$ and $-h(.,\widetilde{y}_i^*)+(.)^TDv^*$ are second-order (F,ρ_i) and (F,ρ_i')-convex, $i=1,2,...,s$ at z^*, respectively, we have

$$f(x^*,\widetilde{y}_i^*)+x^{*T}Bw^*-(f(z^*,\widetilde{y}_i^*)+z^{*T}Bw^*)+$$
$$\frac{1}{2}p^{*T}\nabla^2 f(z^*,\widetilde{y}_i^*)p^*\geq F\Big(x^*,z^*;\{\nabla f(z^*,\widetilde{y}_i^*)+$$
$$Bw^*+\nabla^2 f(z^*,\widetilde{y}_i^*)p^*\}\Big)+\rho_i d_i^2(x^*,z^*), \quad (21)$$
$$-h(x^*,\widetilde{y}_i^*)+x^{*T}Dv^*+h(z^*,\widetilde{y}_i^*)-z^{*T}Dv^*-$$
$$\frac{1}{2}p^{*T}\nabla^2 h(z^*,\widetilde{y}_i^*)p^*\geq F\Big(x^*,z^*;\{-\nabla h(z^*,\widetilde{y}_i^*)$$
$$+Dv^*-\nabla^2 h(z^*,\widetilde{y}_i^*)p^*\}\Big)+\rho_i'(d_i'(x^*,z^*))^2. \quad (22)$$

Multiplying (21) by $t_i^*[h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2}]$ and (22) by $t_i^*[f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2}]$, $i=1,2,...,s$ and then summing up these inequalities and using (8) and sublinearity of F, we obtain

$$\phi_1(x^*)-\phi_1(z^*)+\frac{1}{2}p^{*T}\sum_{i=1}^{s}t_i^*\{(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})$$
$$\nabla^2 f(z^*,\widetilde{y}_i^*)-(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})\nabla^2 h(z^*,\widetilde{y}_i^*)\}p^*$$
$$\geq F\Bigg(x^*,z^*;\sum_{i=1}^{s}t_i^*\{(\nabla f(z^*,\widetilde{y}_i^*)+Bw^*)(h(z^*,\widetilde{y}_i^*)-$$

$$(z^{*T}Dz^*)^{1/2})-(\nabla h(z^*,\widetilde{y}_i^*)-Dv^*)(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2}$$

$$)\}+\sum_{i=1}^{s}t_i^*\{(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})\nabla^2 f(z^*,\widetilde{y}_i^*)-$$

$$(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})\nabla^2 h(z^*,\widetilde{y}_i^*)\}p^*\Bigg)+$$

$$\sum_{i=1}^{s}t_i^*\{(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})\rho_i d_i^2(x^*,z^*)+(f(z^*,\widetilde{y}_i^*)+$$

$$(z^{*T}Bz^*)^{1/2})\rho_i'(d_i'(x^*,z^*))^2\}. \quad (23)$$

Also, the second-order (F,γ_j)-convexity of $g_j(.)$, $j = 1,2,...,m$ at z^*, $\mu_j^* \geq 0$ and feasibility of x^* give

$$-\sum_{j=1}^{m}\mu_j^* g_j(z^*)+\frac{1}{2}p^{*T}\nabla^2\sum_{j=1}^{m}\mu_j^* g_j(z^*)p^*$$

$$\geq F\Bigg(x^*,z^*;\sum_{j=1}^{m}\mu_j^*(\nabla g_j(z^*)+\nabla^2 g_j(z^*)p^*)\Bigg)+$$

$$\sum_{j=1}^{m}\mu_j^*\gamma_j c_j^2(x^*,z^*). \quad (24)$$

Finally, adding (23) and (24) and using (7) and (19), we obtain

$$\phi_1(x^*)-\phi_1(z^*)\geq F\Bigg(x^*,z^*;\sum_{i=1}^{s}t_i^*\{(\nabla f(z^*,\widetilde{y}_i^*)+Bw^*)$$

$$(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})-(\nabla h(z^*,\widetilde{y}_i^*)-Dv^*)(f(z^*,\widetilde{y}_i^*)$$

$$+(z^{*T}Bz^*)^{1/2})\}+\sum_{i=1}^{s}t_i^*\{(h(z^*,\widetilde{y}_i^*)-(z^{*T}Dz^*)^{1/2})$$

$$\nabla^2 f(z^*,\widetilde{y}_i^*)-(f(z^*,\widetilde{y}_i^*)+(z^{*T}Bz^*)^{1/2})\nabla^2 h(z^*,\widetilde{y}_i^*)\}p^*$$

$$+\sum_{j=1}^{m}\mu_j\nabla g_j(z^*)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z^*)p^*\Bigg),$$

which further from (6) implies

$$\phi_1(x^*)\geq\phi_1(z^*).$$

This contradicts (20). Hence the result. □

IV. Duality Model II

In this section, we consider the following dual problem to (P1):

(DM2) $\max\limits_{(s,t,\widetilde{y})\in K(z)} \sup\limits_{(z,\mu,\lambda,w,v,p)\in H_2(s,t,\widetilde{y})} L,$

where $L = \dfrac{\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+\sum_{j=1}^{m}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})}$
and $H_2(s,t,\widetilde{y})$ denotes the set of all $(z,\mu,\lambda,w,v,p) \in R^n\times R_+^m\times R_+\times R^n\times R^n\times R^n$ satisfying

$$\nabla\Bigg(\frac{\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+\sum_{j=1}^{m}\mu_j g_j(z)}{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})}\Bigg)+$$

$$\frac{1}{(\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2}))^2}\Bigg\{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-$$

$$(z^TDz)^{1/2})\Bigg[\sum_{i=1}^{s}t_i\nabla^2 f(z,\widetilde{y}_i)+\sum_{j=1}^{m}\mu_j\nabla^2 g_j(z)\Bigg]-$$

$$\sum_{i=1}^{s}t_i\nabla^2 h(z,\widetilde{y}_i)\Bigg[\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+$$

$$\sum_{j=1}^{m}\mu_j g_j(z)\Bigg]\Bigg\}p=0 \quad (25)$$

$$\frac{1}{2}p^T\Bigg\{\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\Bigg[\sum_{i=1}^{s}t_i\nabla^2 f(z,\widetilde{y}_i)+$$

$$\sum_{j=1}^{m}\mu_j\nabla^2 g_j(z)\Bigg]-\sum_{i=1}^{s}t_i\nabla^2 h(z,\widetilde{y}_i)\Bigg[\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+$$

$$(z^TBz)^{1/2})+\sum_{j=1}^{m}\mu_j g_j(z)\Bigg]\Bigg\}p\leq 0 \quad (26)$$

$$w^TBw\leq 1,\ v^TDv\leq 1,\ (z^TBz)^{1/2}=z^TBw,$$
$$(z^TDz)^{1/2}=z^TDv. \quad (27)$$

If the set $H_2(s,t,\widetilde{y})$ is empty, we define the supremum in (DM2) over $H_2(s,t,\widetilde{y})$ equal to $-\infty$. We use the notation

$$\phi_2(.)=\Bigg[\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-z^TDv)\Bigg]\Bigg[\sum_{i=1}^{s}t_i(f(.,\widetilde{y}_i)+(.)^TBw)$$

$$+\sum_{j=1}^{m}\mu_j g_j(.)\Bigg]-\Bigg[\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+z^TBw)+\sum_{j=1}^{m}\mu_j g_j(z)\Bigg]$$

$$\Bigg[\sum_{i=1}^{s}t_i(h(.,\widetilde{y}_i)-(.)^TDv)\Bigg]$$

Theorem 4.1 (Weak Duality) *Let x and $(z,\mu,\lambda,w,v,s,t,\widetilde{y},p)$ are feasible solutions of (P1) and (DM2) respectively. Suppose that $\phi_2(.)$ is second-order (F,ρ)-convex at z and $\rho_1\geq 0$. Then*

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}\geq L$$

Proof Assume on contrary to the result that

$$\sup_{\widetilde{y}\in Y}\frac{f(x,\widetilde{y})+(x^TBx)^{1/2}}{h(x,\widetilde{y})-(x^TDx)^{1/2}}< L$$

or $(f(x,\widetilde{y}_i)+(x^TBx)^{1/2})\Bigg[\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-(z^TDz)^{1/2})\Bigg]$

$$<(h(x,\widetilde{y}_i)-(x^TDx)^{1/2})\Bigg[\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+(z^TBz)^{1/2})+$$

$$\sum_{j=1}^{m}\mu_j g_j(z)\Bigg],\quad\forall\ \widetilde{y}_i\in Y(z),\ i=1,2,...,s$$

Using $t_i\geq 0,\ i=1,2,...,s$ and (27) in above, we have

$$\sum_{i=1}^{s}t_i(f(x,\widetilde{y}_i)+(x^TBx)^{1/2})\Bigg[\sum_{i=1}^{s}t_i(h(z,\widetilde{y}_i)-z^TDv)\Bigg]$$

$$<\sum_{i=1}^{s}t_i(h(x,\widetilde{y}_i)-(x^TDx)^{1/2})\Bigg[\sum_{i=1}^{s}t_i(f(z,\widetilde{y}_i)+z^TBw)$$

$$+\sum_{j=1}^{m}\mu_j g_j(z)\Bigg],$$

which further from Lemma 2.1 and (27) gives

$$\phi_2(x) \leq \left[\sum_{i=1}^{s} t_i(f(x,\widetilde{y}_i)+(x^T Bx)^{1/2})+\sum_{j=1}^{m}\mu_j g_j(x)\right]$$

$$\sum_{i=1}^{s} t_i(h(z,\widetilde{y}_i) - z^T Dv) - \sum_{i=1}^{s} t_i(h(x,\widetilde{y}_i) - (x^T Dx)^{1/2})$$

$$\left[\sum_{i=1}^{s} t_i(f(z,\widetilde{y}_i) + z^T Bw) + \sum_{j=1}^{m}\mu_j g_j(z)\right]$$

$$< \sum_{i=1}^{s} t_i(h(z,\widetilde{y}_i) - z^T Dv)\sum_{j=1}^{m}\mu_j g_j(x)$$

Since $\sum_{i=1}^{s} t_i(h(z,\widetilde{y}_i) - z^T Dv) > 0$ and $\sum_{j=1}^{m}\mu_j g_j(x) \leq 0$, it follows that

$$\phi_2(x) < 0 = \phi_2(z) \tag{28}$$

Now, by the second-order (F,ρ)-convexity of $\phi_2(.)$ at z, we get

$$\phi_2(x) - \phi_2(z) + \frac{1}{2}p^T\nabla^2\phi_2(z)p \geq F(x,z;\{\nabla\phi_2(z)+$$

$$\nabla^2\phi_2(z)p\}) + \rho_1 d^2(x,z),$$

which by (26) and (27) yield

$$\phi_2(x) - \phi_2(z) \geq F(x,z;\nabla\phi_2(z)+\nabla^2\phi_2(z)p)+\rho_1 d^2(x,z).$$

Finally, using (25), (27), $\rho_1 \geq 0$ and sublinearity of F in above, we have

Hence, $\phi_2(x) \geq \phi_2(z)$,

which contradicts (28). This proves the theorem. □

By a similar way, we can proof the following theorems between (P1) and (DM2):

Theorem 4.2 (Strong Duality) *Let x^* be an optimal solution for (P1) and let $\nabla g_j(x^*)$, $j \in J(x^*)$ be linearly independent. Then, there exist $(s^*, t^*, \widetilde{y}^*) \in K(x^*)$ and $(x^*, \mu^*, \lambda^*, w^*, v^*, p^* = 0) \in H_2(s^*, t^*, \widetilde{y}^*)$ such that $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is feasible solution of (DM2) and the two objectives have same values. If, in addition, the assumption of weak duality hold for all feasible solutions $(x, \mu, \lambda, w, v, s, t, \widetilde{y}, p)$ of (DM2), then $(x^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^* = 0)$ is an optimal solution of (DM2).*

Theorem 4.3 (Strict Converse Duality) *Let x^* be optimal solution to (P1) and $(z^*, \mu^*, \lambda^*, w^*, v^*, s^*, t^*, \widetilde{y}^*, p^*)$ be optimal solution to (DM2). Let $\phi_2(.)$ be second-order (F,ρ)-convex at z^* and $\rho_2 \geq 0$. Further, assume that $\{\nabla g_j(x^*)$, $j \in J(x^*)\}$ are linearly independent. Then $z^* = x^*$, that is z^* is an optimal solution to (P1).*

V. Special cases

(i) If $p = 0$, then the dual model (DM1) reduce to the problems studied in [11,12,15,16].

(ii) If $p = 0$, then the model (DM2) becomes the dual model considered in [11,12,14,16].

(iii) If $p = 0$ and B and D are zero matrices of order n, then (DM1) and (DM2) reduce to the problems studied in [5].

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