Path Embedding in a Faulty Folded Hypercube

Jung-Sheng Fu and Ping-Che Chung

Abstract—Let $F$ denote the faulty vertices in an $n$-dimensional folded hypercube $FQ_n$. In this paper, we show that $FQ_n$ contains a fault-free path with length of at least $2^{n-2|F|}-1$ (respectively, $2^{n-2|F|}-2$) between two arbitrary vertices $x$ and $y$ of odd (respectively, even) Hamming distance in $FQ_n-F$ if $|F| \leq n-1$, where $n \geq 3$. Since $FQ_n$ is $(n+1)$-regular and is bipartite when $n$ is odd, both the number of faults tolerated and the length of a longest fault-free path obtained are worst-case optimal.

Index Terms—Folded hypercubes; bipartite graph; fault tolerant embedding; hypercube; interconnection network

I. INTRODUCTION

The hypercube is one of the most versatile and efficient interconnection networks (networks for short) discovered to date for parallel computation. The hypercube is ideally suited to both special-purpose and general-purpose tasks, and can efficiently simulate many other same sized networks [15]. We usually use $Q_n$ to denote an $n$-dimensional hypercube. Many variants of the hypercube have been proposed. One variant is the folded hypercube [1]. An $n$-dimensional folded hypercube, denoted by $FQ_n$, is an extension of $Q_n$, constructed by adding a link to every pair of nodes with complementary addresses. The folded hypercube is superior to the hypercube in many measurements, such as diameter, fault diameter, connectivity, and so on (see [1], [22]). Previous works relating to the folded hypercube can be found in [1], [4], [8], [10], [11], [12], [13], [16], [17], [18], [19], [22], [24], [25], [26].

An embedding of one guest graph $G$ into another host graph $H$ is a one-to-one mapping $f$ from the node set of $G$ to the node set of $H$ [15]. An edge of $G$ corresponds to a path of $H$ under $f$. Linear arrays and rings, which are two of the most fundamental networks for parallel and distributed computation, are suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on linear arrays and rings for solving various algebraic problems and graph problems can be found in [15]. Linear arrays and rings can also be used as control/data flow structures for distributed computation in arbitrary networks. All of these motivate the embedding of linear arrays and rings in networks.

The fault-tolerant problem has been one of the most important studies on interconnection networks since faults may happen when a network put into use. Some results of fault-tolerant embedding on $Q_n$ or $FQ_n$ can be found in [2], [3], [5], [6], [7], [8], [9], [10], [12], [13], [14], [20], [21], [22], [23]. Let $F$ denote the faulty vertices in an $n$-dimensional folded hypercube $FQ_n$. In this paper, we show that $FQ_n$ contains a fault-free path with length of at least $2^{n-2|F|}-1$ (respectively, $2^{n-2|F|}-2$) between two arbitrary vertices $x$ and $y$ of odd (respectively, even) Hamming distance in $FQ_n-F$ if $|F| \leq n-1$, where $n \geq 3$. Since $FQ_n$ is $(n+1)$-regular and bipartite when $n$ is odd [24], both the number of faults tolerated and the length of a longest fault-free path obtained are worst-case optimal.

II. PRELIMINARIES

Let $G$ be a graph and let $u, v \in V(G)$. We use $(u, v)$ to denote an edge whose endpoints are $u$ and $v$. A path $P_{[x_0, x_1]} = \langle x_0, x_1, \ldots, x_t \rangle$ is a sequence of nodes such that two consecutive nodes are adjacent. Moreover, a path $\langle x_0, x_1, \ldots, x_t \rangle$ may contain other subpaths, denoted as $\langle x_0, x_1, \ldots, y, \ldots, x_t \rangle$, $\langle x_{i0}, x_{i1}, \ldots, x_{ij} \rangle$, where $P_{[x_i, x_j]} = \langle x_{i0}, x_{i1}, \ldots, x_{ij} \rangle$. A cycle is a path with $x_0 = x_t$ and $t \geq 3$.

An $n$-cube is an undirected graph with $2^n$ nodes each labeled with a distinct binary string $b_1b_2b_3 \ldots b_n$. Nodes $b_1 \ldots b_n$ and $b_1' \ldots b_n'$ are joined by an edge along dimension $i$, where $1 \leq i \leq n$ and $b_i'$ represents the one complement of $b_i$. Moreover, suppose $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$. In the rest of the paper, $x^{(i)}$ is used to denote the binary string $x_1 \ldots \widehat{x}_i \ldots x_n$ and $d_0(x, y)$ is used to denote the Hamming distance between $x$ and $y$, namely, the number of different bits between $x$ and $y$.

An $n$-dimensional folded hypercube $FQ_n$ is $Q_n$ augmented by adding more links among its nodes. More specifically, $FQ_n$ is obtained by adding a link between two nodes whose addresses are complementary to each other in $Q_n$, i.e., for a node whose address is $b = b_1b_2 \ldots b_n$, it has one more link to connect to node $\overline{b} = \overline{b}_1\overline{b}_2 \ldots \overline{b}_n$, in addition to its original $n$ links. So $FQ_n$ has $2^n-1$ more links than a regular links. Fig. 1 illustrates a 2-dimensional and a 3-dimensional folded hypercubes.

Fig. 1. The topologies of (a) $FQ_2$ and (b) $FQ_3$. 

Manuscript received Nov. 21, 2011; revised Dec. 08, 2011. This work was supported in part by National Science Council of the Republic of China, Taiwan under under Contract No. NSC 100-2221-E-239-024.

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ISBN: 978-988-19251-1-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)
Conveniently, $F_{Q_n}$ can be represented with $\star \star \ldots \star = x^*$, where $* \in \{0, 1\}$ means “don’t care”. Hence, $x^* 1$ and $x^*0$, which contain the nodes with rightmost bits 1 and 0, respectively. An $j$-partition of $F_{Q_n}$ is $x^*$ partitions $F_{Q_n}$ along dimension $j$ for some $j \in \{1, 2, \ldots, n\}$ into two subcubes $Q_{n,j}^0 = x^*0^*x^*$ and $Q_{n,j}^1 = x^*1^*x^*$, where $Q_{n,j}^0$ (respectively, $Q_{n,j}^1$) is the subgraph of $F_{Q_n}$ induced by $\{b_1b_2\ldots b_n \in V(F_{Q_n}) | b_1 = 0\}$ (respectively, $\{b_1b_2\ldots b_n \in V(F_{Q_n}) | b_1 = 1\}$). Note that $Q_{n,j}^0$ and $Q_{n,j}^1$ are isomorphic to an $(n-1)$-cube $Q_{n-1}$. Note that, for each vertex $b$ in $Q_{n,j}^0$ (respectively, $Q_{n,j}^1$), there are two nodes $b^0$ and $\bar{b}$ in $Q_{n,j}^1$ (respectively, $Q_{n,j}^0$) adjacent to $b$.

The following lemmas, which were shown in [6], [7], will be used in the following section.

**Lemma 1.** [6] Let $F \subset V(Q_n)$ denote the faulty vertices of $Q_n$, where $|F| \leq n - 2$. Suppose that $x$ and $y$ are two arbitrary nodes in $Q_n - F$, where $n \geq 3$. If $d_{Q_n}(x, y)$ is odd (respectively, even), then there exists a path $P[x, y]$ with length of at least $2^n - 2|F| - 1$ (respectively, $2^n - 2|F| - 2$) in $Q_n - F$.

**Lemma 2.** [7] Let $F \subset V(Q_n)$ denote the faulty vertices of $Q_n$, where $|F| \leq n - 1$. Let $x \in V(Q_n) - F$, where $n \geq 3$. Suppose $n - 1$ neighbors of $x$ are in $F$. Then, there exist two paths $P[x, y]$ and $P[x, y]$ with lengths at least $2^n - 2(n - 1)$ in $Q_n - F$ such that $d_{Q_n}(x, y) = d_{Q_n}(x, y) = 2$.

**Lemma 3.** [7] Let $F \subset V(Q_n)$ denote the faulty vertices of $Q_n$, where $|F| \leq n - 1$. Let $x \in V(Q_n) - F$, where $n \geq 3$. Suppose that at least two neighbors of $x$ are in $Q_n - F$. Then there exists a fault-free cycle with length at least $2^n - 2|F|$ that contains $x$ in $Q_n$.

### III. LONGEST FAULT-FREE PATHS WITH NODE FAULTS

In this section, we have the main theorems as follows.

**Theorem 1.** Let $F \subset V(F_{Q_n})$ denote the faulty vertices of $F_{Q_n}$, where $|F| \leq n - 1$ and $n \geq 3$. Suppose that $x$ and $y$ are two arbitrary nodes in $F_{Q_n} - F$. If $d_{F_{Q_n}}(x, y)$ is odd (respectively, even), then there exists a path $P[x, y]$ with length of at least $2^n - 2|F| - 1$ (respectively, $2^n - 2|F| - 2$) in $F_{Q_n} - F$.

**Proof.** By Lemma 1, the theorem holds when $|F| \leq n - 2$ (since $Q_n \subset F_{Q_n}$). In the rest of the proof, we assume that $|F| = n - 1$. We can partition $F_{Q_n}$ over some dimension $j$ into two $(n-1)$-dimension hypercubes $Q_{n-1,j}$ and $Q_{n-1,j}$ such that $|F_{Q_n}| \geq 1$ and $|F_{Q_n}| \geq 1$, where $F_0 = F \cap Q_{n-1,0}$ and $F_1 = F \cap Q_{n-1,1}$. Without loss of generality, we assume that $|F_{Q_n}| \geq |F_{Q_n}|$. Thus, we have $|F_0| \leq \lfloor \frac{n-1}{2} \rfloor$. We have the following case:

**Case 1:** $x, y \in Q_{n-1,0}$. Two cases are further considered:

**Case 1.1:** $|F_0| \leq n - 3$. It is not difficult to see that $n = 5$; for otherwise $|F_0| \leq n - 3 \leq 1 \leq (n - 1) - (n - 3) = 2 = |F_0|$, which contradicts to the assumption that $|F_0| \leq |F_{Q_n}|$. By Lemma 1, there exists a path $P[x, y]$ of length at least $2^{n-1} - 2|F_0| - 1$ (respectively, $2^{n-1} - 2|F_0| - 2$) if $d_{Q_{n-1}}(x, y)$ is odd (respectively, even) in $Q_{n-1}^0 - F_0$. We can choose an edge $(u, v) \in E(P[x, y])$ such that $u^{(0)}, v^{(0)} \notin F_1$. Let $P[u, u]$ and $P[v, y]$ be two subpaths of $P[x, y]$ in $Q_{n-1}$. Also, By Lemma 1, there exists a path $P[u^{(0)}, v^{(0)}]$ of length at least $2^{n-1} - 2|F_0| - 1$ in $Q_{n-1} - F_1$. Thus, $(x, P[x, u], u, u^{(0)}, P[u^{(0)}, v^{(0)}], v^{(0)}, v) \in P[x, y]$ is the desired path of length at least $(2^{n-1} - 2|F_0| - 1) + 2 + 2^{n-1} - 2|F_0| - 1 = 2^{n-1} - 2|F_0| + |F_0| - 1 + 2^{n-1} - 2|F_0| - 1$ (respectively, $2^n - 2(n - 1) - 2$) if $d_{Q_{n-1}}(x, y)$ is odd (respectively, even) (see Fig. 2(a)).

**Case 1.2:** $|F_0| = n - 2$ (i.e., $|F_0| = 1$). We have two scenarios as follows:

**Case 1.2.1:** $n = 3$. We have $|F_0| = |F_1| = 1$. Without loss of generality, let $x = 000$. The desired path $P[x, y]$ are listed below:

<table>
<thead>
<tr>
<th>Node $y$</th>
<th>The node in $F_0$</th>
<th>The node in $F_1$</th>
<th>$P[x, y]$ $(d_{Q_{n-1}}(x, y) = \text{even})$</th>
<th>$P[x, y]$ $(d_{Q_{n-1}}(x, y) = \text{odd})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>001</td>
<td>010</td>
<td>100</td>
<td>$&lt;000, 010, 011&gt;$</td>
<td>$&lt;000, 111, 001, 001&gt;$</td>
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<td></td>
<td>110</td>
<td>101</td>
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<td>011</td>
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<tr>
<td></td>
<td>110</td>
<td>101</td>
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</table>

**Case 1.2.2:** $n \geq 4$. Let $w \in F_0$. Then, $|F_0| = n - 3$. By Lemma 1, there exists a path $P[x, y]$ of length at least $2^{n-1} - 2(n - 3) - 1$ (respectively, $2^n - 2(n - 3) - 2$), if $d_{Q_{n-1}}(x, y)$ is odd (respectively, even) in $Q_{n-1}^0 - (F_0 \setminus \{w\})$.

If $w \notin V(P[x, y])$, let $(u, v) \in E(P[x, y])$ such that $u^{(0)}, v^{(0)} \notin F_1$. Let $P[x, u]$ and $P[v, y]$ are two subpaths of $P[x, y]$. Also, By Lemma 1, there exists a path $P[u^{(0)}, v^{(0)}]$ of length at least $2^{n-1} - 2 \times 1 - 1$ in $Q_{n-1} - F_1$. Thus, $(x, P[x, u], u, u^{(0)}, P[u^{(0)}, v^{(0)}], v^{(0)}, v) \in P[x, y]$ is the desired path with length at least $(2^{n-1} - 2(n - 3) - 1) + 2 + 2^{n-1} - 2 \times 1 = 2^{n-1} - 2(n - 2) - 1 = 2^{n-1} - 2|F_0| - 1$ (respectively, $2^n - 2(n - 2) - 2$) if $d_{Q_{n-1}}(x, y)$ is odd (respectively, even) (see Fig. 2(b)).

If $(u, v)$ does not exist, then $|F_0| = 2 \times 2$ and $|F_0| - 2 = 2^{n-1} - 2|F_0| - 1 \geq 2^{n-1} - (n - 3) - 1 > n - 3$ for $n \geq 5$, which contradicts to the assumption that $|F_0| \leq |F_{Q_n}| \leq n - 3$.
Two cases are further considered:

**Case 2.1:** \(|F_0| \leq n - 3\). It is not difficult to see that \(n \geq 5\); for otherwise \(|F_0| \leq n - 3 \leq 1 \leq (n - 1) - (n - 3) = 2 = |F_1|\), which contradicts to the assumption that \(|F_1| \leq |F_0|\). Let \(u \in Q_{n-1}^0 - \{x\} - F_0\) such that \(d_0(u, x, w^{(0)})\) is odd and \(w^{(0)} \in Q_{n-1}^0 - \{y\}\). By Lemma 1, there exists a path \(P[u, u']\) of length at least \(2^{n-1} - 2|F_0| - 1\) in \(Q_{n-1}^0 - F_0\). Moreover, By Lemma 1 there exists a path \(P[y, u^{(0)}]\) of length at least \(2^{n-1} - 2|F_1| - 1\) (respectively, \(2^{n-1} - 2|F_1| - 2\), if \(d_0(y, u^{(0)})\) is odd (respectively, even) in \(Q_{n-1}^0 - F_1\). Clearly, \(d_0(x, y)\) is odd if and only if \(d_0(y, u^{(0)})\) is odd. Thus, \(d_0(x, y)\) is odd. Therefore, \(x, P[u, u'], u, u^{(0)}, P[u^{(0)}, y, y]\) is a path with length of at least \(2^{n-1} - 2|F_0| - 1 + 1 + (2^{n-1} - 2|F_1| - 1) = 2^n - 2|F_0| - |F_1| - 1 = 2^n - 2|F_1| - 1\) (respectively, \(2^n - 2|F_0| - 1\)) if \(d_0(x, y)\) is odd (respectively, even) (see Fig. 3(a)).

**Case 2.2:** \(|F_0| = n - 2\) (i.e., \(|F_0| = 1\)). We have two scenarios as follows:

**Case 2.2.1:** \(n = 3\). We have \(|F_0| = |F_1| = 1\). Without loss of generality, let \(x = 000\). The desired path \(P[x, y]\) are listed below:

<table>
<thead>
<tr>
<th>Node 0</th>
<th>Node 1</th>
<th>Node 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>010</td>
<td>101</td>
</tr>
</tbody>
</table>

1 Let \(V' = \{z\} \cup\{d_0(x, z)\}\) be odd, \(z \in V(Q_{n+1}^0)\). Note that \(|V'| = 2^n - 2\). If none of the vertices in \(Q_{n+1}^0\) meets the requirements of such a vertex \(z\), then \(V' = F_0 \cup \{w \mid w^{(0)} \in F_1\} \cup \{y^{(0)}\}\).

3 First, consider that only one neighbor of \(x\) in \(Q_{n+1}^0\) is not in \(F_0\); that is, \(n - 2\) neighbors of \(x\) in \(Q_{n+1}^0\) are in \(F_0\). By Lemma 2, there exist two path \(P[x, r]\) and \(P[x, w]\) with length at least \(2^{n-1} - 2(n - 2)\) in \(Q_{n+1}^0 - F_0\) such that \(d_0(x, r) = d_0(x, w) = 2\).

If \(r^{(0)} \neq w\) and \(w^{(0)} \neq r\), then \(|\{r^{(0)}, w^{(0)}\} - F_1\) = 4.

4 We have \(|r^{(0)} \neq r\) and \(|w^{(0)} \neq w\) if \(z \in \{g, y\}\) and let \(z' \in \{z^{(0)}\} - \{g, y\}\). Note that \(d_0(x, z') = 2\) and \(d_0(x, z') = 1\). Thus, we have \(d_0(x, z')\) is odd. Consequently, if \(d_0(x, y)\) is even (respectively, odd), then \(d_0(y, z')\) is odd (respectively, even). By Lemma 1, there exists a path \(P[y, z']\) of length at least \(2^n - 2 \times 1 - 1\) (respectively, \(2^n - 2 \times 2 - 1\)) in \(Q_{n+1}^0 - F_1\). Thus, \(x, P[x, z], z, z', P[z', y], y\) is a path with length of at least \(2^n - 2(n - 1)\).
\[2 + 1 + (2^{n-1} - 2 \times 2 - 1) = 2^n - 2(n - 1) \text{ (respectively, } 2^n - 2(n - 1) - 1\text{) if } d_0(x, y) \text{ is odd (respectively, even) (see Fig. 3(b)).}

If \( r^{(j)} = w \) or \( w^{(j)} = r \), then \( n = 5^2 \). It is not difficult to see that if \( r^{(j)} = w \), then \( w^{(j)} = r \). When \( \{r^{(j)}, w^{(j)}\} \not= \{g, y\} \), the construction is similar as that of Fig. 3(b). When \( \{r^{(j)}, w^{(j)}\} = \{g, y\} \), we have \( y^{(j)} = \{r, w\} \) (thus, \( d_0(x, y) = 2 \) and \( d_0(x, y) \) is odd).

Since \( n = 5 \) and \( d_0(x, y) = 2 \), by Lemma 2, we have a path \( P[x, y]\) of length at least \( 2^n - 2(2n - 2m) = 2^n - 2(2n - 1) = 10 \) in \( Q^0_{n-1} - F_0 \). Since \( |P[x, y]| \geq 10 \), we can choose an edge \((s, t) \in E(P[x, y])\) such that \( (s, t) \cap (r, w) = \emptyset \). Clearly, \( \{s^{(j)}, t^{(j)}\} \cap \{r^{(j)}, w^{(j)}\} = \{y, g\} \). Without loss of generality, let \( P[x, s] \) and \( P[s, y] \) be two subpaths of \( P[x, y] \) in \( Q^0_{n-1} \). By Lemma 1, there exists a path \( P[x^{(j)}, l^{(j)}] \) of length at least \( 2^{n-1} - 2 \times 2 - 1 \) in \( Q^0_{n-1} - \{g, y\} \). Thus, \( P[x, P[x, s], s^{(j)}, t^{(j)}]) \) is odd (respectively, even).

2) Suppose \( r = r_{1i} \ldots r_{is} r_{0i} \ldots r_{0s}, w = w_{1i} \ldots w_{is} w_{0i} \ldots w_{0s}, w_{r} \in \{0, 1\}, \) for \( i \in \{1, 2, \ldots, n\} \). We have \( \{r^{(j)}, w^{(j)}\} \not= \{g, y\} \), and \( \{r^{(j)}, w^{(j)}\} = \{g, y\} \). When \( \{r^{(j)}, w^{(j)}\} = \{g, y\} \), we have \( y^{(j)} = \{r, w\} \) (thus, \( d_0(x, y) = 2 \) and \( d_0(x, y) \) is odd).

\[d_0(x, y) = 2 \text{ or } 4 \text{ if } i^{(j)} = w \text{ or } w^{(j)} = r \text{ then } n = 5^2 \].

When \( d_0(x, w) = 2 \text{ or } n = 3 \), we have \( n = 3 \), which contradicts to the fact that \( n \geq 4 \). When \( d_0(x, w) = 4 \) or \( n = 3 \), we have \( n = 5 \).

IV. DISCUSSION AND CONCLUSION

Fault tolerance is an important research subject in the area of the multi-process computer system, and many studies have focused on the vertex-fault tolerant or edge-fault tolerant properties of various networks. In this paper, we show that \( Q_{n-0} - F \) contains a path \( P[x, y] \) with length at least \( 2^n - 2|F| - 1 \) (respectively, \( 2^n - 2|F| - 2 \)) between two arbitrary vertices \( x \) and \( y \) of odd (respectively, even) Hamming distance, where \( |F| \leq n - 1 \) and \( n \geq 3 \).

REFERENCES