Embedding Two Edge-Disjoint Hamiltonian Cycles and Two Equal Node-Disjoint Cycles into Twisted Cubes

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Abstract—The presence of edge-disjoint Hamiltonian cycles provides an advantage when implementing algorithms that require a ring structure by allowing message traffic to be spread evenly across the network. Edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant Hamiltonicity of an interconnection network. Two node-disjoint cycles in a network are called equal if the number of nodes in the two cycles are the same and every node appears in one cycle exactly once. The presence of two equal node-disjoint cycles provides algorithms that require a ring structure to be preformed in the network simultaneously. The hypercube is one of the most popular interconnection networks since it has simple structure and is easy to implement. The \( n \)-dimensional twisted cube, an important variation of the hypercube, possesses some properties superior to the hypercube. In this paper, we present linear time algorithms to construct two edge-disjoint Hamiltonian cycles and two equal node-disjoint cycles in an \( n \)-dimensional twisted cube.

Index Terms—edge-disjoint Hamiltonian cycles, equal node-disjoint cycles, twisted cubes, parallel computing, inductive construction

I. INTRODUCTION

PARALLEL computing is important for speeding up computation. The topology design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [4], [6], [8], [9], [10], [13], and [18]. Among the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [24]. The topology of an interconnection network is usually modeled by a graph, where nodes represent the processing elements and edges represent the communication links. In this paper, we will use graphs and networks interchangeably.

The \( n \)-dimensional twisted cube \( TQ_n \), an important variation of the hypercube, was first proposed by Hilbers et al. [13] and possesses some properties superior to the hypercube. The twisted cube is derived from the hypercube by twisting some edges. Due to these twisted edges, the diameter, wide diameter, and fault diameter of \( TQ_n \) are about half of those of the comparable hypercube [5]. An \( n \)-dimensional twisted cube is \( (n-3) \)-Hamiltonian connected [16] and \( (n-2) \)-pancyclical [22], whereas the hypercube is not. Moreover, its performance is better than that of the hypercube even if it is asymmetric [1]. Recently, some interesting properties, such as conditional link faults, of the twisted cube \( TQ_n \) were investigated. Yang et al. [27] showed that, with \( n_w+n_v \leq n-2 \), a faulty \( TQ_n \) still contains a cycle of length \( l \) for every \( 4 \leq l \leq |V(TQ_n)| - n_w \), where \( n_w \) and \( n_v \) are the numbers of faulty edges and faulty nodes in \( TQ_n \), respectively, and \( |V(TQ_n)| \) denotes the number of nodes in \( TQ_n \). In [12], Fu showed that \( TQ_n \) can tolerate up to \( 2n-5 \) edge faults, while retaining a fault-free Hamiltonian cycle. Fan et al. [11] showed that the twisted cube \( TQ_n \), with \( n \geq 3 \), is edge-pancyclic and provided an \( O(l \log l + n^2 + nl) \)-time algorithm to find a cycle of length \( l \) containing a given edge of the twisted cube. In [11], the author also asked if \( TQ_n \) is edge-pancyclic with \( (n-3) \) faults for \( n \geq 3 \). Yang [28] answered the question and showed that \( TQ_n \) is not edge-pancyclic with only one faulty edge for any \( n \geq 3 \), and that \( TQ_n \) is node-pancyclic with \( (\lfloor \frac{n}{2} \rfloor - 1) \) faulty edges for every \( n \geq 3 \). Lai et al. [20] embedded a family of 2-dimensional meshes into a twisted cube. A Hamiltonian cycle in a graph is a simple cycle that passes through every node of the graph exactly once. The ring structure is important for distributed computing, and its benefits can be found in [19]. Two Hamiltonian cycles in a graph are said to be edge-disjoint if there exists no common edge in them. The edge-disjoint Hamiltonian cycles can provide an advantage for algorithms that make use of a ring structure [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an \( n \)-node ring that requires \( n-1 \) steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge (link) contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ wormhole or cut-through routing [21]. Further, edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant Hamiltonicity of an interconnected network; that is, when a Hamiltonian cycle of an interconnected network contains one faulty edge, then the other edge-disjoint Hamiltonian cycle can be used to replace it for transmission. The existence of a Hamiltonian cycle in twisted cubes has been verified [16]. However, there has been little work reported so far on edge-disjoint...
properties in the twisted cubes. In this paper, we show that, for any odd integer \( n \geq 5 \), there are two edge-disjoint Hamiltonian cycles in the \( n \)-dimensional twisted cube \( TQ_n \).

Two cycles in a graph are said to be equal and node-disjoint if they contain the same number of nodes, there is no common node in them, and every node of the graph appears in one cycle exactly once. Finding two equal edge-disjoint cycles in an interconnected network is equivalent to decompose the network into two disjoint sub-networks with the same number of nodes such that each sub-network contains a Hamiltonian cycle. Then, algorithms that require a ring structure can be preformed in the two sub-networks simultaneously. In this paper, we show that, for any odd integer \( n \geq 3 \), there exist two equal node-disjoint cycles in the \( n \)-dimensional twisted cube \( TQ_n \).

Related areas of investigation are summarized as follows. The edge-disjoint Hamiltonian cycles in \( k \)-ary \( n \)-cubes and hypercubes has been constructed in [2]. Barth et al. [3] showed that the butterfly network contains two edge-disjoint Hamiltonian cycles. Petrovic et al. [23] characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments. Hisieh et al. [14] constructed edge-disjoint spanning trees in locally twisted cubes. Hisieh et al. [15] investigated the edge-fault tolerant Hamiltonicity of an \( n \)-dimensional locally twisted cube. The existence of a Hamiltonian cycle in locally twisted cubes and twisted cubes has been shown in [26] and [16], respectively. However, there has been little work reported so far on edge-disjoint properties in locally twisted cubes and twisted cubes. In [17], we presented a linear time algorithm to construct two edge-disjoint Hamiltonian cycles in locally twisted cubes. In this paper, we show that there exist two edge-disjoint Hamiltonian cycles and two equal node-disjoint cycles in an \( n \)-dimensional twisted cube \( TQ_n \). Note that for any \( TQ_n \), \( n \) is always an odd integer.

The rest of the paper is organized as follows. In Section II, the structure of the twisted cube is introduced, and some definitions and notations used throughout this paper are given. Section III shows the construction of two edge-disjoint Hamiltonian cycles in the twisted cube. In Section IV, we construct two equal node-disjoint cycles in the twisted cube. Finally, we conclude this paper in Section V.

II. Preliminaries

We usually use a graph to represent the topology of an interconnected network. A graph \( G = (V, E) \) is a pair of the node set \( V \) and the edge set \( E \), where \( V \) is a finite set and \( E \) is a subset of \( \{ (u, v) | (u, v) \text{ is an unordered pair of } V \} \). We will use \( V(G) \) and \( E(G) \) to denote the node set and the edge set of \( G \), respectively. If \( (u, v) \) is an edge in a graph \( G \), we say that \( u \) is adjacent to \( v \) and \( u, v \) are incident to edge \( (u, v) \). A neighbor of a node \( v \) in a graph \( G \) is any node that is adjacent to \( v \). Moreover, we use \( \text{N}(v) \) to denote the set of neighbors of \( v \) in \( G \). The subscript \( G \) of \( \text{N}(v) \) can be removed from the notation if it has no ambiguity.

Let \( G = (V, E) \) be a graph with node set \( V \) and edge set \( E \). A (simple) path \( P \) of length \( \ell \) in \( G \), denoted by \( v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_\ell \), is a sequence \( (v_0, v_1, \cdots, v_\ell, v_\ell) \) of nodes such that \( (v_i, v_{i+1}) \in E \) for \( 0 \leq i \leq \ell - 1 \). The first node \( v_0 \) and the last node \( v_\ell \) visited by \( P \) are denoted by \( \text{start}(P) \) and \( \text{end}(P) \), respectively. Path \( v_\ell \rightarrow v_\ell - 1 \rightarrow \cdots \rightarrow v_1 \rightarrow v_0 \) is called the reversed path, denoted by \( P_\text{rev} \), of path \( P \). That is, \( P_\text{rev} \) visits the nodes of path \( P \) from \( \text{end}(P) \) to \( \text{start}(P) \) sequentially. In addition, \( P \) is a cycle if \( |V(P)| \geq 3 \) and \( \text{end}(P) \) is adjacent to \( \text{start}(P) \). A path \( P = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell - 1} \rightarrow v_\ell \) may contain another subpath \( Q \), denoted as \( v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell - 1} \rightarrow Q \rightarrow v_{\ell + 1} \rightarrow \cdots \rightarrow v_{\ell - 1} \rightarrow v_\ell \), where \( Q = v_\ell \rightarrow v_{\ell + 1} \rightarrow \cdots \rightarrow v_\ell \) for \( 0 \leq i \leq j \leq \ell \). A path (or cycle) in \( G \) is called a Hamiltonian path (or Hamiltonian cycle) if it contains every node of \( G \) exactly once. Two paths (or cycles) \( P_1 \) and \( P_2 \) connecting a node \( u \) to a node \( v \) are said to be edge-disjoint if and only if \( E(P_1) \cap E(P_2) = \emptyset \). Two paths (or cycles) \( Q_1 \) and \( Q_2 \) of graph \( G \) are called node-disjoint if and only if \( V(Q_1) \cap V(Q_2) = \emptyset \). Two node-disjoint paths (or cycles) \( Q_1 \) and \( Q_2 \) of graph \( G \) are said to be equal if and only if \( |V(Q_1)| = |V(Q_2)| \) and \( V(Q_1) \cup V(Q_2) = V(G) \). Two-node-disjoint paths \( Q_1 \) and \( Q_2 \) can be concatenated into a path, denoted by \( Q_1 \Rightarrow Q_2 \), if \( \text{end}(Q_1) \) is adjacent to \( \text{start}(Q_2) \).

Now, we introduce twisted cubes. The node set of the \( n \)-dimensional twisted cube \( TQ_n \) is the set of all binary strings of length \( n \). Note that due to the twisted edge property of a twisted cube, the dimension \( n \) of \( TQ_n \) is always defined as an odd integer. A binary string \( b \) of length \( n \) is denoted by \( b_n \rightarrow b_{n-1} \cdots b_1 b_0 \), where \( b_{n-1} \) is the most significant bit. We denote the complement of bit \( b_j \) by \( \bar{b}_j = 1 - b_j \). To define \( TQ_n \), a \( i \)-th bit parity function \( \text{P}_i(b) \) is introduced. Let \( b = b_n \rightarrow b_{n-1} \cdots b_1 b_0 \) be a binary string. For \( 0 \leq i \leq n - 1 \), \( \text{P}_i(b) = b_i \oplus b_{i-1} \oplus \cdots \oplus b_1 \oplus b_0 \), where \( \oplus \) is the exclusive-or operation. We then give the recursive definition of the \( n \)-dimensional twisted cube \( TQ_n \), for any odd integer \( n \geq 1 \), as follows.

**Definition 1.** [13], [28] \( TQ_1 \) is the complete graph with two nodes labeled by 0 and 1, respectively. For an odd integer \( n \geq 3 \), \( TQ_n \) consists of four copies of \( TQ_{n-2} \). We use \( TQ_{n-2}^p \) to denote an \( (n-2) \)-dimensional twisted cube which is a subgraph of \( TQ_n \) induced by the nodes labeled by \( \bar{i} \rightarrow b_{n-3} \cdots b_1 b_0 \), where \( i, j \in \{0, 1 \} \). Edges that connect these four sub-twisted cubes can be described as follows: Each node \( b = b_n \rightarrow b_{n-1} \cdots b_1 b_0 \) in \( TQ_n \) is adjacent to \( \bar{b}_n \rightarrow b_{n-1} \cdots b_1 b_0 \) and \( \bar{b}_n \rightarrow \bar{b}_{n-1} \cdots \bar{b}_1 \bar{b}_0 \) if \( \text{P}_n(b) = 0 \); and to \( \bar{b}_n \rightarrow b_{n-1} \cdots b_1 b_0 \) and \( b_n \rightarrow \bar{b}_{n-1} \cdots \bar{b}_1 \bar{b}_0 \) if \( \text{P}_n(b) = 1 \).

By the above definition, an \( n \)-dimensional twisted cube \( TQ_n \) is an \( n \)-regular graph with \( 2^n \) nodes and \( 2^{2n-1} \) edges, i.e., each node of \( TQ_n \) is adjacent to \( n \) nodes. The dimension \( n \) of \( TQ_n \) is always an odd integer. In addition, \( TQ_n \) can be decomposed into four sub-twisted cubes \( TQ_{n-2}^0, TQ_{n-2}^{10}, TQ_{n-2}^{11}, TQ_{n-2}^{101} \), where \( TQ_{n-2}^0 \) consists of those nodes \( b = b_n \rightarrow b_{n-1} \cdots b_1 b_0 \) with leading two bits \( b_{n-1} = i \) and \( b_{n-2} = j \). For each \( i, j \in \{0, 10, 01, 11\} \), \( TQ_{n-2}^p \) is isomorphic to \( TQ_{n-2} \). For example, Fig. 1 shows \( TQ_3 \) and Fig. 2 depicts \( TQ_5 \) containing four sub-twisted cubes \( TQ_3^0, TQ_3^{10}, TQ_3^{11}, TQ_3^{101} \).

Let \( b \) be a binary string \( b_{n-1} \rightarrow b_{n-2} \cdots b_1 b_0 \) of length \( t \). We denote \( \bar{b} \) the new binary string obtained by repeating \( b \) string \( \tau \) times. For instance, \( (01)^3 = 010101 \) and \( 0^4 = 0000 \).
III. TWO EDGE-DISJOINT HAMILTONIAN CYCLES

In this section, we first show the existence of two edge-disjoint Hamiltonian cycles of an \( n \)-dimensional twisted cube \( TQ_n \) with odd integer \( n \geq 5 \). We then present a linear time algorithm to construct two such edge-disjoint Hamiltonian cycles of \( TQ_n \). Obviously, a 3-dimensional twisted cube \( TQ_3 \) contains no two edge-disjoint Hamiltonian cycles since each node is incident to three edges. Note that the dimension of \( TQ_n \) is always odd. We will prove the existence of two edge-disjoint Hamiltonian cycles in \( TQ_n \), with \( n \geq 5 \), by induction on \( n \), the dimension of the twisted cube. For any odd integer \( n \geq 5 \), we show that there exist two edge-disjoint Hamiltonian paths \( P \) and \( Q \) in \( TQ_n \) such that \( \text{start}(P) = 00(0)^{n-5}000 \), \( \text{end}(P) = 11(0)^{n-5}000 \), and \( \text{end}(Q) = 01(0)^{n-5}100 \). By the definition of parity function \( P_i(\cdot) \), \( P_{n-3}(\text{end}(P)) = P_{n-3}(11(0)^{n-5}000) = 0 \) and \( P_{n-3}(\text{end}(Q)) = P_{n-3}(01(0)^{n-5}100) = 1 \). By Definition 1, \( \text{start}(P) \in N(\text{end}(P)) \) and \( \text{start}(Q) \in N(\text{end}(Q)) \). Thus, \( P \) and \( Q \) are two edge-disjoint Hamiltonian cycles in \( TQ_n \) for any odd integer \( n \geq 5 \). In the following, we will show how to construct two such edge-disjoint Hamiltonian cycles. We first show that \( TQ_5 \) contains two edge-disjoint Hamiltonian paths as follows.

Lemma 1. There are two edge-disjoint Hamiltonian paths \( P \) and \( Q \) in \( TQ_5 \) such that \( \text{start}(P) = 00000 \), \( \text{end}(P) = 110000 \), \( \text{start}(Q) = 00100 \), and \( \text{end}(Q) = 011000 \).

Proof: We prove this lemma by constructing two such paths \( P \) and \( Q \). Let

\[
P = 00000 \to 00001 \to 00101 \to 00100 \to 10100 \to 10101 \\
\to 10001 \to 10000 \to 11010 \to 10010 \to 00010 \to 00011 \\
\to 01011 \to 10111 \to 00111 \to 01110 \to 01111 \\
\to 11110 \to 11111 \to 11110 \to 11000 \to 00000.
\]

By the definition of parity function \( P_i(\cdot) \), \( P_{k-1}(\text{end}(P)) = P_{k-1}(\text{start}(Q)) = 0 \), \( P_{k-1}(\text{end}(Q)) = 0 \), and \( P_{k-1}(\text{start}(Q)) = 1 \). By Definition 1, we have that \( \text{end}(P_{00}) \in N(\text{end}(P_{10})) \), \( \text{start}(P_{00}) \in N(\text{start}(P_{10})) \), \( \text{end}(P_{11}) \in N(\text{end}(P_{11})) \), \( \text{end}(Q_{00}) \in N(\text{end}(Q_{10})) \), \( \text{start}(Q_{00}) \in N(\text{start}(Q_{10})) \), and \( \text{end}(Q_{11}) \in N(\text{end}(Q_{11})) \).

Let \( P = P_{00} \Rightarrow P_{10} \Rightarrow P_{01} \Rightarrow P_{11} \) and let \( Q = Q_{00} \Rightarrow Q_{10} \Rightarrow Q_{11} \Rightarrow Q_{01} \) in the reversed paths of \( P_{10}, P_{11}, Q_{10}, \) and \( Q_{01} \), respectively. Then, \( P \) and \( Q \) are two edge-disjoint Hamiltonian paths in \( TQ_{k+2} \) such that \( \text{start}(P) = 00(0)^{k-3}000 \), \( \text{end}(P) = 11(0)^{k-3}000 \), and \( \text{start}(Q) = 00(0)^{k-3}100 \), and

\[
\Rightarrow 01010 \Rightarrow 01011 \Rightarrow 11011 \Rightarrow 11011 \Rightarrow 00111 \Rightarrow 01110 \\
\Rightarrow 01110 \Rightarrow 01000 \Rightarrow 01001 \Rightarrow 01101 \Rightarrow 01000 \Rightarrow 10100 \\
\Rightarrow 10010 \Rightarrow 10011 \Rightarrow 11011 \Rightarrow 11010 \Rightarrow 01010 \\
\Rightarrow 01110 \Rightarrow 01101 \Rightarrow 01111 \Rightarrow 01011 \Rightarrow 01010 \\
\Rightarrow 01011 \Rightarrow 01100 \Rightarrow 11000 \Rightarrow 11001 \Rightarrow 01000 \Rightarrow 00100.
\]
end}(Q) = 01(0)^{k−3}100. Fig. 4 depicts the construction of two such edge-disjoint Hamiltonian paths in TQ_{b+2}. Thus, the lemma holds true when \( n = k + 2 \). By induction, the lemma holds true.

Let \( P \) and \( Q \) be two edge-disjoint Hamiltonian paths constructed in Lemma 2. By the definition of parity function, \( \mathcal{P}_{n−3}(end(P)) = \mathcal{P}_{n−3}(11(0)^{n−3}000) = 0 \) and \( \mathcal{P}_{n−3}(end(Q)) = \mathcal{P}_{n−3}(01(0)^{n−5}100) = 1 \). By Definition 1, node \( start(P) \) is adjacent to node \( end(P) \), and node \( start(Q) \) is adjacent to node \( end(\tilde{Q}) \). In addition, the two edges \( (start(P), end(P)) \) and \( (start(Q), end(Q)) \) are distinct. Thus the following theorem holds true.

**Theorem 3.** For any odd integer \( n \geq 5 \), there exist two edge-disjoint Hamiltonian paths (cycles) in an \( n \)-dimensional twisted cube \( TQ_n \).

Based on the proofs of Lemmas 1 and 2, we design a recursive algorithm to construct two edge-disjoint Hamiltonian paths of an \( n \)-dimensional twisted cube. The algorithm typically follows a divide-and-conquer approach [7] and is sketched as follows. It is given by an \( n \)-dimensional twisted cube \( TQ_n \) with odd integer \( n \geq 5 \). If \( n = 5 \), then the algorithm constructs two edge-disjoint Hamiltonian paths according to the proof of Lemma 1. Suppose that \( n > 5 \). It first decomposes \( TQ_n \) into four sub-twisted cubes \( TQ_{n−2}^0, TQ_{n−2}^1, TQ_{n−2}^2, \) and \( TQ_{n−2}^3 \), where \( TQ_{n−2}^1 \) consists of those nodes \( b = b_{n−1}b_{n−2}b_{n−3}...b_0 \) with leading two bits \( b_{n−1} = i \) and \( b_{n−2} = j \). For each \( i,j \in \{00,01,11\} \), \( TQ_{n−2}^1 \) is isomorphic to \( TQ_{n−2} \). Then, the algorithm recursively computes two edge-disjoint Hamiltonian paths of \( TQ_{n−2}^1 \) for \( i,j \in \{00,01,11\} \). Finally, it concatenates these eight paths into two edge-disjoint Hamiltonian paths of \( TQ_n \) according to the proof of Lemma 2, and outputs two such concatenated paths. The algorithm is formally presented as follows.

**Algorithm CONSTRUCTING-2EDHP-TQ.**

**Input:** \( TQ_n \), an \( n \)-dimensional twisted cube with odd integer \( n \geq 5 \).

**Output:** Two edge-disjoint Hamiltonian paths \( P \) and \( Q \) in \( TQ_n \) such that \( start(P) = 00(0)^{n−5}000 \), \( end(P) = 11(0)^{n−5}000 \), \( start(Q) = 00(0)^{n−5}100 \), and \( end(Q) = 01(0)^{n−5}100 \).

**Method:**

1. if \( n = 5 \), then
2. let \( P := 00000 \rightarrow 0001 \rightarrow 00101 \rightarrow 0100 \rightarrow 1001 \rightarrow 10001 \rightarrow 10000 \rightarrow 10010 \rightarrow 00101 \rightarrow 0011 \rightarrow 00011 \rightarrow 01011 \rightarrow 10011 \rightarrow 00110 \rightarrow 00111 \rightarrow 01111 \rightarrow 11111 \rightarrow 01110 \rightarrow 0111 , \)
3. let \( Q := 00010 \rightarrow 00000 \rightarrow 10000 \rightarrow 11000 \rightarrow 10100 \rightarrow 10011 \rightarrow 00111 \rightarrow 00011 \rightarrow 11011 \rightarrow 10110 \rightarrow 11101 \rightarrow 10111 \rightarrow 01101 \rightarrow 01111 \rightarrow 11111 \rightarrow 11110 \rightarrow 11011 \rightarrow 11010 \rightarrow 11001 \rightarrow 11000 \rightarrow 1001 \rightarrow 001 \rightarrow 000 \rightarrow 0 \),
4. output "\( P \) and \( Q \)" as two edge-disjoint Hamiltonian paths of \( TQ_5 \);
5. decompose \( TQ_n \) into four sub-twisted cubes \( TQ_{n−2}^0, TQ_{n−2}^1, TQ_{n−2}^2 \), and \( TQ_{n−2}^3 \), where \( TQ_{n−2}^0 \) consists of those nodes \( b = b_{n−1}b_{n−2}b_{n−3}...b_0 \) with leading two bits \( b_{n−1} = i \) and \( b_{n−2} = j \).
6. for \( i,j \in \{00,01,11\} \) do
7. call Algorithm CONSTRUCTING-2EDHP-TQ given \( TQ_{n−2}^i \) to compute two edge-disjoint Hamiltonian paths \( P_{\overline{i}} \) and \( Q_{\overline{i}} \) of \( TQ_{n−2}^i \), where \( start(P_{\overline{i}}) = ij00(0)^{n−5}000 \), \( end(P_{\overline{i}}) = ij11(0)^{n−5}000 \), \( start(Q_{\overline{i}}) = ij00(0)^{n−5}100 \), and \( end(Q_{\overline{i}}) = ij11(0)^{n−5}100 \).
8. compute \( P := P_{\overline{0}} \rightarrow P_{\overline{1}} \rightarrow P_{\overline{2}} \rightarrow P_{\overline{3}} \) and \( Q := Q_{\overline{0}} \rightarrow Q_{\overline{1}} \rightarrow Q_{\overline{2}} \rightarrow Q_{\overline{3}} \), where \( P_{\overline{0}} = Q_{\overline{0}}, P_{\overline{1}} = Q_{\overline{1}}, Q_{\overline{2}} , \) and \( Q_{\overline{3}} \) are the reversed paths of \( P_{\overline{0}}, P_{\overline{1}}, Q_{\overline{0}} \), and \( Q_{\overline{1}} \), respectively.
9. output "\( P \) and \( Q \)" as two edge-disjoint Hamiltonian paths of \( TQ_n \).

The correctness of Algorithm CONSTRUCTING-2EDHP-TQ follows from Lemmas 1 and 2. Now, we analyze its time complexity. Let \( m \) be the number of nodes in \( TQ_n \). Then, \( m = 2^n \). Let \( TQ(m) \) be the running time of Algorithm CONSTRUCTING-2EDHP-TQ given \( TQ_n \). It is easy to verify from lines 2 and 3 that \( TQ(m) = O(1) \) if \( n = 5 \). Suppose that \( n > 5 \). By visiting every node of \( TQ_n \) once, decomposing \( TQ_n \) into \( TQ_{n−2}^0, TQ_{n−2}^1, TQ_{n−2}^2 \), and \( TQ_{n−2}^3 \) can be done in \( O(m) \) time, where each node in \( TQ_{n−2}^i, ij \in \{00,01,11\} \), is labeled with leading two bits \( ij \). Thus, line 5 of the algorithm runs in \( O(m) \) time. Then, our division of the problem yields four subproblems, each of which is \( 1/4 \) the size of the original. It takes time \( TQ(\frac{m}{4}) \) to solve one subproblem, and so it takes time \( 4 \cdot TQ(\frac{m}{4}) \) to solve the four subproblems. In addition, concatenating eight paths into two paths (last 8) can be easily done in \( O(m) \) time. Thus, we get the following recurrence equation:

\[
TQ(m) = \begin{cases} 
O(1), & \text{if } n = 5; \\
4 \cdot TQ(\frac{m}{4}) + O(m), & \text{if } n > 5.
\end{cases}
\]

The solution of the above recurrence is \( TQ(m) = \text{imecs2012} \).
$O(m \log m) = O(n^{2n})$. Thus, the running time of Algorithm CONSTRUCTING-2EDHP-TQ given $TQ_n$ is $O(n^{2n})$. Since an $n$-dimensional twisted cube $TQ_n$ contains $2^n$ nodes and $n2^{n-1}$ edges, the algorithm is a linear time algorithm. Let $P$ and $Q$ be two edge-disjoint Hamiltonian paths output by Algorithm CONSTRUCTING-2EDHP-TQ given $TQ_n$. By Definition 1, $\text{start}(P) \in N(\text{end}(P))$ and $\text{start}(Q) \in N(\text{end}(Q))$. In addition, the edge connecting $\text{start}(P)$ with $\text{end}(P)$ is different from the edge connecting $\text{start}(Q)$ with $\text{end}(Q)$. Thus, $P$ and $Q$ are two edge-disjoint Hamiltonian cycles of $TQ_n$. We then conclude the following theorem.

**Theorem 4.** Algorithm CONSTRUCTING-2EDHP-TQ correctly constructs two edge-disjoint Hamiltonian cycles (paths) of an $n$-dimensional twisted cube $TQ_n$, with odd integer $n \geq 5$, in $O(n^{2n})$-linear time.

### IV. TWO EQUAL NODE-DISJOINT CYCLES

In this section, we will construct two equal node-disjoint cycles $P$ and $Q$ in a $n$-dimensional twisted cube $TQ_n$, for any odd integer $n \geq 3$. Our method for constructing two equal node-disjoint cycles of $TQ_n$ is also based on an inductive construction. For any odd integer $n \geq 3$, we will construct two equal node-disjoint paths $P$ and $Q$ in $TQ_n$ such that $\text{start}(P) = 00(0)^{n-3}1$, $\text{end}(P) = 01(0)^{n-3}1$, $\text{start}(Q) = 00(0)^{n-3}0$, and $\text{end}(Q) = 11(0)^{n-3}0$. The basic idea is similar to that of constructing two edge-disjoint Hamiltonian paths and is described as follows. Initially, we construct two equal node-disjoint paths $P$ and $Q$ in $TQ_3$ such that $\text{start}(P) = 001$, $\text{end}(P) = 011$, $\text{start}(Q) = 000$, and $\text{end}(Q) = 110$. By Definition 1, $P$ and $Q$ are also node-disjoint cycles with the same length. Consider that $n$ is an odd integer with $n \geq 5$. We first partition $TQ_n$ into four subtwisted cubes $TQ_{n-2}$, $TQ_{n-2}$, $TQ_{n-2}$, $TQ_{n-2}$. Assume that $P^{ij}$ and $Q^{ij}$ are two equal node-disjoint paths in $TQ_n$, for $i, j \in \{0, 1\}$, such that $\text{start}(P^{ij}) = ij00(0)^{n-5}1$, $\text{end}(P^{ij}) = ij01(0)^{n-5}1$, $\text{start}(Q^{ij}) = i0j0(0)^{n-5}0$, and $\text{end}(Q^{ij}) = i1j1(0)^{n-5}0$. We then concatenate them into two equal node-disjoint paths $P$ and $Q$ such that $\text{start}(P) = 00(0)^{n-3}1$, $\text{end}(P) = 01(0)^{n-3}1$, $\text{start}(Q) = 00(0)^{n-3}0$, and $\text{end}(Q) = 11(0)^{n-3}0$. By Definition 1, $P$ and $Q$ are also two equal node-disjoint cycles of $TQ_n$, since $\text{start}(P) \in N(\text{end}(P))$ and $\text{start}(Q) \in N(\text{end}(Q))$. The concatenating process will be presented in Lemma 6.

For $TQ_n$, let $P = 001 \rightarrow 101 \rightarrow 111 \rightarrow 011$ and let $Q = 000 \rightarrow 100 \rightarrow 010 \rightarrow 110$. Then, $P$ and $Q$ are two equal node-disjoint paths in $TQ_n$. By Definition 1, $\text{start}(P) \in N(\text{end}(P))$ and $\text{start}(Q) \in N(\text{end}(Q))$. Thus, the following lemma holds true.

**Lemma 5.** There are two equal node-disjoint paths (cycles) $P$ and $Q$ in $TQ_n$ such that $\text{start}(P) = 001$, $\text{end}(P) = 011$, $\text{start}(Q) = 000$, and $\text{end}(Q) = 110$.

Based on Lemma 5, we prove the following lemma.

**Lemma 6.** For any odd integer $n \geq 3$, there exist two equal node-disjoint paths $P$ and $Q$ in $TQ_n$ such that $\text{start}(P) = 00(0)^{n-3}1$, $\text{end}(P) = 01(0)^{n-3}1$, $\text{start}(Q) = 00(0)^{n-3}0$, and $\text{end}(Q) = 11(0)^{n-3}0$.

**Proof:** We prove this lemma by induction on $n$, the dimension of $TQ_n$. By Lemma 5, the lemma holds true when $n = 3$. Assume that the lemma holds when $n \geq k$. We will prove that the lemma holds true for $n = k + 2$. We first partition $TQ_{k+2}$ into four subtwisted cubes $TQ_{k+2}^{00}$, $TQ_{k+2}^{01}$, $TQ_{k+2}^{10}$, $TQ_{k+2}^{11}$. By the induction hypothesis, there are two equal node-disjoint paths $P^{ij}$ and $Q^{ij}$, for $i, j \in \{0, 1\}$, in $TQ_{k+2}^{ij}$ such that $\text{start}(P^{ij}) = ij00(0)^{k-3}1$, $\text{end}(P^{ij}) = i0j0(0)^{k-3}0$, $\text{start}(Q^{ij}) = i0j1(0)^{k-3}0$, and $\text{end}(Q^{ij}) = ij11(0)^{k-3}0$. By the definition of parity function $P_{k-1}(\cdot)$, $P_{k-1}(\text{end}(P^{ij})) = 0$, $P_{k-1}(\text{start}(P^{ij})) = 1$, and $P_{k-1}(\text{end}(Q^{ij})) = P_{k-1}(\text{start}(Q^{ij})) = 0$. According to Definition 1, we have that $\text{end}(P^{00}) \in N(\text{end}(P^{10}))$, $\text{start}(P^{00}) \in N(\text{start}(P^{10}))$, $\text{end}(Q^{00}) \in N(\text{end}(Q^{10}))$, $\text{start}(Q^{10}) \in N(\text{start}(Q^{10}))$, and $\text{end}(Q^{11}) \in N(\text{end}(Q^{11}))$.

Let $P = P^{00} \Rightarrow P^{10} \Rightarrow P^{11} \Rightarrow P^{01}$, and let $Q = Q^{00} \Rightarrow Q^{10} \Rightarrow Q^{11}$, $P_{rev}^{00}$, $P_{rev}^{10}$, $Q_{rev}^{00}$, $Q_{rev}^{10}$, and $Q_{rev}^{11}$ are the reversed paths of $P^{10}$, $P^{11}$, $Q^{10}$, and $Q^{11}$, respectively. Then, $P$ and $Q$ are two equal node-disjoint paths in $TQ_{k+2}$ such that $\text{start}(P) = 00(0)^{k-1}1$, $\text{end}(P) = 01(0)^{k-1}1$, $\text{start}(Q) = 00(0)^{k-1}0$, and $\text{end}(Q) = 11(0)^{k-1}0$. Fig. 5 depicts the constructions of two such equal node-disjoint paths $P$ and $Q$ in $TQ_{k+2}$. Thus, the lemma holds true when $n = k + 2$. By induction, the lemma holds true.

**Corollary 7.** For any odd integer $n \geq 3$, there exist two equal node-disjoint cycles in $TQ_n$.

By the same arguments and analysis in constructing two edge-disjoint Hamiltonian cycles of $TQ_n$, we can easily construct two equal node-disjoint cycles of $TQ_n$ in linear time. We then conclude the following theorem.

**Theorem 8.** There exists an algorithm such that it correctly constructs two equal node-disjoint cycles (paths) of an $n$-dimensional twisted cube $TQ_n$, with odd integer $n \geq 3$, in $O(n^{2n})$-linear time.
V. CONCLUDING REMARKS

In this paper, we construct two edge-disjoint Hamiltonian cycles (paths) of a \(n\)-dimensional twisted cubes \(TQ_n\), for any odd integer \(n \geq 3\). Furthermore, we construct two equal node-disjoint cycles (paths) of \(TQ_n\), for any odd integer \(n \geq 3\). Note that due to the twisted edge property of a twisted cube, the dimension \(n\) of \(TQ_n\) is always an odd integer. In the construction of two edge-disjoint Hamiltonian cycles (paths) of \(TQ_n\), some edges are not used. It is interesting to see if there are more edge-disjoint Hamiltonian cycles of \(TQ_n\) with odd dimension \(n \geq 7\). We would like to post this as an open problem to interested readers.

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