On the Optimality of Plug-In Optimal Control Systems

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Abstract—We consider a plug-in control system via a concept of Implicit and Explicit Controls as the two stage controllers. We first show two conditions under which the plug-in control system can be optimized by a suitable design of explicit control in the 2nd stage, when an implicit control is given in the 1st stage. We then show another condition under which the resultant optimal cost can be minimized by a suitable design of the 1st stage implicit control, which is characterized as a feedback control that allocates all the closed-loop poles onto the imaginary axis. These two results clarify the system theoretic meaning of the implicit control from the viewpoint of inverse optimal control problem.

Index Terms—A plug-in control system, implicit and explicit controls, optimal control, inverse optimal control problem

I. INTRODUCTION

A concept of “Implicit Control” and “Explicit Control” has been proposed recently in [1] by the second author. To explain this concept, we consider for example a motor control system which consists of a motor and a servo-driver. As is often the case, the servo-driver has a certain built-in feedback control loop. Designers then regard the motor and the servo-driver as a controlled plant, and design a control law for the new controlled plant. We call the former inner control loop “Implicit Control Law” and the latter outer control loop “Explicit Control Law”. Similar situations can be found in various control systems such as a brain nervous system of living things.

We can regard that the above two control systems are designed by a common design method. That is, firstly a certain feedback control law is constructed, and then an outer feedback control law is added. In this paper, we focus on such a design method and call the system as a plug-in control system. Especially, we adopt a concept of Implicit Control and Explicit Control. By using an inverse optimal control approach, we first show two equivalent n.a.c.s. conditions under which the plug-in control system can be optimized by a certain plug-in optimal control law. We then show n.a.c.s. conditions for a class of single input systems under which the resultant optimal cost can be minimized by some implicit control embedded in the 1st stage, which is characterized as a feedback control that allocates all the closed-loop poles onto the imaginary axis. These results together clarify the system theoretic meaning of implicit control characterized as above from the viewpoint of inverse optimal control problem.

II. PROBLEM STATEMENT

A. Plug-In Control System

At first, we define a Plug-in Control System based on the concept of Implicit and Explicit Control as proposed by the second author [1] in the following.

Definition 1 (Plug-In Control System)

Suppose that a controllable plant

\[ S_0: \dot{x} = Ax + Bu \]

is constructed by a feedback control law named “Implicit Control Law” \( u_I = -K_I x \) from the original controlled plant

\[ S: \dot{x} = Ax + Bv \]

as shown in Fig. 1 (a). That is,

\[ \dot{x} = A_I x + Bu = (A - BK_I) x + Bu. \] (3)

Here, we regard that the Implicit Control Law appears due to the interaction between plant and field.

(a) Implicit Control System

![Fig. 1 Implicit Control System](image)

(b) Plug-in Control System

![Fig. 1 Plug-in Control System](image)

Fig. 1 Implicit Control and Plug-In Control Systems

We then consider an outer loop feedback control law

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\[ u = u_e = -K_x x \]  
(4)
as shown in Fig.1(b). We named this control law as Explicit Control Law in [1]. Similarly, we name this two stage design method shown in the figure as a plug-in design method, and name the resultant double loop control system as a Plug-In Control System.

B. Plug-In Optimal Control System

Consider the Plug-In Control System shown in Fig.1(b).
\[ \dot{x} = (A - BK) x + Bu_e = Ax + Bv, \quad v = u + u_e \]  
(5)
where \( A, B \) and the Implicit Control Law \( u_e = -K_x x \) are given, while the Explicit Control Law \( u_e = -K_x x \) is unspecified. It is well known that we can always design \( u_e \) as an optimal control for \( S_0 \) by LQ optimal control theory; we call the resultant optimal control system “Plug-In Optimal Control System.” However, it is not trivial that the overall control \( v = u + u_e \) also becomes an optimal control for the original system \( S \). In view of this, we first consider the following optimality problem of Plug-In Optimal Control System in the usual sense with the above setting.

**Problem 1**

Can you find conditions and design \( u_e \) such that the following two specifications are satisfied.

S1) The input \( u = u_e \) is an optimal control for the system \( S_0 \).

S2) The combined input \( v = u + u_e \) is also an optimal control for the system \( S \).

Note that not every Plug-In Optimal Control System is optimal for \( S \). In other words, the specification S1 does not necessarily mean the specification S2, which is important for optimal design by the plug-in design method. To satisfy this specification, some constraints may be required on the quadratic weights used in LQ design of \( u_e \). This problem is a kind of inverse problem of optimal control [2].

In Problem 1 the Implicit Control Law \( K \) is specified. So, there remains a possibility of strengthening the optimality of the Plug-In Optimal Control System, even if it satisfies S2, by proper choice of \( K \). This observation allows us to consider the following optimality problem of Plug-In Optimal Control System in a stronger sense.

**Problem 2**

Consider the Plug-In Optimal Control System which is optimal in the sense of satisfying the specification S2. Can you find conditions and design \( K \) such that it minimizes the resultant optimal cost.

This problem is defined in more detail in the next section so that we can obtain a meaningful solution.

### III. SOLUTIONS TO THE PROBLEMS

In this section we solve the above inverse optimal control problems. In order to understand Problem 1 intuitively, we first show a solution for a scalar system [3] in which \( A \) and \( B \) are scalars with \( A = a, \ B = 1 \).

#### A. Optimality of Plug-in Optimal Control System for a scalar system

Suppose that we design the Explicit Control Law \( u_e = -k_e x \) for the system
\[ \dot{x} = a x + u_e, \quad a = a - k_e \leq 0 \]  
(6)
as an optimal control law that minimizes the cost function.

\[ J_e = \int_0^\infty \left( q_e x^2 + u_e^2 \right) dt \quad q_e \geq 0. \]  
(7)
The resultant optimal feedback gain of the Explicit Control Law is given by
\[ k_e = a_e + \sqrt{a_e^2 + q_e} \]  
(8)
Then, we can easily derive conditions and the weight \( q_e \) such that the combined input \( v = u + u_e \) becomes an optimal control minimizing the cost function
\[ J_v = \int_0^\infty \left( q_e x^2 + v^2 \right) dt \quad q_e \geq 0 \]  
(9)
The answer to the question is the following.

**Result 1** (Optimality Problem)

Firstly, set the weight \( q_e \) in the cost function \( J_v \) as
\[ q_e = k_e^2 - 2ak_e + q_e = (k_e - a)^2 + q_e - a^2 \]  
(10)
and define
\[ D = q_e - a^2 \]  
(11)
Then we have the following result.

**Case 1** If \( D > 0 \) (i.e., \( q_e > a^2 \)), then for an arbitrary \( k_e \), the overall input \( v = -(k_e + k_e)x \) is always an optimal control for the cost \( J_v \).

**Case 2** If \( D < 0 \) (i.e., \( q_e \leq a^2 \)), then for an arbitrary \( k_e \), satisfying \( k_e \geq a + \sqrt{a^2 - q_e} \), the overall control \( v = -(k_e + k_e)x \) is an optimal control for the cost \( J_v \).

#### B. Optimality of Plug-in Optimal Control System for a general single input system

The above result shows that a certain condition must be satisfied by the weight in the cost \( J_v \) chosen in the 2nd stage and the gain \( K \) determined in the 1st stage, in order that the overall control \( v = -(K_e + K_e)x \) is also an optimal control for the original system \( S \). We show this condition in the next theorem in two ways. One is given in the form of frequency domain condition, and the other in the form of Linear Matrix Inequality (LMI).

**Theorem 1** Let \( K_e \) be an optimal feedback control law for the system \( S_0 \) that minimizes the standard quadratic cost:
Then the combined feedback control law \( v = -(K_x + K_e)x \) is an optimal control for the system \( S \) that minimizes the cost
\[
J_v = \int_0^\infty (Q_x x^2 + u^2) \, dt \quad Q_x \geq 0
\]  
(12)

for some weight \( Q_x \geq 0 \) if and only if the weight \( Q_x \) and the gain \( K_e \) satisfies the following two equivalent conditions:

**C1)** \( T(-j\omega)^T T(j\omega) \geq I - B^T (j\omega I - A)^T Q_x (j\omega I - A)^T B \quad \text{a.e.} \quad \omega \in \mathbb{R} \)  
(14)

\[
T(s) := I + K_e(sI - A)^{-1} B
\]

**C2)** The following LMI has a real symmetric solution \( P \):
\[
\begin{bmatrix}
PA + A^T P - K_e^T K_e - Q_x & PB - K_e^T \\
B^T P - K_e & 0
\end{bmatrix} \leq 0
\]
(15)

Moreover, if the condition C2 holds, the overall control law \( v = -(K_x + K_e)x \) minimizes the cost \( J_v \) for the weight \( Q_x \) given by
\[
Q_x = Q_x = -PA + A^T P + K_e^T K_e
\]
(16)

**Proof** Since \( A_x - BK_e = A - B(K_x + K_e) \) is stable by the assumption on \( K_x \), the overall control law \( K_x + K_e \) is a stabilizing control law for \( S \). Thus it follows from the Inverse LQ theory [2] that the control law \( K_x + K_e \) is an optimal control law for \( S \) if and only if the following LMI has a real symmetric solution \( X \geq 0 \):
\[
\begin{bmatrix}
X A + A^T X - (K_x + K_e)^T (K_x + K_e) & XB - (K_x + K_e)^T \\
B^T X - (K_x + K_e) & 0
\end{bmatrix} \leq 0
\]
(17)

Since \( K_x \) is an optimal control law for \( S \) that minimizes the cost (13), there exists some real symmetric \( P \geq 0 \) such that
\[
PA + A^T P - P A A^T P + Q_x = 0
\]
(18)

Let \( P = X - P_e \). Substituting \( X = P + P_e \) into (17) and using (18) we see that the left side of (17) is the same as that of (15), and moreover applying some feedback transformation to (17) yields
\[
X (A_x - BK_e) + (A_x - BK_e)^T X + (K_x + K_e)^T (K_x + K_e) \leq 0
\]

from which \( X \geq 0 \) follows by Lyapunov theorem as well as the stability of \( A_x - BK_e \) as stated above. Finally by a well-known result on LMI [4] two conditions C1 and C2 are equivalent. This completes the proof of Theorem 1 except the last part. Since (1,1) block of the left side matrix of (17) is equal to that of (15) and hence to \( -Q_x \) by (16), the last part is obvious from LQ theory.

**C. Strong optimality of the Plug-In Optimal Control System for a general single input system**

In the previous section we have characterized the weighting matrix \( Q_x \) of those cost \( J_v \) that is minimized by the overall control law \( K_x + K_e \) for a given \( K_e \). In this section we minimize the resultant optimal cost \( J_v^* \) further by proper choice of \( K_e \). This minimization is equivalent to that of the weighting matrix \( Q_x \) since the optimal cost \( J_v^* \) reduces if so does the \( Q_x \) due to a well-known monotonicity property of maximal solutions of Riccati equations [4]. Here we are concerned only with the \( Q_x \) of diagonal form, and if all diagonal elements of \( Q_x \) is minimized by \( K_e \), then we say that the associated Plug-In Optimal Control System is **strongly optimal**. With regard to this problem we are interested in those \( K_e \) for which all the eigenvalues of \( A_x - BK_e \) are on the imaginary axis, since the Implicit Control Law \( K_x \) is one of such control laws [1]. With these settings, we then define the following detailed version of Problem 2, and show necessary and sufficient conditions under which the optimal cost \( J_v^* \) can be minimized by some implicit control law embedded in the 1st stage, thereby clarify a merit of the Implicit Control Law.

**Problem 2** (Strong Optimality of Optimal \( S_{opt} \))

Consider the Plug-In Optimal Control System:
\[
\dot{x} = Ax + Bu, \quad A_x = A - BK_e, \quad \text{Re}(A_x) \leq 0
\]
(20)

where \( u = -K_e x \) is an optimal control law minimizing the cost:
\[
J_v = \int_0^\infty (Q_x x^2 + u^2) \, dt \quad Q_x = \text{diag}(q_i) \geq 0
\]
(21)

and the overall control \( v = -(K_x + K_e)x \) is also an optimal control minimizing the cost \( J_v \) with the weight \( Q_x \) given in Theorem 1. Can you find conditions and design \( K_e \) such that the following three specifications are satisfied.

1. **S1** The weighting matrix \( Q_x \) is diagonal for some symmetric solution \( P \) to the LMI of (15).
2. **S2** The gain matrix \( K_e \) minimizes the diagonal weighting matrix \( Q_x \).
3. **S3** All the eigenvalues of \( A_x = A - BK_e \) are on the imaginary axis.

To solve this problem, we consider the single input system \( S \) in the phase variable canonical form:

\[
S : \dot{x} = Ax + Bv
\]

where
\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

(22)
For such a system we can always obtain a diagonal weighting matrix $Q_v$ for some real symmetric solution $P$ of the LMI, and express it in terms of the elements of $A, Q_v$ and $K_i$. However, these expressions are complicated in general, so we consider only the 2nd order system:

$$S: \dot{x} = Ax + Bv, \ A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ (23)

Let $K_i = \{k_1, k_2\}$. Then by (15) we have

$$B^TP = K_i = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$ (24)

and so by $B^t = [0 \ 1]$ we obtain

$$P = \begin{bmatrix} p_1 & k_1 \\ k_1 & k_2 \end{bmatrix}$$ (25)

given by (16) gives

$$Q_v = Q_v - PA - A^t P + K_i$$

$$= \begin{bmatrix} q_1 + k_1^2 - 2ak_2 \\ k_1 \end{bmatrix} - \begin{bmatrix} k_2 - a_1k_1 - a_2p_1 & \frac{q_2 - 2ak_1 - 2k}{2} \\ k_1 & k_2 \end{bmatrix}$$ (26)

To meet the specification S1 for $Q_v$, therefore, it is to choose $p_1$ uniquely as follows:

$$p_1 = k_1k_2 - a_1k_1 - a_2k_2 = (k_1 - a_1)(k_2 - a_2) - a_1a_2$$ (27)

Then we can obtain the following diagonal weighting matrix

$$Q_v = \begin{bmatrix} (k_1 - a_1)^2 + q_1 - a_1^2 \\ 0 \\ (k_2 - a_2)^2 + q_2 - a_2^2 \end{bmatrix}$$ (28)

This expression allows us to give a solution to this problem as follows:

**Theorem 2**

Let $n = 2$, $Q_v = \text{diag} \{q_1, q_2\}$ and $K_i^0 = \{k_1^0, k_2^0\}$ in Problem 2. Then there exists a control law $K^o_i$ satisfying the specifications S2 and S3 if and only if

$$q_1 > 0, \quad q_2 \geq a_2^2 + 2a_2 + 2\sqrt{\max \{0, a_1^2 - q_1\}}$$ (29)

Under this condition the minimizing control law $K^o_i = \{k_1^0, k_2^0\}$ with $Re \lambda(A) \leq 0$ is given by

$$k_1^0 = a_1 + \sqrt{\max \{0, a_1^2 - q_1\}}, \quad k_2^0 = a_2$$ (30)

and the minimized weighting matrix $Q_v = \text{diag} \{Q^o_{11}, Q^o_{22}\}$ is given by

$$Q^o_{11} = \max \{0, q_1 - a_1^2\}, \quad Q^o_{22} = q_2 - a_2^2$$ (31)

**Remark 1**: By the phase variable canonical form of (23) it is easy to see that $k^o_i = a_i$ means that $A = A - BK^o_i$ has complex conjugate eigenvalues $\pm j \sqrt{k_i^2 - a_i}$, thereby satisfying the specification S3. We should note, however, that S3 does not imply S2 in general. In other words, not all $K_i$ that allocates all the closed-loop poles onto the imaginary axis guarantees the strong optimality of the Plug-In Optimal Control System. This is obvious from the expression of the minimizing control law $k_i^0$ given by (30). Obviously this expression suggests the existence of both lower and upper bounds of $k_i^0$. For example, in the case of $a_1 = 2, a_2 = 1$, $k_2^0$ takes only the values between 2 and 4.

**Remark 2**: By Theorem 2 we can conclude that if we choose the weight $q_2$ alone larger than a certain value, then the resultant Plug-In Optimal Control System with $Re \lambda(A) = 0$ becomes optimal for the original system $S$. This observation clarifies an important role played by Implicit Control in the plug-in design method, in the sense that it achieves the strong optimality of Plug-In Optimal Control System.

**IV. CONCLUSION**

In this paper, we have considered a plug-in control system via a concept of Implicit Control Law and Explicit Control Law. First, we showed that the plug-in control system can be optimized by designing a plug-in optimal control law suitably if and only if the quadratic weight and the Implicit Control Law associated with the plug-in optimal control system satisfy a certain condition at a time. We then showed that the resultant optimal cost can be minimized further by choosing an Implicit Control Law suitably if and only if the quadratic weight alone satisfies a certain condition, thereby we clarified an important role which the Implicit Control Law plays in the plug-in design method. Although we treat only the 2nd order system for simplicity in the latter half, we can extend the results in the same way to a higher order system up to 5th order system.

**REFERENCES**


