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Bilevel Linear Programming Problems with Quadratic Membership Functions of Fuzzy Parameters

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***Abstract*—This article considers bilevel linear programming problems where the coefficients of the objective functions in the problem are given as a possibilistic variable characterized by a quadratic membership function. An extended Stackelberg solution is defined by incorporating the notions of possibility theory into the original concept of Stackelberg solutions. The characteristic of the proposed model is that the corresponding Stackelberg problem is exactly solved by using nonlinear bilevel programming techniques.**

***Index Terms*—Bilevel linear programming, fuzzy parameters, possibilistic variable, Stackelberg solutions, quadratic membership function, possibility theory.**

I. Introduction

Bilevel programming problems (BLPPs) are hierarchical optimization problems in which there exist two decision makers (DMs) who have different priorities on decision. It is assumed that the DM at the upper level, who has higher priority than the other, first specifies a strategy, and then the DM at the lower level chooses a strategy so as to optimize its own objective with full knowledge of the action of the DM at the upper level.

Bilevel or multilevel optimization is closely related to the economic problem of Stackelberg [1] in the field of game theory. In conventional bilevel or multilevel mathematical programming models employing the solution concept of Stackelberg equilibrium, it is assumed that there is no communication among DMs, or they do not make any binding agreement even if there exists such communication. Bilevel programs were initially considered by Bracken and McGill [2], [3], [4] as applications in the military fields as well as in production and marketing decision making, although they did not use the terms *bilevel* and *multilevel programming*, which were introduced later by Candler and Norton [5].

Bilevel or multilevel programming models have been applied to various hierarchical decision making situations such as oligopolistic market supplying a homogeneous product [6], principal-agent problem [7], traffic planning [8], pricing and fare optimization in the airline industry [9], management of hazardous materials [10], aluminum production process [11], pollution control policy determination [12], tax credits determination for biofuel producers [13], pricing in competitive electricity markets [14], flow shop scheduling [15], supply chain planning [16], facility location [17], [18], [19], defense problem [3], [20] and so forth.

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From a viewpoint of ambiguity or fuzziness involved in human's judgments, rather than randomness caused by stochastic events, the concept of fuzzy sets [21] was applied to decision making problems or optimization problems under fuzziness [22], [23], [24], [25] including a bilevel or multi-level problem [26], [27].

Under these circumstances, this article firstly tackles a noncooperative BLPP with possibilistic variables where the ambiguity of coefficient values in problems are mutually dependent. In particular, we focus on the case where the possibilistic variables involved in bilevel problems are assumed to be characterized as a possibilistic distribution defined by a quadratic membership function. In order to consider ambiguity involved in the bilevel programming problem, we consider the concepts of Stackelberg solutions under fuzziness by incorporating possibility theory into the original Stackelberg solution concept.

This paper is organized as follows. Section 2 formulates a BLPP with possibilistic variables and proposes a decision making model using a possibility measure. In Section 3, we show the original problem involving ambiguity can be transformed into a deterministic nonlinear BLPP which is exactly solved by nonlinear bilevel programming techniques. In Section 4, we conclude this paper and discuss future studies.

II. Bilevel programming problems with possibilistic variables

Consider the bilevel linear programming problems formulated as

$$
\left.
\begin{array}{ll}
\underset{\boldsymbol{x}_1,\boldsymbol{x}_2}{\text{maximize}} & z_1(\boldsymbol{x}_1,\boldsymbol{x}_2)=\tilde{\boldsymbol{C}}_{11}\boldsymbol{x}_1+\tilde{\boldsymbol{C}}_{12}\boldsymbol{x}_2 \\
\text{where } \boldsymbol{x}_2 \text{ solves} & \\
\underset{\boldsymbol{x}_2}{\text{maximize}} & z_2(\boldsymbol{x}_1,\boldsymbol{x}_2)=\tilde{\boldsymbol{C}}_{21}\boldsymbol{x}_1+\tilde{\boldsymbol{C}}_{22}\boldsymbol{x}_2 \\
\text{subject to} & \tilde{\boldsymbol{A}}_{i1}\boldsymbol{x}_1+\tilde{\boldsymbol{A}}_{i2}\boldsymbol{x}_2\le\tilde{B}_i, \\
 & \qquad \forall i\in I \stackrel{\triangle}{=} \{1,2,\ldots,r\} \\
 & \boldsymbol{a}_{i1}\boldsymbol{x}_1+\boldsymbol{a}_{i2}\boldsymbol{x}_2\le b_i, \\
 & \qquad i=r+1,r+2,\ldots,v \\
 & \boldsymbol{x}_1\ge\boldsymbol{0},\ \boldsymbol{x}_2\ge\boldsymbol{0},
\end{array}
\right\} \quad (1)
$$

where $\boldsymbol{x}_1$ is an n_1 dimensional decision variable column vector for the DM at the upper level (DM1), $\boldsymbol{x}_2$ is an n_2 dimensional decision variable column vector for the DM at the lower level (DM2), and $z_l(\boldsymbol{x}_1,\boldsymbol{x}_2)$, $l=1,2$ are the objective functions for DMl, $l=1,2$, respectively.

We assume that each of $\tilde{C}_{ljk}$, $k=1,2,\ldots,n_j$ of $\tilde{\boldsymbol{C}}_{lj}$, $l=1,2$, $j=1,2$ is a possibilistic variable characterized as a possibility distribution defined by the following quadratic

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membership function [28], [29], [30], [31], [32]:

$$\pi_{\tilde{\boldsymbol{C}}_{lj}}(\boldsymbol{c}_{lj}) = \mu_{\tilde{\boldsymbol{C}}_{lj}}(\boldsymbol{c}_{lj}) = L\left((\boldsymbol{c}_{lj} - \boldsymbol{d}^c_{lj})(U^c_l)^{-1}(\boldsymbol{c}_{lj} - \boldsymbol{d}^c_{lj})\right), \tag{2}$$

where $\boldsymbol{d}^c_{lj} = (d^c_{lj1}, d^c_{lj2}, \ldots, d^c_{ljn})$ is the most conceivable vector for $\boldsymbol{c}_{lj}$, U^c_l is an $n \times n$ $(n = n_1 + n_2)$ symmetrical positive-definite matrix representing the interactions among the coefficients of the lth objective function. $(U^c_l)^{-1}$ is the inverse matrix of U^c_l. L is a reference or shape function which is a nonnegative continuous function satisfying the following condition:

1) $L(t)$ is nonincreasing for any $t > 0$.
2) $L(0) = 1$.
3) $L(t) = L(-t)$ for any $t \in \mathbb{R}$.
4) There exists a $t^L_0 > 0$ such that $L(t) = 0$ for any t larger than t^L_0.

Possibilistic variables with quadratic membership functions have been applied to linear regression [28], identification of linear systems [29], evidence theory [30], portfolio section [32] and so forth. As far as we know, this paper is the first study to consider bilevel linear programming problem with possibilistic variables characterized by quadratic membership functions.

In problem (1), $\tilde{A}_{ijk}$, $\forall i \in I$, $j = 1, 2$, $k = 1, 2, \ldots, n_j$ and $\tilde{B}_i$ are possibilistic variables which are expressed as L-L fuzzy numbers and L-R fuzzy numbers characterized by the following membership functions:

$$\mu_{\tilde{A}_{ijk}}(a_{ijk}) = \begin{cases} L_a\left(\dfrac{m^a_{ijk} - a_{ijk}}{\alpha^a_{ijk}}\right) & \text{if } m^a_{ijk} \geq a_{ijk} \\ L_a\left(\dfrac{a_{ijk} - m^a_{ijk}}{\beta^a_{ijk}}\right) & \text{if } m^a_{ijk} < a_{ijk} \end{cases} \tag{3}$$

and

$$\mu_{\tilde{B}_i}(b_i) = \begin{cases} L_b\left(\dfrac{m^b_i - b_i}{\alpha^b_i}\right) & \text{if } m^b_i \geq b_i \\ R_b\left(\dfrac{b_i - m^b_i}{\beta^b_i}\right) & \text{if } m^b_i < b_i, \end{cases} \tag{4}$$

where L_a, L_b and R_b are reference functions satisfying the same conditions of L.

It should be noted here that problem (1) is an ill-defined problem because neither the meaning of minimizing the objective function nor that of constraints is well defined. In other words, some interpretation of the problem is needed so that the original problem can be reformulated as a well-defined one. In the following subsection, we shall discuss this issue and show how to transform the original problem into a well-defined one.

III. Possibilistic bi-level programming model

A. Possibilistic constraint

In this subsection, at the first step to transform the original problem (1) into a well-defined one, we focus only on the following constraints involving possibilistic variables:

$$\tilde{\boldsymbol{A}}_{i1}\boldsymbol{x}_1 + \tilde{\boldsymbol{A}}_{i2}\boldsymbol{x}_2 \leq \tilde{B}_i, \ \forall i \in I.$$

For simplicity, instead of the above constraint, we consider

$$\tilde{\boldsymbol{A}}_i\boldsymbol{x} \leq \tilde{B}_i,$$

where $\tilde{\boldsymbol{A}}_i = \left(\tilde{\boldsymbol{A}}_{i1}, \ \tilde{\boldsymbol{A}}_{i2}\right)$ and $\boldsymbol{x} = (\boldsymbol{x}^t_1, \ \boldsymbol{x}^t_2)^t$.

Since both sides of the above constraint involves possibilistic variables, the meaning of inequality sign $\leq$ is not uniquely determined, which means that some interpretation of the above constraint is necessary.

As one of reasonable and useful tools for decision making under fuzziness, possibility theory [33] has been widely used to deal with constraints involving possibilistic variables. Possibility theory is a mathematical theory for dealing with certain types of uncertainty. Zadeh [34] firstly introduced possibility theory in 1978 as an extension of fuzzy sets and fuzzy logic [35].

Along the line of possibilistic constraints in the framework of possibilistic programming [32], we consider the following constraint:

$$\Pi\left\{\tilde{\boldsymbol{A}}_i\boldsymbol{x} \leq \tilde{B}_i\right\} \geq \hat{h}^{cst}_i, \tag{5}$$

where Π denotes a possibility measure and $\hat{h}^{cst}_i$ is an aspiration level given by a DM. When the membership function of $\tilde{\boldsymbol{A}}_i\boldsymbol{x}$ and $\tilde{B}_i$ are given, on the basis of ranking of fuzzy number using possibility theory [36], the left-hand side of (5) is defined as

$$\Pi\left\{\tilde{\boldsymbol{A}}_i\boldsymbol{x} \leq \tilde{B}_i\right\} \triangleq \sup_{u^a_i \leq b} \min\left\{\pi_{\tilde{\boldsymbol{A}}_i\boldsymbol{x}}\left(u^a_i\right), \pi_{\tilde{B}_i}\left(b_i\right)\right\}, \tag{6}$$

where $\pi_{\tilde{\boldsymbol{A}}_i\boldsymbol{x}}\left(u^a_i\right)$ and $\pi_{\tilde{B}_i}\left(b_i\right)$ are possibilistic distribution functions of $\tilde{\boldsymbol{A}}_i\boldsymbol{x}$ and $\tilde{B}_i$, respectively.

In general, membership functions can be regarded as possibilistic distribution functions. Through the Zadeh's extension principle, the membership functions of possibilistic variables corresponding to $\tilde{\boldsymbol{A}}_i\boldsymbol{x}$ is calculated as

$$\begin{aligned} \pi_{\tilde{\boldsymbol{A}}_i\boldsymbol{x}}\left(u^a_i\right) &= \mu_{\tilde{\boldsymbol{A}}_i\boldsymbol{x}}\left(u^a_i\right) \\ &= \begin{cases} L_a\left(\dfrac{\boldsymbol{m}^a_i\boldsymbol{x} - u^a_i}{\boldsymbol{\alpha}^a_i\boldsymbol{x}}\right) & \text{if } \boldsymbol{m}^a_i\boldsymbol{x} \geq u^a_i \\ L_a\left(\dfrac{u^a_i - \boldsymbol{m}^a_i\boldsymbol{x}}{\boldsymbol{\beta}^a_i\boldsymbol{x}}\right) & \text{if } \boldsymbol{m}^a_i\boldsymbol{x} < u^a_i. \end{cases} \end{aligned} \tag{7}$$

In order to describe how likely an event occurs, possibility theory deals with not only the possibility of event using possibility measures but also the necessity of the event using necessity measures. Whereas possibility measures are used by optimistic DMs, necessity measures are recommended to pessimistic DMs. Therefore, it is worth introducing the constraint using a necessity measure because the DM may consider that some of constraints is necessarily satisfied. Then, we consider the following constraint:

$$N\left\{\tilde{\boldsymbol{A}}_i\boldsymbol{x} \leq \tilde{B}_i\right\} \geq \hat{h}^{cst}_i, \tag{8}$$

where N is a necessity measure and $\hat{h}^{cst}_i$ is an aspiration level specified by a DM. For any set or event U, necessity measures are defied by

$$N\{U\} = 1 - \Pi\{\overline{U}\},$$

where $\overline{U}$ denotes the complement of U. Hence, the left-hand side of (8) is defined as

$$N\left\{\tilde{\boldsymbol{A}}_i\boldsymbol{x} \leq \tilde{B}_i\right\} \triangleq \inf_{u^a_i \leq b} \max\left\{1 - \pi_{\tilde{\boldsymbol{A}}_i\boldsymbol{x}}\left(u^a_i\right), \pi_{\tilde{B}_i}\left(b_i\right)\right\}. \tag{9}$$

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By applying the outcomes obtained by previous studies on possibilistic programming [32] to the constraints using possibility and necessity measures, constraint (5) is transformed into

$$\boldsymbol{m}_i^a \boldsymbol{x} - L_a^* \left(\hat{h}_i^{cst}\right) \boldsymbol{\alpha}_i^a \boldsymbol{x} \le m_i^b + R_b^* \left(\hat{h}_i^{cst}\right) \beta_i^b. \tag{10}$$

Similarly, constraint (8) is written as

$$\boldsymbol{m}_i^a \boldsymbol{x} + L_a^* \left(1 - \hat{h}_i^{cst}\right) \boldsymbol{\beta}_i^a \boldsymbol{x} \ge m_i^b - L_b^* \left(1 - \hat{h}_i^{cst}\right) \alpha_i^b. \tag{11}$$

B. Possibilistic Stackelberg problem

In the previous subsection, we give an interpretation of the constraints with possibilistic variables on the basis of possibility theory and transform the original possibilistic constraints (5) and (8) into deterministic linear constraints (10) and (11), respectively.

It should be noted here that (1) is still an ill-defined problem because the objective function of each DM involves possibilistic variables. In other words, the Stackelberg solution of (1) has not been clearly defined yet.

Therefore, we consider the following Stackelberg problem as one of the reasonable decision making models for BLPP with possibilistic variables.

$$\left.\begin{array}{ll}
\underset{\boldsymbol{x}_1, \boldsymbol{x}_2, f_1, f_2}{\text{maximize}} & f_1 \\
\text{where } \boldsymbol{x}_2 \text{ and } f_2 \text{ solve} & \\
\underset{\boldsymbol{x}_2, f_2}{\text{maximize}} & f_2 \\
\text{subject to} & N\left\{\tilde{\boldsymbol{C}}_1 \boldsymbol{x} \ge f_1\right\} \ge \hat{h}_1^{obj} \\
& N\left\{\tilde{\boldsymbol{C}}_2 \boldsymbol{x} \ge f_2\right\} \ge \hat{h}_2^{obj} \\
& \Pi\left\{\tilde{\boldsymbol{A}}_i \boldsymbol{x} \le \tilde{B}_i\right\} \ge \hat{h}_i^{cst}, \ \forall i \in I_{pos} \\
& N\left\{\tilde{\boldsymbol{A}}_i \boldsymbol{x} \le \tilde{B}_i\right\} \ge \hat{h}_i^{cst}, \ \forall i \in I_{nec} \\
& \boldsymbol{a}_{i1}\boldsymbol{x}_1 + \boldsymbol{a}_{i2}\boldsymbol{x}_2 \le b_i, \\
& \qquad i = r+1, r+2, \ldots, v \\
& \boldsymbol{x}_1 \ge \boldsymbol{0}, \ \boldsymbol{x}_2 \ge \boldsymbol{0},
\end{array}\right\} \tag{12}$$

where I_{pos} and I_{nec} are index sets satisfying $I_{pos} \cup I_{nec} = I$ and $I_{pos} \cap I_{nec} = \emptyset$.

It should be noted here that the Stackelberg problem to be solved, which is an interpretation of the original ill-defined problem (1), is clearly defined, which means that the Stackelberg solution of (1) is defined as the Stackelberg solution of (12).

Since we have already obtained (10) and (11), the remaining task is to transform the following constraint into deterministic ones:

$$N\left\{\tilde{\boldsymbol{C}}_1 \boldsymbol{x} \ge f_1\right\} \ge \hat{h}_1^{obj},$$
$$N\left\{\tilde{\boldsymbol{C}}_2 \boldsymbol{x} \ge f_2\right\} \ge \hat{h}_2^{obj}.$$

Through the Zadeh's extension principle, the membership function of a possibilistic variable corresponding to each of objective functions $z_l(\boldsymbol{x}_1, \boldsymbol{x}_2)$, $l = 1, 2$ is given as

$$\pi_{\tilde{\boldsymbol{C}}_l \boldsymbol{x}}\left(u_l^c\right) = \mu_{\tilde{\boldsymbol{C}}_l \boldsymbol{x}}\left(u_l^c\right) = L\left(\frac{(u_l^c - \boldsymbol{d}_l^c \boldsymbol{x})^2}{\boldsymbol{x}^t U_l^c \boldsymbol{x}}\right). \tag{13}$$

Then, we transform the constraint

$$N\left\{\tilde{\boldsymbol{C}}_l \boldsymbol{x} \ge f_l\right\} \ge \hat{h}_l^{obj}$$

as the following deterministic nonlinear constraint [31]:

$$\boldsymbol{d}_l^c \boldsymbol{x} - \sqrt{L^*\left(1 - \hat{h}_l^{obj}\right) \boldsymbol{x}^t U_l^c \boldsymbol{x}} \ge f_l. \tag{14}$$

It should be noted here that the maximization of f_l under the constraint (14) is equivalent to the maximization of

$$\boldsymbol{d}_l^c \boldsymbol{x} - \sqrt{L^*\left(1 - \hat{h}_l^{obj}\right) \boldsymbol{x}^t U_l^c \boldsymbol{x}}.$$

Therefore, (12) is equivalently transformed into the following problem:

$$\left.\begin{array}{l}
\underset{\boldsymbol{x}_1, \boldsymbol{x}_2, f_1, f_2}{\text{maximize}} \ \boldsymbol{d}_1^c \boldsymbol{x} - \sqrt{L^*\left(1 - \hat{h}_1^{obj}\right) \boldsymbol{x}^t U_1^c \boldsymbol{x}} \\
\text{where } \boldsymbol{x}_2 \text{ and } f_2 \text{ solve} \\
\underset{\boldsymbol{x}_2, f_2}{\text{maximize}} \ \boldsymbol{d}_2^c \boldsymbol{x} - \sqrt{L^*\left(1 - \hat{h}_2^{obj}\right) \boldsymbol{x}^t U_2^c \boldsymbol{x}} \\
\text{subject to } \boldsymbol{m}_i^a \boldsymbol{x} - L_a^*\left(\hat{h}_i^{cst}\right) \boldsymbol{\alpha}_i^a \boldsymbol{x} \\
\qquad \le m_i^b + R_b^*\left(\hat{h}_i^{cst}\right) \beta_i^b, \ \forall i \in I_{pos} \\
\qquad \boldsymbol{m}_i^a \boldsymbol{x} + L_a^*\left(1 - \hat{h}_i^{cst}\right) \boldsymbol{\beta}_i^a \boldsymbol{x} \\
\qquad \ge m_i^b - L_b^*\left(1 - \hat{h}_i^{cst}\right) \alpha_i^b, \ \forall i \in I_{nec} \\
\qquad \boldsymbol{a}_{i1}\boldsymbol{x}_1 + \boldsymbol{a}_{i2}\boldsymbol{x}_2 \le b_i, \\
\qquad\qquad i = r+1, r+2, \ldots, v \\
\qquad \boldsymbol{x}_1 \ge \boldsymbol{0}, \ \boldsymbol{x}_2 \ge \boldsymbol{0}.
\end{array}\right\} \tag{15}$$

It should be emphasized that problem (15) is a deterministic problem that is obtained from the original possibilistic BLPP (1) through the proposed decision making model expressed by (12).

IV. Solution procedure

For the resulting bilevel programming problem (15) which has nonlinear objective functions and linear constraints, recall that DM1 first makes a decision $\boldsymbol{x}_1$, and then DM2 makes a decision $\boldsymbol{x}_2$ so as to optimize the objective function with full knowledge of decision $\boldsymbol{x}_1$ of DM1. In other words, DM2 optimally responses for a given decision of DM1 by solving the mathematical programming problem for DM2. To be more precise, when we consider a Stackelberg problem for (15), it is assumed that DM1 selects a decision $\boldsymbol{x}_1$ such that his/her objective function is optimized on the assumption that DM2 chooses $\boldsymbol{x}_2$ as a rational reaction to $\boldsymbol{x}_1$, denoted by $\boldsymbol{x}_2(\boldsymbol{x}_1)$. The solution obtained by such a procedure is called a Stackelberg solution. It should be noted here that $\boldsymbol{x}_2(\boldsymbol{x}_1)$ is not always uniquely determined because there may be a lot of solutions $\boldsymbol{x}_2$ that optimize the DM2's objective function for a given $\boldsymbol{x}_1$.

Now we discuss how to obtain a Stackelberg solution to (15). Let S be a set of feasible solutions $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ of problem (15). Also, let $Z_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $Z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ be

$$\begin{aligned}
& Z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
& \quad = \ \boldsymbol{d}_1^c \boldsymbol{x} - \sqrt{L^*\left(1 - \hat{h}_1^{obj}\right) \boldsymbol{x}^t U_1^c \boldsymbol{x}} \\
& \quad = \ \boldsymbol{d}_{11}^c \boldsymbol{x}_1 + \boldsymbol{d}_{12}^c \boldsymbol{x}_2 - \sqrt{L^*\left(1 - \hat{h}_1^{obj}\right)} \\
& \qquad \times \sqrt{\boldsymbol{x}_1^t U_{11}^c \boldsymbol{x}_1 + 2\boldsymbol{x}_1^t U_{12}^c \boldsymbol{x}_2 + \boldsymbol{x}_2^t U_{13}^c \boldsymbol{x}_2}
\end{aligned} \tag{16}$$

and

$$
\begin{aligned}
& Z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
&= \boldsymbol{d}_2^c \boldsymbol{x} - \sqrt{L^*\left(1-\hat{h}_2^{obj}\right) \boldsymbol{x}^t U_2^c \boldsymbol{x}} \\
&= \boldsymbol{d}_{21}^c \boldsymbol{x}_1 + \boldsymbol{d}_{22}^c \boldsymbol{x}_2 - \sqrt{L^*\left(1-\hat{h}_2^{obj}\right)} \\
&\quad \times \sqrt{\boldsymbol{x}_1^t U_{21}^c \boldsymbol{x}_1 + 2\boldsymbol{x}_1^t U_{22}^c \boldsymbol{x}_2 + \boldsymbol{x}_2^t U_{23}^c \boldsymbol{x}_2}, \quad (17)
\end{aligned}
$$

where

$$
U_1^c = \begin{pmatrix} U_{11}^c & U_{12}^c \\ (U_{12}^c)^t & U_{13}^c \end{pmatrix}, \quad
U_2^c = \begin{pmatrix} U_{21}^c & U_{22}^c \\ (U_{22}^c)^t & U_{23}^c \end{pmatrix}.
$$

Then, a Stackelberg solution to the bilevel programming problem (15) is defined as:

$$
\left\{ (\boldsymbol{x}_1, \boldsymbol{x}_2) \;\middle|\; (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \arg \max_{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in IR} Z_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \right\},
$$

where IR is an inducible region defined by

$$
IR = \{(\boldsymbol{x}_1, \boldsymbol{x}_2) \mid (\boldsymbol{x}_1, \boldsymbol{x}_2) \in S, \ \boldsymbol{x}_2 \in R(\boldsymbol{x}_1)\}.
$$

Here, $R(\boldsymbol{x}_1)$ is a set of rational response of DM2 to a given $\boldsymbol{x}_1$, defined by

$$
R(\boldsymbol{x}_1) = \left\{ \boldsymbol{x}_2 \;\middle|\; \boldsymbol{x}_2 \in \arg \max_{\boldsymbol{x}_2 \in S(\boldsymbol{x}_1)} Z_2(\boldsymbol{x}_1, \boldsymbol{x}_2) \right\}
$$

and $S(\boldsymbol{x}_1)$ is a feasible solution set of $\boldsymbol{x}_2$ for a fixed $\boldsymbol{x}_1$.

In other words, IR is obtained by calculating $\boldsymbol{x}_2$ of the following lower-level problem for each of given $\check{\boldsymbol{x}}_1$.

$$
\left.
\begin{aligned}
& \underset{\boldsymbol{x}_2}{\text{maximize}} \ \boldsymbol{d}_{22}^c \boldsymbol{x}_2 - \sqrt{L^*\left(1-\hat{h}_2^{obj}\right)} \\
& \qquad \times \sqrt{\check{\boldsymbol{x}}_1^t U_{21}^c \check{\boldsymbol{x}}_1 + 2\check{\boldsymbol{x}}_1^t U_{22}^c \boldsymbol{x}_2 + \boldsymbol{x}_2^t U_{23}^c \boldsymbol{x}_2} \\
& \text{subject to } \boldsymbol{m}_{i2}^a \boldsymbol{x}_2 - L_a^*\left(\hat{h}_i^{cst}\right) \boldsymbol{\alpha}_{i2}^a \boldsymbol{x}_2 \\
& \qquad \le m_i^b + R_b^*\left(\hat{h}_i^{cst}\right) \beta_i^b \\
& \qquad -\boldsymbol{m}_{i1}^a \check{\boldsymbol{x}}_1 + L_a^*\left(\hat{h}_i^{cst}\right) \boldsymbol{\alpha}_{i1}^a \check{\boldsymbol{x}}_1, \ \forall i \in I_{pos} \\
& \quad \boldsymbol{m}_{i2}^a \boldsymbol{x}_2 + L_a^*\left(1-\hat{h}_i^{cst}\right) \boldsymbol{\beta}_{i2}^a \boldsymbol{x}_2 \\
& \qquad \ge m_i^b - L_a^*\left(1-\hat{h}_i^{cst}\right) \alpha_i^b - \boldsymbol{m}_{i1}^a \check{\boldsymbol{x}}_1 \\
& \qquad -L_a^*\left(1-\hat{h}_i^{cst}\right) \boldsymbol{\beta}_{i1}^a \check{\boldsymbol{x}}_1, \ \forall i \in I_{nec} \\
& \quad \boldsymbol{a}_{i2} \boldsymbol{x}_2 \le b_i - \boldsymbol{a}_{i1} \check{\boldsymbol{x}}_1, \ i = r+1, r+2, \ldots, v \\
& \quad \boldsymbol{x}_2 \ge \boldsymbol{0}.
\end{aligned}
\right\} \quad (18)
$$

It is very important to check whether or not $R(\boldsymbol{x}_1)$ is a singleton for any fixed $\boldsymbol{x}_1$. If $R(\boldsymbol{x}_1)$ is not a singleton, then DM1 has to select one solution in $R(\boldsymbol{x}_1)$ as a rational reaction of DM2 to $\boldsymbol{x}_1$. In this case, the concept of weak/strong (or optimistic/pessimistic) Stackelberg solution [37] is necessary. Fortunately, we do not need to introduce weak/strong Stackelberg solution to (15) because $R(\boldsymbol{x}_1)$ is proved to be a singleton for any fixed $\boldsymbol{x}_1$.

Theorem 1: $R(\boldsymbol{x}_1)$ is a singleton for any fixed $\boldsymbol{x}_1$.

Proof: Since U_2^c is positive definite, $\boldsymbol{x}_2^t U_{23}^c \boldsymbol{x}$ is a strictly convex function. The constraint is linear and then problem (18) is a strictly convex programming problem, which means that $R(\boldsymbol{x}_1)$ is a singleton for any fixed $\boldsymbol{x}_1$. ■

Note that the objective function of DM1 defined by (16) is strictly concave for any fixed rational response $R(\boldsymbol{x}_1)$ of DM2 that is a singleton. In other words, the upper-level problem to be solved for DM1 is also a strictly concave programming problem. From this fact together with the above theorem, the Stackelberg solution of (15) is uniquely determined. Thus, the Stackelberg solution of (15) is exactly obtained by existing computational methods for obtaining a Stackelberg solution to nonlinear BLPPs [38], [39]. Edmunds and Bard [40] introduced a solution algorithm using branch-and-bound techniques which does not guarantee global optimality but assures ϵ-optimality. Savard and Gauvin [39] developed a descent direction method for nonlinear BLPPs using the property that the steepest descent direction coincides with the optimal solution of the linear-quadratic bilevel program. Gümüs and Floudas [38] constructed an exact solution algorithm for nonlinear BLPPs. Falk and Liu [41] presented a bundle method using subdifferential information obtained from the lower-level problem. Colson et al. [42] developed a trust-region method for solving nonlinear BLPPs. If readers are interested in various solution algorithms for BLPPs, refer to bibliography and/or overview of BLPPs [43], [44], [45].

V. Conclusion

In this paper, assuming noncooperative behavior of the two DMs, we have considered a possibilistic bilevel linear programming problem. In order to properly handle possibilistic information involved in the problem, we have developed a novel decision making model. Though the proposed decision making model, we have transformed the original possibilistic bilevel programming problem into a deterministic nonlinear bilevel programming problem. Using the convexity property of the resulting problem, we have shown that the Stackelberg solution of the problem is obtained by using conventional nonlinear bilevel programming techniques. In the future, we will apply the proposed model to real-world hierarchical decision making problems. Extensions of the proposed model in this paper to cooperative cases [46] will be considered elsewhere.

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