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A Hybrid Extragradient Method for Solving Ky Fan Inequalities, Variational Inequalities and Fixed Point Problems

Supak Phiangsungnoen and Poom Kumam, *Member, IAENG.*

***Abstract*—The purpose of this paper is to introduce a new hybrid extragradient iterative method for finding a common element of the set of points satisfying the Ky Fan inequalities, variational inequality and the set of fixed points of a strict pseudocontraction mapping in Hilbert spaces. Consequently, we obtain the strong convergence of an iterative algorithm generated by the hybrid extragradient projection method, under some suitable assumptions the function associated with Ky Fan inequality is pseudomonotone and weakly continuous.**

***Index Terms*—Extragradient Method, Fixed point problems, Hybrid projection method,Variational inequality problems, $\xi-$strict pseudocontraction, Lipschitz continuity.**

I. Introduction

THROUGHOUT this paper, we always assume that H be a real Hilbert space, whose inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a closed convex subset of H and $F : C \times C \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real number, a bifunction. We concider the Ky Fan inequality which was first introduced by Ky Fan [25] as follows:

$$\text{find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of such that $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Numerous problems in physics, optimization, and economics reduce to find a solution of (1). Some methods have been proposed to solve the Ky Fan inequality (or some time called equilibrium problem, see [1, 3, 8, 14]).

In 2005, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by $Fix(S)$ the set of fixed points of, S i.e., $Fix(S) = \{x \in C : Sx = x\}$. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $Fix(S)$ is nonempty (see [6]).

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Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand.
Emails: supuk_piang@hotmail.com (S. Phiangsungnoen) and poom.kum@kmutt.ac.th (P. Kumam)

We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0.$$

The classical *variational inequality problem* is to find $u \in C$ such that $\langle v - u, Au\rangle \geq 0$ for all $v \in C$. We denoted by $VI(A, C)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g. [19, 20] and the references therein. A mapping A of C into H is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{2}$$

for all $u, v \in C$. It is obvious that any α -inverse-strongly-monotone mappings A is monotone and Lipschitz continuous.

In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \tag{3}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [11].

In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [4]. Attempts to modify the Mann iteration method (3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [9] proposed the following modification of the Mann iteration method (3):

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{4}$$

where P_C is metric projection on C.

For finding an element of $Fix(S) \cap VI(A, C)$, Takahashi and Toyoda [18] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{5}$$

for every $n = 0, 1, 2, ...$, where P_C is the metric projection on the set C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved a weak convergence theorem in a Hilbert space.

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On the other hand, for finding an element of $Fix(S)$, Takahashi et al., [17] introduced the following iteration procedure which is usually called the shrinking projection method. Let C be nonempty closed convex subset of a real Hilbert space H. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1, \end{cases} \tag{6}$$

where P_{C_n} is the metric projection of H onto C_n and $\{S_n\}$ is a family of nonexpansive mappings. They proved that the sequence $\{x_n\}$ generated by (6) converges strongly to $z = P_{Fix(S)}x_0$, where $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$.

Moreover, in 2012, Phan Tu Vuong, et. al., [26], they considered the sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y\in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)Sz_n], \\ x_{n+1} = t_n, \end{cases} \tag{7}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1[$, $\{\beta_n\} \subset]0,1[$, and $\{\lambda_n\} \subset]0,1]$.

In this paper, we consider the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ and $\{t_n\}$ generated by $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y\in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ w_n = P_C(z_n - \lambda_n A z_n), \\ t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n], \\ C_{n+1} = \{z \in C_n : \|t_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \end{cases} \tag{8}$$

for every $n \in \mathbb{N}$, where μ be a constant in $(0,1)$, $\{\alpha_n\} \subset [0,1), \{\beta_n\} \subset (0,1), \{\lambda_n\} \subset (0,1]$. We combine the equations (5), (6) and (7) above for solving the $EP(F)$ with a fixed point and variational inequality problems. Consequencely, we obtained the strong convergent theorem for solving fixed point problems, Ky Fan inequality and variational inequality problems.

II. Preliminaries

Let C be a closed convex subset of a Hilbert space H. Then the following hold:y

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{9}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{10}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the *Opial's condition* [10], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{11}$$

holds for every $y \in H$ with $y \neq x$. Also Hilbert space H satisfies the *Kadec-Klee property* [5, 15], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ satisfying the property

$$\|x - P_C x\| \leq \|x - y\|. \tag{12}$$

For a given $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{13}$$

It is well known that P_C is *firmly nonexpansive* of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2. \tag{14}$$

In the context of the variational inequality problem, this implies that

$$u \in VI(A, C) \Leftrightarrow u = P_C(u - \lambda Au), \quad \forall \lambda > 0. \tag{15}$$

We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &\leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Au - Av\|^2. \end{aligned} \tag{16}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Let us recall that S is a ξ-strict pseudocontraction mapping if there exists a scalar $\xi \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \xi\|(x - Sx) - (y - Sy)\|^2 \tag{17}$$

for every $x, y \in C$. It is easy to see that a nonexpansive mapping on C is also a 0-strict pseudocontraction mapping. Furthermore [24], if S is a ξ-strict pseudocontraction mapping, then the fixed point set $Fix(S)$ is closed and convex and the mapping $I - S$ is demiclosed at zero, i.e., satisfies the property

$$x_n \rightharpoonup x \quad \text{and} \quad Sx_n - x_n \to 0 \Rightarrow Sx = x.$$

A set valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - h\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(A, C)$; see [12, 13].

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Proposition 1. *([21], Lemma 3.1) For every $x^* \in EP(F)$, and every $n \in \mathbb{N}$, one has*

(i) $\langle x_n - y_n, y - y_n\rangle \le \lambda_n F(x_n, y) - \lambda_n F(x_n, y_n)$, $\forall y \in C$;

(ii) $\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1-2\lambda_n c_1)\|y_n - x_n\|^2 - (1-2\lambda_n c_2)\|z_n - y_n\|^2$.

Lemma 2. *[28] Let C be a closed convex subset of H and let $\{x_n\}$ be a bounded sequence in H. Assume that*

1) *the weak $\omega -$ limit set $\omega_w(x_n) \subset C$,*
2) *for each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

Proposition 3. *[24] Let K be a nonempty closed and convex subset of H. Let $u \in H$ and let $\{x_n\}$ be a sequence in H. If any weak limit point of $\{x_n\}$ belongs to K, and $\|x_n - u\| \le \|u - P_K u\|$ for all $n \in \mathbb{N}$, then $x_n \to P_K u$.*

In order to apply this Proposition, we set $K := EP(F) \cap Fix(S) \cap VI(A, C)$ and $u = x_0$. Furthermore, we impose that the sequence $\{x_n\}$ generated by our algorithms satisfies, for all $n \in \mathbb{N}$, $\|x_n - x_0\| \le \|x_{n+1} - x_0\| \le \|\tilde{x_0} - x_0\|$
where $\tilde{x_0} = P_K x_0$. In that case, the sequence $\{\|x_n - x_0\|\}$ is convergent and the sequence $\{x_n\}$ is bounded. These properties will be useful to prove that any weak limit point of $\{x_n\}$ belongs to K.

For solving the Ky Fan inequality or equilibrium problem, let us assume that the following conditions (A1) - (A5) are satisfied on the bifunction $F : C \times C \to R$.

(A1) $F(x, x) = 0$ for every $x \in C$;
(A2) F is pseudomonotone on C, i.e.,
$F(x, y) \ge 0 \Rightarrow F(y, x) \le 0, \quad \forall x, y \in C$;
(A3) F is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y, respectively, then $F(x_n, y_n) \to F(x, y)$;
(A4) $F(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C foe very $x \in C$;
(A5) F satisfies the Lipschitz-type condition, there exist positive integer c_1 and c_2, for every $x, y, z \in C$,
$F(x, y) + F(y, z) \ge F(x, z) - c_1\|y - x\|^2 - c_2\|z - y\|^2$.

If F satisfies the properties (A1)-(A4), then the set $EP(F)$ of solutions to the Ky Fan inequality is closed and convex.

Remark 4. A first example of function F satisfying assumption $(A5)$, is given by

$$F(x, y) = \langle G(x), y - x\rangle \quad \forall x, y \in C,$$

where $G : C \to H$ is Lipschitz continuous on C (with constant $L > 0$) (see[22]). In that example, $c_1 = c_2 = L\backslash 2$. Another example, related to the Cournot-Nash equilibrium model, is described in [23]. The function $F : C \times C \to \mathbb{R}$ is defined, for every $x, y \in C$, by

$$F(x, y) = \langle G(x) + Q(y) + q, y - x\rangle,$$

with $C = \{x \in \mathbb{R}^n : Ax \le b\}, F : C \to \mathbb{R}^n, Q \in \mathbb{R}^{n\times n}$, a symmetric positive semidefinite matrix, and $q \in \mathbb{R}$. If F is Lipschitz continuous on C (with constant $L > 0$) then F satisfies $(A5)$ with $c_1 > 0, c_2 > 0$ such that $2\sqrt{c_1 c_2} \ge L + \|Q\|$.

III. Main Result

A. An Algorithm

Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C\times$ C into $\mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C.

Algorithm 5. Choose the sequences $\{\alpha_n\} \subset [0, 1), \{\beta_n\} \subset (0, 1), \{\lambda_n\} \subset (0, 1]$ and μ be a constant in $(0, 1)$.

Assume that the following 6 Steps S(1) - S(6)

S(1) Let $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$. Set $n = 0$.
S(2) Solve successively the strongly convex programs
$\arg\min_{y\in C}\{\lambda_n F(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}$ and
$\arg\min_{y\in C}\{\lambda_n F(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}$ to obtain the unique optimal solution y_n and z_n, respectively.
S(3) Compute $w_n = P_C(z_n - \lambda_n A z_n)$.
S(4) Compute
$t_n = \alpha_n x_n + (1-\alpha_n)[(1-\mu)S w_n + \mu P_C(1-\beta_n) w_n]$.
If $y_n = x_n$ and $t_n = x_n$, then STOP:
$x_n \in EP(F) \cap Fix(S) \cap VI(A, C)$. Otherwise, go to Step 5.
S(5) Compute $x_{n+1} = P_{C_{n+1}} x_0$,
where $C_{n+1} = \{z \in C_n : \|t_n - z\| \le \|x_n - z\|\}$,
S(6) Set $n := n + 1$, and go to Step 2.

In the sequel, we also suppose that the sequences of parameters $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \xi$ and μ satisfy the following conditions: (1) - (4)

(1) $\{\lambda_n\} \subset [\lambda_{\min}, \lambda_{\max}]$, where $0 < \lambda_{\min} \le \lambda_{\max} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$;
(2) $\{\alpha_n\} \subset [0, c]$ for some $c < 1$;
(3) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(4) ξ and μ be constant, where $0 \le \xi < \mu < 1$.

Now, let $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{w_n\}$ be the sequences generated by combination of the hybrid extragradient method, variational inequality by projection method and the fixed point method described at the beginning of this section.

B. A strong convergence theorem

Here we start our main theorem.

Theorem 6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C\times C$ into $\mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C and such that $\Omega := EP(F) \cap Fix(S) \cap VI(A, C) \ne \emptyset$. Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and μ satisfying the conditions $(1) - (4)$. Then the sequence $\{x_n\}$ generated by Algorithm 5 converges strongly to the projection of x_0 on to the set Ω.*

Proof. ***Step 1.*** We show that $\{x_n\}$ is well defined.
First, we show that C_n is closed and convex for each $n \ge 1$. Indeed, it is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \ge 1$.

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Next, we show that C_{k+1} is closed and convex for the same k. Let $z_1, z_2 \in C_{k+1}$ and $z = tz_1 + (1-t)z_2$, where $t \in (0,1)$. Notice that $\|t_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|t_n - x_n\|^2 + 2\langle t_n - x_n, x_n - z\rangle \leq 0$$

Thus C_{k+1} is closed and convex. Then, C_n is closed and convex for any $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Step 2. Next, we show by induction that $\Omega \subset C_n$ for each $n \geq 1$. Let $x^* \in \Omega$. Then $x^* = P_C(x^* - \lambda_n Ax^*)$.
Since $w_n = P_C(z_n - \lambda_n Az_n)$, we consider

$$\begin{aligned}
&\|w_n - x^*\|^2 \\
&\quad = \|[P_C(z_n - \lambda_n Az_n)] - [P_C(x^* - \lambda_n Ax^*)]\|^2 \\
&\quad \leq \|(z_n - \lambda_n Az_n) - (x^* - \lambda_n Ax^*)\|^2 \\
&\quad = \|z_n - x^*\|^2 - 2\lambda_n\langle z_n - x^*, Az_n - Ax^*\rangle \\
&\qquad + \lambda_n^2\|Az_n - Ax^*\|^2 \\
&\quad \leq \|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Az_n - Ax^*\|^2 \qquad (18) \\
&\quad \leq \|z_n - x^*\|^2.
\end{aligned}$$

By Proposition 1 (ii), we have

$$\begin{aligned}
\|z_n - x^*\|^2 \leq\ & \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 \\
& -(1 - 2\lambda_n c_2)\|z_n - y_n\|^2 \qquad (19)
\end{aligned}$$

that is, we obtain $\|z_n - x^*\| \leq \|x_n - x^*\|$
and $\|w_n - x^*\| \leq \|x_n - x^*\|$.
Set $u_n := (1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n$, for all $n \geq 0$.
Then, we have $t_n = \alpha_n x_n + (1-\alpha_n)u_n$, It follows that

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&\quad = \|(1-\mu)Sw_n + \mu P_C(1-\beta_n)w_n - x^*\|^2 \\
&\quad \leq \|(1-\mu)(Sw_n - x^*) + \mu\big[(1-\beta_n)w_n - x^*\big]\|^2 \\
&\quad \leq \|w_n - x^*\|^2 - (1-\mu)(\mu - \xi)\|Sw_n - w_n\|^2 \\
&\qquad -\beta_n\mu^2\|w_n\|^2 \qquad (20) \\
&\quad \leq \|w_n - x^*\|^2 \leq \|x_n - x^*\|^2
\end{aligned}$$

that is, $\|u_n - x^*\| \leq \|x_n - x^*\|$.

Since $t_n = \alpha_n x_n + (1-\alpha_n)u_n$ for every $x^* \in \Omega$, we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|\alpha_n x_n + (1-\alpha_n)u_n - x^*\|^2 \\
&= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned}$$

Hence $\|t_n - x^*\| \leq \|x_n - x^*\|$. It follows that $x^* \in C_{n+1}$.
This implies that

$$\Omega := EP(F) \cap Fix(S) \cap VI(A,C) \subset C_n, \quad \forall n \in \mathbb{N}.$$

Step 3. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|t_n - x_n\| = 0$.
From $x_n = P_{C_n}x_0$, we have

$$\langle x_0 - x_n, x_n - v\rangle \geq 0$$

for each $v \in C_n$. Using $\Omega \subset C_n$ for each we also have

$$\langle x_0 - x_n, x_n - z\rangle \geq 0, \quad \forall z \in \Omega, n \in \mathbb{N}.$$

So, for $z \in \Omega$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - z\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - z\rangle \\
&= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - z\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - z\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - z\|, \quad \forall z \in \Omega, n \in \mathbb{N}.$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \qquad (21)$$

From (21), we have, for $n \in \mathbb{N}$,

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.
\end{aligned}$$

It follows that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Thus the sequence $\{\|x_n - x_0\|\}$ is bounded and nonincreasing sequence, so $\lim_{n\to\infty}\|x_n - x_0\|$ exists, that is

$$\lim_{n\to\infty}\|x_n - x_0\| = m. \qquad (22)$$

Indeed, (21), we get

$$\begin{aligned}
&\|x_n - x_{n+1}\|^2 \\
&\quad = \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&\quad = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle \\
&\qquad + \|x_0 - x_{n+1}\|^2 \\
&\quad = -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle \\
&\qquad + \|x_0 - x_{n+1}\|^2 \\
&\quad \leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

From (22), we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \qquad (23)$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned}
C_n = \{z \in C : \|t_n - z\| \leq \|x_n - z\|\}; \\
\|t_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_n - t_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \\
&\leq 2\|x_n - x_{n+1}\|.
\end{aligned}$$

By (23), we obtain

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \qquad (24)$$

Step 4. We will show that $\lim_{n\to\infty}\|Sw_n - w_n\| = 0$. For $x^* \in \Omega$, from (18), (19) and (20), we can choose a constant $M > 0$ such that,

$$\sup_n\{\|w_n\|^2\} \leq M.$$

We observe that

$$\begin{aligned}
&\|t_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\big[\|w_n - x^*\|^2 \\
&\quad -(1-\mu)(\mu-\xi)\|Sw_n - w_n\|^2 - \beta_n\mu^2\|w_n\|^2\big] \\
&= \|x_n - x^*\|^2 - (1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \\
&\quad -(1-\alpha_n)(1-2\lambda_n c_2)\|z_n - y_n\|^2 \\
&\quad +(1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Az - Ax^*\|^2 \\
&\quad -(1-\alpha_n)(1-\mu)(\mu-\xi)\|Sw_n - w_n\|^2 \\
&\quad -(1-\alpha_n)\beta_n\mu^2 M.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&(1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 - (1-\alpha_n)\beta_n\mu^2 M \\
&= \big[\|x_n - x^*\| + \|t_n - x^*\|\big]\big[\|x_n - t_n\| \\
&\quad -(1-\alpha_n)\beta_n\mu^2 M\big].
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - t_n\| = 0, \lim_{n\to\infty}\beta_n = 0$, $1-\alpha_n \geq 1-c > 0, 1-2\lambda_n c_1 > 1-2\lambda_{\max}c_1 > 0$ and the sequence $\{x_n\}, \{t_n\}$ are bounded, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

By similar way since $\lim_{n\to\infty}\beta_n = 0$, $1-\alpha_n \geq 1-c > 0$ and $1-2\lambda_n c_2 > 1-2\lambda_{\max}c_2 > 0$, we have

$$\lim_{n\to\infty}\|z_n - y_n\| = 0.$$

Since $\lim_{n\to\infty}\beta_n = 0, 1-\alpha_n \geq 1-c > 0$ and $-\lambda_n(\lambda_n - 2\alpha) > 0$, we obtain

$$\lim_{n\to\infty}\|Az - Ax^*\| = 0.$$

By $\lim_{n\to\infty}\beta_n = 0, \quad 1-\alpha_n \geq 1-c > 0$ and $(1-\mu)(\mu-\xi) > 0$, we have

$$\lim_{n\to\infty}\|Sw_n - w_n\| = 0.$$

Step 5. We will show that $\tilde{x} \in \Omega$.

(5.1). We will show that $\tilde{x} \in EP(F)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i} \rightharpoonup \tilde{x}$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have that $y_{n_i} \rightharpoonup \tilde{x}$. On the other hand, by using Proposition 1 (i), we have, for every $y \in C$ and for every $i \in \mathbb{N}$, that

$$\langle x_{n_i} - y_{n_i}, y - y_{n_i}\rangle \leq \lambda_{n_i}F(x_{n_i}, y) - \lambda_{n_i}F(x_{n_i}, y_{n_i}).$$

Since $\|x_{n_i} - y_{n_i}\| \to 0$ and $y - y_{n_i} \rightharpoonup y - \tilde{x}$ as $i \to \infty$ and since $\forall i \in \mathbb{N}, \quad 0 < \lambda_{\min} \leq \lambda_{n_i} \leq \lambda_{\max}$, as $i \to \infty$, we get

$$F(\tilde{x}, y) \geq 0, \forall y \in C.$$

It means that $\tilde{x} \in EP(F)$.

(5.2). We will show that $\tilde{x} \in Fix(S)$. Since $\{w_n\}$ is bounded then there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which converges weakly to $\tilde{x}$. Since S is a ξ-strict pseudocontraction mapping, we know that the mapping $I - S$ is demiclosed at zero. From $\|Sw_n - w_n\| \to 0$ as $n \to \infty$ and $\{w_{n_i}\} \rightharpoonup \tilde{x}$. Thus, we obtain that $\tilde{x} \in Fix(S)$.

(5.3). Finally, we show that $\tilde{x} \in VI(A, C)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, we have that T is maximal monotone operator. Since $w \in Tv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \geq 0, \quad (n = 1, 2, 3, \ldots).$$

We also have $w_n = P_C(z_n - \lambda_n A z_n)$ and $\forall v \in C$, we get

$$\big\langle v - w_n, \frac{w_n - z_n}{\lambda_n} + Az_n\big\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
&\langle v - w_{n_i}, w\rangle \\
&\geq \langle v - w_{n_i}, Av\rangle \\
&\geq \langle v - w_{n_i}, Av\rangle - \big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}} + Az_{n_i}\big\rangle \\
&= \langle v - w_{n_i}, Av - Aw_{n_i}\rangle + \langle v - w_{n_i}, Aw_{n_i} - Az_{n_i}\rangle \\
&\quad -\big\langle v - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{\lambda_{n_i}}\big\rangle.
\end{aligned}$$

Using $w_{n_i} \to \tilde{x}$ and $\|w_{n_i} - z_{n_i}\| \to 0$ which A is Lipshitz continuous implies that

$$\langle v - \tilde{x}, w\rangle \geq 0, \quad \text{as} \quad i \to \infty.$$

Since T is maximal monotone, we have $\tilde{x} \in T^{-1}(0)$, and hence $\tilde{x} \in VI(A, C)$. Thus is clear that $\tilde{x} \in \Omega$.

Step 6. Finally, we show that $x_n \to \tilde{x}$, where $\tilde{x} = P_\Omega x_0$. Since Ω is nonempty closed convex subset of H, there exists a unique $x^* \in \Omega$ such that $x^* = P_\Omega x_0$. Since $x^* \in \Omega \subset C_n$ and $x_n = P_{C_n}x_0$, we have $\|x_n - x_0\| \leq \|\tilde{x} - x_0\|$.
It follows from $x^* = P_\Omega x_0$ and the lower semicontinuity of norm that

$$\begin{aligned}
\|x^* - x_0\| &\leq \|\tilde{x} - x_0\| \leq \lim_{i\to\infty}\inf\|x_{n_i} - x_0\| \\
&\leq \lim_{i\to\infty}\sup\|x_{n_i} - x_0\| \leq \|x^* - x_0\|.
\end{aligned}$$

Thus, we obtain that

$$\lim_{i\to\infty}\|x_{n_i} - x_0\| = \|\tilde{x} - x_0\| = \|x^* - x_0\|.$$

Using the Kadec-Klee property of H, we obtain that

$$\lim_{i\to\infty}x_{n_i} = \tilde{x} = x^*.$$

Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_\Omega x_0$.

C. *Reduced theorem*

Corollary 7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, let F be a bifunction $F : C\times C$ into $\mathbb{R}$ satisfying conditions $(A1) - (A5)$, let A be an α-inverse-strongly monotone mapping of C into H. Let S be a ξ-strict pseudocontraction mapping from C to C and such that $\Omega := EP(F) \cap Fix(S) \cap VI(A, C) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ and μ satisfying the condition $(i) - (iv)$. Then the sequences $\{x_n\}$ generated by Algorithm 5 converges strongly to the projection of x_0 on to the set Ω.*

Proof. Setting $\xi = 0$ in Theorem 6, we obtain the result directly.

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