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A Model for Adding an Efficient Relation to an Organization Structure with Different Numbers of Subordinates at Each Level

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***Abstract*—This study aims at revealing optimal additional relations to a pyramid organization such that the communication of information between every member in the organization becomes the most efficient. This paper proposes a model of adding a relation between two members in the same level of the organization structure in which each member of m-th level below the top has $m+2$ subordinates. The total shortening distance which is the sum of shortening lengths of shortest paths between every pair of all nodes is formulated to obtain an optimal pair of two members between which a relation is added.**

***Index Terms*—organization structure, adding relation, total shortening distance.**

I. INTRODUCTION

OUR studies aim at revealing optimal additional relations to a pyramid organization such that the communication of information between every member in the organization becomes the most efficient. We have obtained an optimal set of additional edges to a complete K-ary tree of height $H(H=1,2,\ldots)$ minimizing the sum of lengths of shortest paths between every pair of all nodes in the complete K-ary tree for the following three models in [1]: (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth, and (iii) a model of adding edges between every pair of siblings with the same depth. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have $K(K=2,3,\ldots)$ children [5].

The complete K-ary tree expresses a pyramid organization in which every member except the top should have a single superior. Nodes and edges in the complete K-ary tree correspond to members and relations between members in the organization respectively. Then the pyramid organization structure is characterized by the number of subordinates of each member [6], [7], that is, K which is the number of children of each node and the number of levels in the organization, that is, H which is the height of the complete K-ary tree. Moreover, the path between each node in the complete K-ary tree is equivalent to the route of communication of information between each member in the organization, and adding edges to the complete K-ary tree is equivalent to forming additional relations other than that between each superior and his subordinates.

The above models give us optimal additional relations to the organization structure of a complete K-ary tree, but these models cannot be applied to adding relations to an organization structure which is not a complete K-ary tree. This paper expands the above model (i) into a model of adding an edge between two nodes with the same depth in a rooted tree with different numbers of children at each depth, that is, a model of adding a ralation between two members of the same level in a pyramid organization structure with different numbers of subordinates at each level. This paper assumes that each node with a depth m has $m+2$ children.

If $l_{i,j}(=l_{j,i})$ denotes the distance, which is the number of edges in the shortest path from a node v_i to a node v_j in the rooted tree, then $\sum_{i<j} l_{i,j}$ is the total distance. Furthermore, if $l'_{i,j}$ denotes the distance from v_i to v_j after adding an edge, $l_{i,j}-l'_{i,j}$ is called the shortening distance between v_i and v_j, and $\sum_{i<j}(l_{i,j}-l'_{i,j})$ is called the total shortening distance. Minimizing the total distance is equivalent to maximizing the total shortening distance.

In Section II we formulate the total shortening distance of the above model. In Section III we show an optimal adding edge at each depth N and illustrate an optimal depth N^* which maximizes the total shortening distance with numerical examples.

II. FORMULATION OF TOTAL SHORTENING DISTANCE

This section formulates the total shortening distance when a new edge between two nodes with the same depth $N(N=1,2,\ldots,H)$ is added to a rooted tree of height $H(H=1,2,\ldots)$ in which each node with a depth $m(m=0,1,\ldots,H-1)$ has $m+2$ children.

We can add a new edge between two nodes with the same depth N in the above rooted tree in N ways that lead to non-isomorphic graphs. Let $S_H(N,L)$ denote the total shortening distance by adding the new edge, where $L(L=0,1,2,\ldots,N-1)$ is the depth of the deepest common ancestor of the two nodes on which the new edge is incident.

We formulate $S_H(N,L)$ in the following.

Let v_0^X and v_0^Y denote the two nodes on which the adding edge is incident. Let v_k^X and v_k^Y denote ancestors of v_0^X and v_0^Y, respectively, with depth $N-k$ for $k=1,2,\ldots,N-L-1$. The sets of descendants of v_0^X and v_0^Y are denoted by V_0^X and V_0^Y respectively. (Note that every node is a descendant of itself [5].) Let V_k^X denote the set obtained by removing the descendants of v_{k-1}^X from the set of descendants of v_k^X and let V_k^Y denote the set

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obtained by removing the descendants of v_{k-1}^Y from the set of descendants of v_k^Y, where $k = 1, 2, \ldots, N-L-1$.

Since addition of the new edge doesn't shorten distances between pairs of nodes other than between pairs of nodes in $V_k^X(k = 0, 1, 2, \ldots, N-L-1)$ and nodes in $V_k^Y(k = 0, 1, 2, \ldots, N-L-1)$, the total shortening distance can be formulated by adding up the following three sums of shortening distances:
(i) the sum of shortening distances between every pair of nodes in V_0^X and nodes in V_0^Y,
(ii) the sum of shortening distances between every pair of nodes in V_0^X and nodes in $V_k^Y(k = 1, 2, \ldots, N-L-1)$ and between every pair of nodes in V_0^Y and nodes in $V_k^X(k = 1, 2, \ldots, N-L-1)$ and
(iii) the sum of shortening distances between every pair of nodes in $V_k^X(k = 1, 2, \ldots, N-L-1)$ and nodes in $V_k^Y(k = 1, 2, \ldots, N-L-1)$.

The sum of shortening distances between every pair of nodes in V_0^X and nodes in V_0^Y is given by

$$\begin{aligned} & A_H(N,L) \\ &= \left(\sum_{i=N}^{H-1}\prod_{j=N}^{i}(j+2)+1\right)^2(2N-2L-1) \\ &= \left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)^2(2N-2L-1) \end{aligned} \quad (1)$$

where we define $\sum_{i=h}^{h-1}\cdot = 0$. The sum of shortening distances between every pair of nodes in V_0^X and nodes in $V_k^Y(k = 1, 2, \ldots, N-L-1)$ and between every pair of nodes in V_0^Y and nodes in $V_k^X(k = 1, 2, \ldots, N-L-1)$ is given by

$$\begin{aligned} & B_H(N,L) \\ &= 2\left(\sum_{i=N}^{H-1}\prod_{j=N}^{i}(j+2)+1\right) \\ &\quad \times \sum_{i=L+1}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\prod_{k=i+1}^{j}(k+2)+1\right)+1\right\} \\ &\quad \times(2i-2L-1) \\ &= 2\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right) \\ &\quad \times \sum_{i=L+1}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\} \\ &\quad \times(2i-2L-1) \end{aligned} \quad (2)$$

and the sum of shortening distances between every pair of nodes in $V_k^X(k = 1, 2, \ldots, N-L-1)$ and nodes in $V_k^Y(k = 1, 2, \ldots, N-L-1)$ is given by

$$\begin{aligned} & C_H(N,L) \\ &= \sum_{i=L+2}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\prod_{k=i+1}^{j}(k+2)+1\right)+1\right\} \\ &\quad \times \sum_{j=N+L-i+1}^{N-1}\left\{(j+1)\left(\sum_{k=j+1}^{H-1}\prod_{l=j+1}^{k}(l+2)+1\right)\right. \\ &\quad \left.+1\right\}(2i+2j-2N-2L-1) \\ &= \sum_{i=L+2}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\} \\ &\quad \times \sum_{j=N+L-i+1}^{N-1}\left\{(j+1)\left(\sum_{k=j+1}^{H-1}\frac{(k+2)!}{(j+2)!}+1\right)\right. \\ &\quad \left.+1\right\}(2i+2j-2N-2L-1) \end{aligned} \quad (3)$$

where we define $\sum_{i=h}^{h-2}\cdot = 0$.

From the above equations, the total shortening distance $S_H(N,L)$ is given by

$$\begin{aligned} & S_H(N,L) \\ &= A_H(N,L)+B_H(N,L)+C_H(N,L) \\ &= \left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)^2(2N-2L-1) \\ &\quad +2\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right) \\ &\quad \times \sum_{i=L+1}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\} \\ &\quad \times(2i-2L-1) \\ &\quad + \sum_{i=L+2}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\} \\ &\quad \times \sum_{j=N+L-i+1}^{N-1}\left\{(j+1)\left(\sum_{k=j+1}^{H-1}\frac{(k+2)!}{(j+2)!}+1\right)\right. \\ &\quad \left.+1\right\}(2i+2j-2N-2L-1). \end{aligned} \quad (4)$$

III. An Optimal Depth

This section shows an optimal depth L^* of the deepest common ancestor of the two nodes on which the adding edge is incident for each depth N of the two nodes and illustrates an optimal depth N^* which maximizes the total shortening distance with numerical examples.

Theorem 1: $L^* = 0$ maximizes $S_H(N,L)$ for each N.

Proof: If $N = 1$, then $L^* = 0$ trivially. If $N \geq 2$, then $L^* = 0$ for each N since

$$\begin{aligned} & S_H(N,L+1)-S_H(N,L) \\ &= -2\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)^2 \\ &\quad -2\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right) \\ &\quad \times\left\{(L+2)\left(\sum_{j=L+2}^{H-1}\frac{(j+2)!}{(L+3)!}+1\right)+1\right\} \end{aligned}$$

TABLE I
TOTAL SHORTENING DISTANCE $\hat{S}_H(N)$

N	$H=1$	$H=2$	$H=3$	$H=4$	$H=5$	$H=6$	$H=7$	$H=8$	$H=9$	$H=10$
1	1	16	256	5776	190096	8737936	534349456	41843157136	4076183329936	483004176750736
2	–	9	185	4425	147465	6793545	415584585	32544495945	3170363017545	375669895715145
3	–	–	67	1837	62857	2909197	178089517	13947480397	1358725421197	161001366609997
4	–	–	–	538	19870	931354	57121306	4474677274	435923030554	51654590148634
5	–	–	–	–	4992	244210	15070434	1181504850	115112690610	13640376725970
6	–	–	–	–	–	54087	3418791	268847367	26202610887	3105011340807
7	–	–	–	–	–	–	679033	54120403	5282749783	626103554563
8	–	–	–	–	–	–	–	9759060	959761540	113836297620
9	–	–	–	–	–	–	–	–	158609342	18890857844
10	–	–	–	–	–	–	–	–	–	2882681413

$$
\begin{aligned}
&-4\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)\\
&\times\sum_{i=L+2}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\}\\
&-\sum_{i=L+3}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\}\\
&\times\Bigg[\Bigg\{(N+L-i+2)\\
&\times\left(\sum_{k=N+L-i+2}^{H-1}\frac{(k+2)!}{(N+L-i+3)!}+1\right)+1\Bigg\}\\
&+2\sum_{j=N+L-i+2}^{N-1}\Bigg\{(j+1)\left(\sum_{k=j+1}^{H-1}\frac{(k+2)!}{(j+2)!}+1\right)\\
&+1\Bigg\}\Bigg]+t_H(N,L)\\
<\;\; & 0
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
&t_H(N,L)\\
&\quad=\; -\left\{(L+3)\left(\sum_{j=L+3}^{H-1}\frac{(j+2)!}{(L+4)!}+1\right)+1\right\}\\
&\qquad\times\left\{N\left(\sum_{k=N}^{H-1}\frac{(k+2)!}{(N+1)!}+1\right)+1\right\}
\end{aligned}
\tag{6}
$$

for $L=0,1,2,\dots,N-3$ and

$$
t_H(N,L)=0 \tag{7}
$$

for $L=N-2$. The proof is complete.

Theorem 1 shows that the most efficient additional relation between two members in the same level N is that between two members which doesn't have common superiors except the top.

Let $\hat{S}_H(N)$ denote the total shortening distance when $L=0$, then $\hat{S}_H(N)$ becomes

$$
\begin{aligned}
&\hat{S}_H(N)\\
&=\; S_H(N,0)\\
&=\; \left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)^2(2N-1)\\
&\quad+2\left(\sum_{i=N}^{H-1}\frac{(i+2)!}{(N+1)!}+1\right)\\
&\quad\times\sum_{i=1}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\}\\
&\quad\times(2i-1)\\
&\quad+\sum_{i=2}^{N-1}\left\{(i+1)\left(\sum_{j=i+1}^{H-1}\frac{(j+2)!}{(i+2)!}+1\right)+1\right\}\\
&\quad\times\sum_{j=N-i+1}^{N-1}\left\{(j+1)\left(\sum_{k=j+1}^{H-1}\frac{(k+2)!}{(j+2)!}+1\right)+1\right\}\\
&\quad\times(2i+2j-2N-1).
\end{aligned}
\tag{8}
$$

Table I shows numerical examples of the total shortening distance $\hat{S}_H(N)$ in the case of $H=1,2,\dots,10$ and $N=1,2,\dots,H$.

Table I reveals that $N^*=1$ maximizes $\hat{S}_H(N)$ irrespective of H when $H=1,2,\dots,10$. This means that the most efficient level of adding a relation to the organization structure in this model is the first level below the top when the organization structure has few levels.

IV. CONCLUSIONS

This study considered revealing an optimal additional relation to a pyramid organization such that the communication of information between every member in the organization becomes the most efficient. For a model of adding an edge between two nodes with the same depth of the rooted tree in which each node with a depth m has $m+2$ children, we formulated the total shortening distance and showed an optimal adding edge at each depth N in Theorem 1. Furthermore, we illustrated an optimal depth N^* which maximizes the total shortening distance with numerical examples.

Theorem 1 and numerical examples reveal that the most efficient manner of adding a relation between two members in the same level of the organization structure in this model is to add the relation between two members in the first level below the top when the organization structure has few levels.

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