Cost Analysis of a Discrete-time Queue

Tsung-Yin Wang, Fu-Min Chang and Jau-Chuan Ke

Abstract—**This paper considers a Geo/G/1 queue, in which the server operates a single vacation at end of each consecutive service period. After all the customers in the system are served exhaustively, the server immediately leaves for a vacation. Upon returning from the vacation, the server inspects the queue length. If there are customers waiting in the queue, the server either resumes serving the waiting customers (with probability *p*) or remains idle in the system (with probability 1-*p*) until the next customer arrives; and if no customer presents in the queue, the server stays dormancy in the system until at least one customer arrives. Using the generating function technique, the system state evolution is analyzed. The probability generating functions of the system size distributions in various states are obtained. The waiting time distribution is also derived. With the vacation of fixed length time (say *T*), the long run average cost function per unit time is analytically developed to determine the joint optimal values of *T* and *p* at a minimum cost.**

Keywords—**Cost, Discrete time queue, Randomized vacation, Direct search method**

I. Introduction

THE modelling analysis for the queueing systems with vacations has been done by a considerable amount of work in the past. A comprehensive and excellent study on the vacation models, including some applications such as production/inventory system and communication/computer systems, can be found in Takagi [4] and Tian and Zhang [7]. On the other hand, along with the advent of computer and communication technologies, the analysis of discrete-time queueing systems has received more attention in the scientific literatures over the past years—Hunter [2], Bruneel and Kim [1], Takagi [5], and Woodward [9]. The reason for this is that discrete-time systems are more appropriate than their continuous-time counterparts in their applicability for the study of many computer and communication systems applications in which time is divided into fixed-length time intervals ('slots'). The applications to communication and computer systems include asynchronous transfer mode multiplexers in the broadband integrated services digital network, slotted carrier-sense multiple access protocols, and time-division multiple access schemes.

An excellent study on discrete-time queueing systems with vacations has been presented by Takagi [5]. Zhang and Tian [10] investigated a Geo/G/1 queue with multiple adaptive vacations. Tian and Zhang [6] analyzed a GI/Geo/1 queueing system with multiple vacations by matrix-geometric solution method and Li and Tian [3] used the same method to study a GI/Geo/1 queueing system with working vacation and vacation interruption. Wang et al. [8] investigated the discrete-time Geo/G/1 queue with randomized vacations and at most J vacations. Zhang and Tian [10], Tian and Zhang [6], and Li and Tian [3], they gave the stochastic results for the queue length and waiting time. We should note that in Zhang and Tian [10], Zhang and Tian [6], and Li and Tian [3], no optimal vacation policies are obtained.

In this paper, a Geo/G/1 system with a single vacation policy and randomized activation (namely <*V*, *p*> policy) was considered. The <*V*, *p*> policy is performed under the following conditions: (i) the server leaves for a single vacation when the system is empty, (ii) if the server returns from the vacation and at least one customer is waiting in the queue, the server may either activates with probability *p* or stays dormancy in the system with probability $1-p$, and (iii) when the server is dormant in the system, he activates to serve the waiting customers as the next customer arrives.

Such a model has a potential application in wireless local area networks (WLANs). Access Points (APs) are specially configured nodes on WLANs and act as a central transmitter and receiver of WLAN radio signals. To keep the APs functioning well, some maintenance activities are needed. For example, virus scan is an important maintenance activity for the APs. It can be performed when the AP is idle and be programmed to perform on a regular basis. After finishing the maintenance activity, AP can enter the sleep mode when there is no radio signal to be transmitted for power saving. It can also enter the sleep mode after finishing the some kinds of maintenance activities such as refreshing AP current status. AP will awake from sleep mode and begin to serve when the new radio signal arrives.

II. Model Description

Let the time axis be marked by 0, 1,…, *n*,…. Assume that a potential arrival occurs within (n^-, n) and a potential departure occurs (n, n^+) .When a customer arrives in the *n*th slot and the system is empty, the service is started in the (*n*+1)th slot. This type rule of arrivals and departures is called late arrival system (LAS) with delay access. Customers arrive according to a Bernoulli process with rate λ . The service times of the customers are independent and identically distributed according to a general probability mass function $\{b_i\}_{i=1}^{\infty}$ with probability generating function

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(PGF) $B(u)=\sum_{i=1}^{\infty} b_i u^i$ and jth factorial moments B_j, $j=1,2$. After all the customers in the system are served exhaustively, the server operates a <V, p> policy. As soon as the system becomes empty, the server immediately takes a single vacation, where the vacation time is a discrete random variable, denoted by V, with probability mass function $\{v_i\}_{i=1}^{\infty}$ having PGF $V(u)=\sum_{i=1}^{\infty} v_i u^i$ and jth factorial moments V_j, $j=1,2$. At the vacation completion instant, the server checks the system to see if there is any waiting customer and decides the action to take one of the following two cases according to the state of the system:

Case 1: If there is any customer waiting in the queue, the server will resume serving the queue with probability p or to stay dormancy in the system with probability $1-p$ until at least one customer arrives.

Case 2: If there is no customer waiting in the queue, the server remains idle in the system until the next customer arriving

Arriving customers form a single waiting line based on the order of their arrivals. The server can serve only one customer at a time. If the server is busy, arriving customer has to wait in the queue until the server is available. All customers arriving to the system are assumed to be eventually served, i.e. $\lambda B_1 < 1$. Furthermore, various stochastic processes involved in the system are independent of each other. We use the symbol $\bar{x}=1-x$, for $0\le x\le 1$.

III. Model Formulation and Stationary Distribution

Let Φ_n denote the state of the server,

$$\Phi_n=\begin{cases}0, & \text{if the server is on vacation at time } n^+;\\ 1, & \text{if the server is idle at time } n^+;\\ 2, & \text{if the server is busy at time } n^+.\end{cases}$$

Let L_n indicate the number of customers in the system at time n^+.

Define

$$\xi_n=\begin{cases}\text{remaining vacation time at } n^+, & \text{if } \Phi_n=0,\\ \text{remaining service time at } n^+, & \text{if } \Phi_n=2.\end{cases}$$

Let us define the following limiting probabilities

$\tilde{\pi}_{k,i}=\lim_{n\to\infty}\Pr[\Phi_n=0, L_n=k, \xi_n=i]$, $k\ge 0$, $i\ge 1$;

$\Omega_k=\lim_{n\to\infty}\Pr[\Phi_n=1, L_n=k]$, $k\ge 0$;

$\pi_{k,i}=\lim_{n\to\infty}\Pr[\Phi_n=2, L_n=k, \xi_n=i]$, $k\ge 0$, $i\ge 1$.

The Kolmogorov equations for the stationary distribution are given by

$$\tilde{\pi}_{0,i}=\pi_{1,1}\bar{\lambda}v_i+\tilde{\pi}_{0,i+1}\bar{\lambda},\ i\ge 1 \tag{1}$$

$$\tilde{\pi}_{k,i}=\tilde{\pi}_{k,i+1}\bar{\lambda}+\tilde{\pi}_{k-1,i+1}\lambda,\ k\ge 1,\ i\ge 1 \tag{2}$$

$$\Omega_0=\tilde{\pi}_{0,1}\bar{\lambda}+\Omega_0\bar{\lambda} \tag{3}$$

$$\Omega_k=\bar{p}\tilde{\pi}_{k,1}\bar{\lambda}+\bar{p}\tilde{\pi}_{k-1,1}\lambda+\Omega_k\bar{\lambda},\ k\ge 1 \tag{4}$$

$$\begin{aligned}\pi_{1,i}&=\Omega_0\lambda b_i+p\tilde{\pi}_{0,1}\lambda b_i+p\tilde{\pi}_{1,1}\bar{\lambda}b_i+\pi_{1,i+1}\bar{\lambda}\\&\quad+\pi_{2,1}\bar{\lambda}b_i+\pi_{1,1}\lambda b_i,\ i\ge 1\end{aligned} \tag{5}$$

$$\begin{aligned}\pi_{k,i}&=\Omega_{k-1}\lambda b_i+p\tilde{\pi}_{k-1,1}\lambda b_i+p\tilde{\pi}_{k,1}\bar{\lambda}b_i+\pi_{k,i+1}\bar{\lambda}\\&\quad+\pi_{k+1,1}\bar{\lambda}b_i+\pi_{k-1,i+1}\lambda+\pi_{k,1}\lambda b_i,\ k\ge 2,\ i\ge 1.\end{aligned} \tag{6}$$

To resolve (1)-(6), we use the following generating functions:

$G_V(u,z)=\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}z^k u^i\tilde{\pi}_{k,i}$, $\varphi_V(z)=\sum_{k=0}^{\infty}z^k\tilde{\pi}_{k,1}$,

$G_I(z)=\sum_{k=0}^{\infty}z^k\Omega_k$, $G_B(u,z)=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}z^k u^i\pi_{k,i}$,

$\varphi_B(z)=\sum_{k=1}^{\infty}z^k\pi_{k,1}$, ($|z|\le 1$ and $|u|\le 1$).

Multiplying (1) and (2) by z^k and summing over k after multiplying (1) and (2) by u^i and summing over i, it finally yields

$$\frac{u-(\bar{\lambda}+\lambda z)}{u}G_V(u,z)=\bar{\lambda}\pi_{1,1}V(u)-(\bar{\lambda}+\lambda z)\varphi_V(z). \tag{7}$$

Multiplying (5) and (6) by z^k and summing over k after multiplying (5) and (6) by u^i and summing over i, we have

$$\begin{aligned}\frac{u-(\bar{\lambda}+\lambda z)}{u}G_B(u,z)&=B(u)[\lambda zG_I(z)+p(\bar{\lambda}+\lambda z)\varphi_V(z)]\\&\quad-p\bar{\lambda}B(u)\tilde{\pi}_{0,1}+\frac{(B(u)-z)(\bar{\lambda}+\lambda z)}{z}\varphi_B(z)-\bar{\lambda}B(u)\pi_{1,1}.\end{aligned} \tag{8}$$

Inserting $u=(\bar{\lambda}+\lambda z)$ in (7) and (8), respectively, we obtain

$$\varphi_V(z)=\frac{\bar{\lambda}V(\bar{\lambda}+\lambda z)}{(\bar{\lambda}+\lambda z)}\pi_{1,1}, \tag{9}$$

$$\begin{aligned}\varphi_B(z)&=\frac{\bar{\lambda}zB(\bar{\lambda}+\lambda z)}{(\bar{\lambda}+\lambda z)\left[B(\bar{\lambda}+\lambda z)-z\right]}\times\\&\quad\left\{p(1-z)\tilde{\pi}_{0,1}+\left[1-(p+\bar{p}z)V(\bar{\lambda}+\lambda z)\right]\pi_{1,1}\right\}.\end{aligned} \tag{10}$$

Multiplying (3) and (4) by z^k and summing over k, we obtain

$$G_I(z)=p\bar{\lambda}\tilde{\pi}_{0,1}+\bar{p}(\bar{\lambda}+\lambda z)\varphi_V(z). \tag{11}$$

Substituting (9) into (7) and (11), it gives

$$G_I(z)=\frac{\bar{\lambda}}{\lambda}\left(p\tilde{\pi}_{0,1}+\bar{p}V(\bar{\lambda}+\lambda z)\pi_{1,1}\right), \tag{12}$$

$$G_V(u,z)=\frac{\bar{\lambda}u\left(V(u)-V(\bar{\lambda}+\lambda z)\right)}{u-(\bar{\lambda}+\lambda z)}\pi_{1,1}. \tag{13}$$

Substituting (9), (10) and (12) into (8), we obtain

$$\begin{aligned}G_B(u,z)&=\frac{\bar{\lambda}uz\left(B(u)-B(\bar{\lambda}+\lambda z)\right)}{\left[u-(\bar{\lambda}+\lambda z)\right]\left(B(\bar{\lambda}+\lambda z)-z\right)}\times\\&\quad\left\{p(1-z)\tilde{\pi}_{0,1}+\left[1-(p+\bar{p}z)V(\bar{\lambda}+\lambda z)\right]\pi_{1,1}\right\}.\end{aligned} \tag{14}$$

The PGF of the number of customers in the system is given by

$$\begin{aligned}L(z)&=G_I(z)+G_V(1,z)+G_B(1,z)=\frac{\bar{\lambda}B(\bar{\lambda}+\lambda z)}{\lambda\left(B(\bar{\lambda}+\lambda z)-z\right)}\times\\&\quad\left\{p(1-z)\tilde{\pi}_{0,1}+\left[1-(p+\bar{p}z)V(\bar{\lambda}+\lambda z)\right]\pi_{1,1}\right\}.\end{aligned} \tag{15}$$

Setting $z=0$ in (9) and using the normalization condition, $L(1)=1$, we obtain

$$\tilde{\pi}_{0,1}=\frac{\lambda(1-\rho)V(\bar{\lambda})}{\bar{\lambda}\left(\bar{p}+\lambda V_1+pV(\bar{\lambda})\right)}, \tag{16}$$

$$\pi_{1,1}=\frac{\lambda(1-\rho)}{\bar{\lambda}\left(\bar{p}+\lambda V_1+pV(\bar{\lambda})\right)}. \tag{17}$$

Substitution $\tilde{\pi}_{0,1}$ and $\pi_{1,1}$ into (15) gives

$$L(z)=\frac{(1-z)(1-\rho)B(\bar{\lambda}+\lambda z)}{\left(B(\bar{\lambda}+\lambda z)-z\right)}\times \frac{\left\{p(1-z)V(\bar{\lambda})+\left[1-(p+\bar{p}z)V(\bar{\lambda}+\lambda z)\right]\right\}}{(1-z)\left(\bar{p}+\lambda V_1+pV(\bar{\lambda})\right)}. \tag{18}$$

Differentiating $L(z)$ and setting $z=1$, the expected number of customers in the system is given by

$$E[L]=\rho+\frac{\lambda^2 B_2}{2(1-\rho)}+\frac{2\bar{p}\lambda V_1+\lambda^2 V_2}{2\left(\bar{p}+\lambda V_1+pV(\bar{\lambda})\right)}. \tag{19}$$

IV. The turned-off period and turned-on period

A. The turned-off period

The turned-off period is comprised of vacation period and idle period. Hence the PGF of the server turned-off period is given by

$$I_V(u)=\left(p+\bar{p}I(u)\right)\left(V(u)-V(\bar{\lambda}u)\right)+V(\bar{\lambda}u)I(u), \tag{20}$$

which leads to the expected length of the turned-off period as

$$E[S_{off}]=V_1+\frac{pV(\bar{\lambda})+q}{\lambda}. \tag{21}$$

From (21), we obtain the expected lengths of the vacation period and idle period are

$$E[S_V]=V_1, \tag{22}$$

$$E[S_I]=\frac{pV(\bar{\lambda})+\bar{p}}{\lambda}. \tag{23}$$

B. The turned-on (busy) period

From Takagi [8], we have the PGF of busy period for the classical Geo/G/1 with late arrive delay access

$$\Psi(z)=B(\lambda z\Psi(z)+\bar{\lambda}z). \tag{24}$$

The busy period begins as one of the following three cases:

Case 1: j messages arrive during the vacation period which vacation time is k slots. After the vacation completion instant, the server begins service with probability p. Such event occurs with probability $pv_k^{(j)}, k=1,2\cdots, j=1,2,\cdots k$, where $v_k^{(j)}=C_j^k\lambda^j\bar{\lambda}^{k-j}v_k$.

Case 2: j messages arrive during the vacation period which vacation time is k slots. At the end of the vacation, the server remains idle in the system with probability $\bar{p}$. In this case, the server begins providing service as next message arrives. Such event occurs with probability

$$\bar{p}v_k^{(j)}\sum_{m=1}^{\infty}\bar{\lambda}^{m-1}\lambda=\bar{p}v_k^{(j)}, k=1,2\cdots, j=1,2,\cdots k.$$

Case 3: No message arrives during the vacation period which vacation time is k slots. In this case, the server starts providing service as a message arrives. Such event occurs with probability $\bar{\lambda}^k v_k\sum_{m=1}^{\infty}\bar{\lambda}^{m-1}\lambda=\bar{\lambda}^k v_k, k=1,2\cdots$.

The PGF of the sub-busy period is extended by Case 1-3 as

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k}pv_k^{(j)}\sum_{n=1}^{\infty}b_n z^n\sum_{l=0}^{n}C_l^n\lambda^l\bar{\lambda}^{n-l}\left(\Psi(z)\right)^{l+j-1}+ \sum_{k=1}^{\infty}\sum_{j=1}^{k}\bar{p}v_k^{(j)}\sum_{n=1}^{\infty}b_n z^n\sum_{l=0}^{n}C_l^n\lambda^l\bar{\lambda}^{n-l}\left(\Psi(z)\right)^{l+j}+. \sum_{k=1}^{\infty}\bar{\lambda}^k v_k\sum_{n=1}^{\infty}b_n z^n\sum_{l=0}^{n}C_l^n\lambda^l\bar{\lambda}^{n-l}\left(\Psi(z)\right)^{l}.$$

Hence the PGF of the busy period for the <V, p> policy Geo/G/1 queueing system is given by

$$\bar{\Psi}(z)=B\left(\bar{\lambda}z+\lambda z\Psi(z)\right)\times \left\{\left(\frac{p}{\Psi(z)}+\bar{p}\right)\left[V(\bar{\lambda}+\lambda\Psi(z))-V(\bar{\lambda})\right]+V(\bar{\lambda})\right\} \tag{25}$$

and the expected length of the busy period is given by

$$E[S_{on}]=\bar{\Psi}'(1)=\frac{\rho\left[\bar{p}+\lambda V_1+pV(\bar{\lambda})\right]}{\lambda(1-\rho)}. \tag{26}$$

From (21) and (26), we obtain the expected length of busy cycle

$$E[C_V]=E[S_{off}]+E[S_{on}]=\frac{\bar{p}+\lambda V_1+pV(\bar{\lambda})}{\lambda(1-\rho)}. \tag{27}$$

V. Waiting Time in the Queue

Let us define the following PGFs:

$W_V(z\,|\,vacation)\equiv$ the PGF of the waiting time in the queue of a customer conditioning that the server state is on vacation;

$W_I(z\,|\,idle)\equiv$ the PGF of the waiting time in the queue of a customer conditioning that the server state is idle;

$W_B(z\,|\,busy)\equiv$ the PGF of the waiting time in the queue of a customer conditioning that the server is busy;

$W_Q(z)\equiv$ the PGF of waiting time in the queue of a customer;

For the Geo/G/1 system with <V, p> policy, an arrival may occur as one of the following three cases:

Case 1: A customer arrives while the server is on vacation and find k customer($k\geq 0$) in the system: (i) while the vacation is just end, the server is switched to busy period with probability p, the customer must wait the service time of the preceding k customers; and (ii) while the vacation is just end, the server is switched to idle period with probability $\bar{p}$ until the next customer arrives, the customer must wait the time for the next arrival plus the service time of the preceding k customers.

Case 2: There are exactly k customers in the queue and the server is idle in the system when the customer arrives.

Case 3: A customer arrives while the server is busy and finds k customers in the system. In this case, the waiting time in the queue of the customer consists of: (i) the remaining service time of the customer being served at time n^+; and (ii) the service time of the k-1 customers in the queue at time n^+.

From Case 1 yields

$$W_V(z\,|\,vacation)=\frac{p\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\tilde{\pi}_{k,i}z^{i-1}\left[B(z)\right]^k+\bar{p}\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\tilde{\pi}_{k,i}z^{i-1}[B(z)]^k\left(\frac{\lambda z}{1-\bar{\lambda}z}\right)}{1-\rho-P_I} \tag{28}$$

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From Case 2, it gives

$$W_I(z \mid idle) \equiv \frac{\sum_{k=0}^{\infty} \Omega_k \left[B(z)\right]^k}{P_I}. \tag{29}$$

From Case 3, we have

$$W_B(z \mid busy) = \frac{\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \pi_{k,i} z^{i-1} \left[B(z)\right]^{k-1}}{\rho}. \tag{30}$$

Finally, the PGF of the waiting time in the queue of a test customer is given by

$$W_Q(z) = (1-\rho-P_I)W_V(z) + P_I W_I(z) + \rho W_B(z \mid busy)$$

which implies

$$E[W_Q] = \frac{2\bar{p}V_1 + \lambda V_2}{2\left(\bar{p} + \lambda V_1 + pV(\bar{\lambda})\right)} + \frac{\lambda B_2}{2(1-\rho)}. \tag{31}$$

VI. Optimization Analysis

As a particular case, the Geo/G/1 queueing system with <T, p> policy, in which the server takes a vacation of fixed length T at the ending of the busy period and the server begins service with probability p if customers present in the queue at vacation completion instant. We construct the total expected cost function per unit time for the <T, p> policy system. The main objective of this study is to determine the discrete time T, say T^*, and the probability p, say p^*, simultaneously so that the expected cost function is minimized. To do this, we define the following cost elements:

$C_h \equiv$ cost per unit time per customer present in the system,

$C_s \equiv$ cost per unit time for a cycle,

$C_r \equiv$ profit per unit time due to vacation.

Using these cost elements listed above and the corresponding system characteristics, the expected cost function $F(T,p)$ per customer per unit time is given by

$$F(T,p) = C_h E[L] + C_s \frac{1}{E[C_V]} - C_r \frac{E[S_V]}{E[C_V]}$$
$$= \frac{\lambda}{\bar{p} + \lambda T + p\bar{\lambda}^T}\left\{C_h \frac{\left[2\bar{p}T + \lambda T(T-1)\right]}{2} + C_s(1-\rho)\right\} \tag{32}$$
$$- C_r \frac{\lambda(1-\rho)T}{\bar{p} + \lambda T + p\bar{\lambda}^T}.$$

The cost function in (32) would have been a hard task to develop analytic results for the optimum value (T^*, p^*) because one is discrete variable T and one is continuous variable p. We first use direct search method to find the discrete variable, say T^* when p is fixed. Next, we fix T^* and derive the continuous value of p, say p^*.

A. Direct search method

In practical use, the discrete variable T is bounded by a positive integer T_U. Under a given p, we successively use direct substitution of ascendant values of $T = 1, 2, \ldots, T_U$ into the cost function. The optimum value T^* could be determined by the following

$$F(T^* \mid p) = \underset{\rho<1}{Minimize}\, F(T \mid p),\ T \in \{1, 2, \ldots, T_U\}. \tag{33}$$

Some numerical examples are presented to demonstrate that the cost function is really convex in T and the solution gives a minimum. For convenience, the numerical experiments are performed by considering $B_1 = 1.0$ and the following with cost parameter elements: C_h =\$20/customer/unit time, C_s =\$1000/unit time, C_r =\$50/unit time, λ =0.1 and vary the values of p and T. The numerical results are displayed in Figure 1 and show the global optimal value can be obtained.

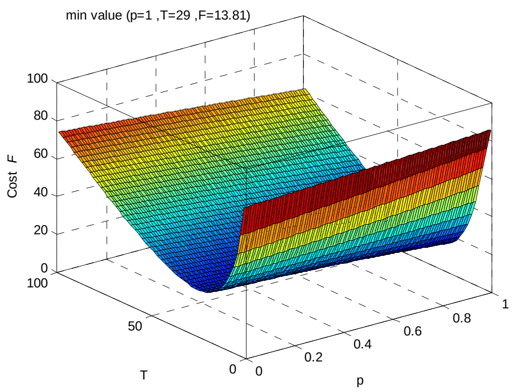

Fig. 1. Cost with different values of T and p (λ =0.1, B_1 =0.1, C_h =\$20, C_s =\$1000, and C_r =\$50).

B. Optimize p

After we find T^*, we will find p such that the minimum value of $F(T^*, p)$ is achieved, say $F(T^*, p)$. The cost minimization problem can be illustrated mathematically as

$$F(T^*, p^*) = \underset{\rho<1}{Minimize}\, F(T^*, p) \tag{34}$$

The study notes that the derivative of the cost function F with respect to P indicates the direction which cost function increases.

$$\frac{dF(p)}{dp} = \frac{\lambda}{(\bar{p} + \lambda T^* + p\bar{\lambda}^{T^*})^2} \times$$
$$\left\{-\frac{1}{2} C_h T^* \left[\lambda T^* + \lambda + 2\bar{\lambda}^{T^*} + \lambda\bar{\lambda}^{T^*}(T^*-1)\right]\right.$$
$$\left. + (1-\bar{\lambda}^{T^*})(1-\rho)(C_s - C_r T^*)\right\}. \tag{35}$$

The first derivative test supposes that p is a critical number on the interval [0, 1] of the continuous cost function F with respect to p. From equation (35), $dF(p)/dp$ does not change sign at p, then F has no local maximum or minimum at p.

If $C_s < C_r T^* + \dfrac{C_h T^* \left[\lambda T^* + \lambda + \bar{\lambda}^{T^*} + \lambda\bar{\lambda}^{T^*}(T^*-1)\right]}{2(1-\bar{\lambda}^{T^*})(1-\rho)}$, then $dF(p)/dp$ is negative, and shows that cost function F is decreases on the interval [0, 1] of p. Thus, the cost function F has an absolute minimum at $p = 1$.

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If $C_s > C_r T^* + \dfrac{C_h T^* \left[\lambda T^* + \lambda + \bar{\lambda}^{T^*} + \lambda \bar{\lambda}^{T^*} (T^* - 1) \right]}{2(1-\bar{\lambda}^{T^*})(1-\rho)}$, then $dF(p)/dp$ is positive on the interval [0, 1] of *p*, and shows that *F* cost function is increases. Thus, the cost function *F* has an absolute minimum at $p = 0$.

A numerical illustration is provided by consider the following cases.

Case 1: C_h =\$20, C_s =\$1000, C_r =\$50, and vary the values of λ and B_1.

Case 2: λ =0.5, B_1 =1.0, C_s =\$1000, C_r =\$20, and vary the values of C_h =\$1, \$10, \$100, \$1000, \$10000.

For illustrative purpose, we present the two cases listed above to illustrate the optimization procedure shown in Tables 1-2, respectively. The results are in accordance with the analysis listed above. From Tables 1 and 2, it is seen that (i) T^* increases as λ or B_1 decreases; (ii) T^* increases as C_h decreases.

Table 1. The illustration of the implement process of p^*

(λ, B_1)	(0.1,0.2)	(0.1,0.5)	(0.1,0.8)	(0.1,1.0)	(0.5,1.0)	(0.9,1.0)
T^*	31	30	30	30	10	3
p^*	1	1	1	1	1	1
$F^*(T^*, p^*)$	12.46	12.98	13.48	13.8	69.99	46.32
$\frac{dF(p)}{dp}$ in (0,1)	-15.94	-15.63	-15.48	-15.38	-6.02	-2.86
	<0	<0	<0	<0	<0	<0

Table 2. The illustration of the implement process of p^*

C_h	1	10	100	1000	10000
T^*	45	14	4	1	1
p^*	1	1	1	1	1
$F^*(T^*, p^*)$	12.11	58.21	184.24	245	245
$\frac{dF(p)}{dp}$ in (0,1)	-0.46	-1.68	-13.22	-377.5	-4877.5
	<0	<0	<0	<0	<0

VII. CONCLUSION

The study introduces the <*V*, *p*> policy for a discrete-time Geo/G/1 queueing system, in which a single server randomly reactivates when some customers present in the queue at ending of vacation completion instant. Some important system characteristics are derived, including the system length distribution and waiting time distribution. The study finally develops efficient methods to find the optimal <*T*, *p*> policy that minimizes the expected cost function.

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