Numerical Analysis for Time-dependent Machine Repair Model with Threshold Recovery Policy and Server Vacations

Dong-Yuh Yang, Ching-Ho Yen and Ya-Chu Chiang

***Abstract*—We consider a machine repair problem with server breakdowns, in which an unreliable server (repairman) operates the threshold recovery policy. When there are no failed machines in the system, the server leaves for a vacation. If the server returns from a vacation to find no failed machines in the system, he/she immediately takes another vacation. And the server continues in this manner until he/she finds at least one failed machine waiting in the queue upon returning from a vacation. It is assumed that the failure and service times of each machine are exponentially distributed. The server breaks down with a constant failure rate. The repair and vacation times obey exponential distributions. Using the Runge–Kutta method of fourth order, we solve the differential equations for this machine repair model numerically. Machine availability is developed in terms of transient probabilities. Finally, a sensitivity analysis is conducted to investigate the effects of system parameters on the machine availability with respect to time.**

***Keywords*—Machine repair problem, Runge–Kutta method, server vacation, threshold recovery policy**

I. Introduction

Over the years, a number of attempts have been made to study the machine repair problems due to applications in a variety of fields, such as computer systems, industrial systems, inventory systems, and so on. For a complete survey of the machine repair problems, we refer to Stecke and Aronson [16] and Haque and Armstrong [8]. Gupta and Srinivasa Rao [7] applied a recursive method to analyze the M/G/1 machine repair problem with spares. Jain et al. [10] gave the reliability characteristics of a machine repairable system with spares under *N*-policy. Using a mathematical programming approach, Chen [2] constructed the membership function of the performance measure of the machine repair model, where the machine breakdown rate and the service rate are fuzzy numbers.

The machine repair models mentioned above are all assumed that the server (repairman) is reliable, but in many practical situations, the server is unreliable. That is the server is subject to breakdowns and repairs. Wang [19] studied the M/M/1 machine repair problem with two types of server breakdowns. The M/E_k/1 machine repair problem with an unreliable server was considered by Wang and Kuo [21]. Later, Wang et al. [22] investigated reliability characteristics of a repairable system with warm standbys and server breakdowns. Further recent results on the machine repair problems with server breakdowns can be found in Ke and Lin [12] and Lv et al. [15].

Server vacation models are useful for a wide variety of applications including computer networks and manufacturing systems, as well as many others. In a machine repair system with server vacations, the repairman can spend his/her idle time on other tasks. Various excellent surveys on the server vacation models in queueing literature can be found in Doshi [3], Takagi [17], Tian and Zhang [18] and the references therein. Gupta [6] presented an efficient algorithm to calculate the steady-state probability distribution of number of failed machines in the machine interference problem with warm spares, server vacations and exhaustive service. Ke and Wang [13] obtained the steady-state solutions for M/M/R machine repair problems with two types of spares, where the servers operate two vacation policies (multiple vacations vs. single vacation). Recently, the model of Ke and Wang [13] was generalized by Jain and Upadhyaya [11] to include multiple types of spares, heterogeneous servers, common cause failure, degraded failure and threshold *N*-policy. For more related works, see Wang et al. [20] and Ke and Wu [14].

Efrosini and Semenova [4] was the first to introduced the concept of threshold recovery policy, in which the server may break down only if operating, and the repair can only be performed when there are $q \geq 1$ or more customers in the system. An M/M/1 retrial queueing system with constant retrial rate, an un-reliable server and a threshold recovery policy was discussed by Efrosini and Winkler [5]. Recently, Jain and Bhagat [9] focused on the transient state solution of the finite population retrial queueing model with geometric arrivals, second optional service, threshold recovery policy and impatient customers. To the best of our knowledge, there are only a few works dealing with threshold recovery policy for the machining system. This motivates us to investigate an unreliable machine repair model by considering the threshold recovery policy. To makes the system more practical, we

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further incorporate the concept of the server vacations into our model.

The rest of this paper is organized as follows: in Section 2, it is described the mathematical model of the machining system. Section 3 formulates the Kolmogorov equations for the transient distribution of the number of failed machines in the system. In Section 4, we present the machine availability in terms of transient probabilities as well as numerical examples. A sensitivity analysis for the transient machine availability is also performed. Finally, we conclude the paper in Section 5.

II. The Model Description

The system we consider here consists of M identical operating machines which are maintained by an unreliable server (repairman). The following assumptions and notations are used:

- The lifetime distributions of operating machines are assumed to be exponentially distributed with parameter λ. As soon as an operating machine fails, it is immediately sent to repair facility and repaired based on the order of their breakdowns, i.e., the first-in, first-out (FIFO) discipline.
- The server can repair only one failed machine at a time, and the service times follow an exponential distribution with mean $1/\mu$.
- As soon as the system becomes empty, the server leaves for a vacation. If the server returns from a vacation to find no failed machines waiting in the queue, he/she immediately takes another vacation. Otherwise, if there is at least one failed machine in the repair facility, the server starts to repair the failed machines waiting in the queue upon returning from a vacation. The duration of the server vacation is an exponential distribution with rate ν.
- The server breaks down at any time with breakdown rate α. If the server breaks down, the server can not be repaired until that the number of the failed machines in the system reaches a specified threshold q ($1 \le q \le M$). Repair times of the server are assumed to be exponentially distributed with mean $1/\beta$.
- Various stochastic processes involved in the system are mutually independent of each other.

III. The Mathematical Model

For the machine repair problem with an unreliable server, threshold recovery policy and server vacations, we define some notations in the following:

$N(t) \equiv$ the number of failed machines in the system at time t,

$Y(t) \equiv$ the server state at time t,

where

$$Y(t) = \begin{cases} 0, & \text{if the server is on vacation,} \\ 1, & \text{if the server is working,} \\ 2, & \text{if the server is broken down.} \end{cases}$$

Then $\{Y(t), N(t); t \ge 0\}$ is a continuous time Markov process associated with the state space

$$S = \left\{(0,n) \middle| n = 0,1,2,...,M\right\} \cup \left\{(1,n) \middle| n = 1,2,...,M\right\} \cup \left\{(2,n) \middle| n = 1,2,...,M\right\}.$$

Let

$$P_{0,n}(t) = \Pr\left\{Y(t) = 0, N(t) = n\right\}, \quad 0 \le n \le M$$

and

$$P_{i,n}(t) = \Pr\left\{Y(t) = i, N(t) = n\right\}, \quad i = 1,2\ , \ 1 \le n \le M\ ,$$

The differential equations governing the system are as follows:

$$\frac{d}{dt}P_{0,0}(t) = -M\lambda P_{0,0}(t) + \mu P_{1,1}(t), \tag{1}$$

$$\frac{d}{dt}P_{0,n}(t) = -[(M-n)\lambda + \nu]P_{0,n}(t) + [M-(n-1)]\lambda P_{0,n-1}(t), \quad 1 \le n \le M-1, \tag{2}$$

$$\frac{d}{dt}P_{0,M}(t) = -\nu P_{0,M}(t) + \lambda P_{0,M-1}(t), \tag{3}$$

$$\frac{d}{dt}P_{1,1}(t) = -[(M-1)\lambda + \mu + \alpha]P_{1,1}(t) + \mu P_{1,2}(t) + \nu P_{0,1}(t), \tag{4}$$

$$\frac{d}{dt}P_{1,n}(t) = -[(M-n)\lambda + \mu + \alpha]P_{1,n}(t) + \mu P_{1,n+1}(t) + \nu P_{0,n}(t) + [M-(n-1)]\lambda P_{1,n-1}(t), \quad 2 \le n \le q-1, \tag{5}$$

$$\frac{d}{dt}P_{1,n}(t) = -[(M-n)\lambda + \mu + \alpha]P_{1,n}(t) + \mu P_{1,n+1}(t) + \nu P_{0,n}(t) + [M-(n-1)]\lambda P_{1,n+1}(t) + \beta P_{2,n}(t), \quad q \le n \le M-1, \tag{6}$$

$$\frac{d}{dt}P_{1,M}(t) = -(\mu + \alpha)P_{1,M}(t) + \nu P_{0,M}(t) + \lambda P_{1,M-1}(t) + \beta P_{2,M}(t), \tag{7}$$

$$\frac{d}{dt}P_{2,1}(t) = -(M-1)\lambda P_{2,1}(t) + \alpha P_{1,1}(t), \tag{8}$$

$$\frac{d}{dt}P_{2,n}(t) = -(M-n)\lambda P_{2,n}(t) + \alpha P_{1,n}(t) + [M-(n-1)]\lambda P_{2,n-1}(t), \quad 2 \le n \le q-1, \tag{9}$$

$$\frac{d}{dt}P_{2,n}(t) = -[(M-n)\lambda + \beta]P_2(n) + \alpha P_{1,n}(t) + [M-(n-1)]\lambda P_{2,n-1}(t), \ q \le n \le M-1, \tag{10}$$

$$\frac{d}{dt}P_{2,M}(t) = -\beta P_{2,M}(t) + \alpha P_{1,M}(t) + \lambda P_{2,M-1}(t). \tag{11}$$

IV. Transient Machine Availability Analysis

This section aims to develop the machine availability in terms of transient probabilities. Following Benson and Cox [1], the machine availability is defined as the ratio of the average number of machines running to the total number of machines. Let $MA(t)$ be the machine availability at time t, we obtain

$$MA(t) = 1 - \frac{E[N(t)]}{M} = 1 - \frac{\sum_{n=1}^{M}\sum_{i=0}^{2} nP_{i,n}(t)}{M}, \tag{12}$$

where $E[N(t)]$ is the expected number of failed machines in the system at time t.

Numerical results

Since analytical results are not available, we employ a numerical technique based on the fourth order Runge–Kutta method to solve the differential equations (1)-(11) with the initial condition $P_{0,0}(0)=1$. Once the $MA(t)$ can be obtained through the transient solutions. In our computations, the Runge–Kutta method was implemented using Matlab software. We fix $M=16$ and consider various values of the parameters $\lambda,\ \mu,\ \alpha,\ \beta,\ \nu,\ q$. The effect of different parameters on the machine availability is shown in Tables I-VI. In the numerical results, one can also describe the variation of the machine availability with respect to time. The default parameters for Tables I-VI are set as $\lambda=1.0$, $\mu=3.0$, $\alpha=0.05$, $\beta=6.0$, $\nu=2.0$ and $q=8$. From Tables I-II, it can be found that the $MA(t)$ decreases as λ increases or μ decreases. We observe from Tables III-IV that $MA(t)$ decreases with increasing values of α or decreasing values of β. Tables V-VI show that $MA(t)$ increases when ν increases or q decreases. One can easily from Tables I-VI that $MA(t)$ decreases with the increasing value of t. Moreover, it reveals that $\beta,\ \nu$ and q have slight effects on the $MA(t)$. This means that the effect of $\lambda,\ \mu$ and α on the $MA(t)$ is larger than that of $\beta,\ \nu$ and q.

Table I.
Effect of λ on the machine availability under different values of t.
($\mu=3.0$, $\alpha=0.05$, $\beta=6.0$, $\nu=2.0$, $q=8$)

t	λ		
	0.8	1.0	1.2
0	1.00000	1.00000	1.00000
0.2	0.85515	0.82210	0.79028
0.4	0.74007	0.68512	0.63408
0.6	0.64899	0.57980	0.51784
0.8	0.57633	0.49830	0.43095
1	0.51784	0.43482	0.36569
2	0.35132	0.27024	0.21237
4	0.25560	0.19686	0.15981
6	0.23706	0.18741	0.15539
8	0.23334	0.18614	0.15500
10	0.23259	0.18597	0.15496

Table II.
Effect of μ on the machine availability under different values of t.
($\lambda=1.0$, $\alpha=0.05$, $\beta=6.0$, $\nu=2.0$, $q=8$)

t	μ		
	3.0	4.0	5.0
0	1.00000	1.00000	1.00000
0.2	0.82210	0.82314	0.82414
0.4	0.68512	0.68968	0.69406
0.6	0.57980	0.58948	0.59877
0.8	0.49830	0.51377	0.52868
1	0.43482	0.45616	0.47682
2	0.27024	0.31439	0.35777
4	0.19686	0.25617	0.31512
6	0.18741	0.24899	0.31035
8	0.18614	0.24803	0.30974
10	0.18597	0.24791	0.30966

Table III.
Effect of α on the machine availability under different values of t.
($\lambda=1.0$, $\mu=3.0$, $\beta=6.0$, $\nu=2.0$, $q=8$)

t	α		
	0.01	0.05	0.1
0	1.00000	1.00000	1.00000
0.2	0.82211	0.82210	0.82209
0.4	0.68519	0.68512	0.68503
0.6	0.58003	0.57980	0.57951
0.8	0.49875	0.49830	0.49774
1	0.43549	0.43482	0.43399
2	0.27137	0.27024	0.26886
4	0.19809	0.19686	0.19534
6	0.18865	0.18741	0.18589
8	0.18739	0.18614	0.18462
10	0.18721	0.18597	0.18444

Table VI.
Effect of β on the machine availability under different values of t.
($\lambda=1.0$, $\mu=3.0$, $\alpha=0.05$, $\nu=2.0$, $q=8$)

t	β		
	3.0	6.0	9.0
0	1.00000	1.00000	1.00000
0.2	0.82210	0.82210	0.82210
0.4	0.68511	0.68511	0.68511
0.6	0.57978	0.57979	0.57980
0.8	0.49824	0.49829	0.49832
1	0.43466	0.43481	0.43490
2	0.26935	0.27023	0.27058
4	0.19542	0.19685	0.19734
6	0.18590	0.18741	0.18792
8	0.18462	0.18614	0.18665
10	0.18444	0.18597	0.18648

Table V.
Effect of ν on the machine availability under different values of t.
($\lambda = 1.0$, $\mu = 3.0$, $\alpha = 0.05$, $\beta = 6.0$, $q = 8$)

t	ν		
	2.0	4.0	6.0
0	1.00000	1.00000	1.00000
0.2	0.82210	0.82481	0.82699
0.4	0.68512	0.69450	0.70066
0.6	0.57980	0.59530	0.60368
0.8	0.49830	0.51766	0.52649
1	0.43482	0.45581	0.46406
2	0.27024	0.28428	0.28781
4	0.19686	0.19923	0.19971
6	0.18741	0.18774	0.18781
8	0.18614	0.18619	0.18620
10	0.18597	0.18598	0.18598

Table VI.
Effect of q on the machine availability under different values of t.
($\lambda = 1.0$, $\mu = 3.0$, $\alpha = 0.05$, $\beta = 6.0$, $\nu = 2.0$)

t	q		
	4	8	12
0	1.00000	1.00000	1.00000
0.2	0.82210	0.82210	0.82210
0.4	0.68514	0.68512	0.68511
0.6	0.57990	0.57980	0.57977
0.8	0.49854	0.49830	0.49816
1	0.43519	0.43482	0.43441
2	0.27059	0.27024	0.26808
4	0.19692	0.19686	0.19567
6	0.18743	0.18741	0.18677
8	0.18615	0.18614	0.18561
10	0.18598	0.18597	0.18546

V. CONCLUSIONS

In this paper, we analyzed a machine interference model with an unreliable server, threshold recovery policy and server vacations. The differential equations governing the system were established and solved by the Runge-Kutta method. Using the transient solutions, we obtained the machine availability with respect to time. Numerical results were given to illustrate the effects of the system parameters on the transient machine availability. It would be useful to extend the analysis to the steady-state solutions, which deserves further investigation.

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REFERENCES

[1] F. Benson and D. R. Cox, "The productivity of machines requiring attention at random intervals," *Journal of the Royal Statistical Society, Series B*, vol. 13, pp. 65-82. 1951.
[2] S.-P. Chen, "A mathematical programming approach to the machine interference problem with fuzzy parameters," *Applied Mathematics and Computation*, vol. 174, pp. 374-387, 2006.
[3] B. T. Doshi, "Queueing systems with vacations-a survey," *Queueing Systems*, vol. 1, pp. 29-66. 1986.
[4] D. V. Efrosinin and O. V. Semenova, "An M/M/1 system with an unreliable device and a threshold recovery policy," *Journal of Communications Technology and Electronics*, vol. 55, pp. 1526-1531, 2010.
[5] D. Efrosinin and A. Winkler, "Queueing system with a constant retrial rate, non-reliable server and threshold-based recovery," *European Journal of Operational Research*, vol. 210, pp. 594-605, 2011.
[6] S. M. Gupta, "Machine interference problem with warm spares, server vacations and exhaustive service," *Performance Evaluation*, vol. 29, pp. 195-211, 1997.
[7] U. C. Gupta and T. S. S. Srinivasa Rao, "On the M/G/1 machine interference model with spares," *European Journal of Operational Research*, vol. 89, pp. 164-171. 1996.
[8] L. Haque and M. J. Armstrong, "A survey of the machine interference problem," *European Journal of Operational Research*, vol. 179, pp. 469-482. 2007.
[9] M. Jain and A. Bhagat, "Finite population retrial queueing model with threshold recovery, geometric arrivals and impatient customers," *Journal of Information and Operations Management*, vol. 3, pp. 162-165. 2012.
[10] M. Jain, Rakhee and S. Maheshwari, "*N*-policy for a machine repair system with spares and reneging," *Applied Mathematical Modelling*, vol. 28, pp. 513-531. 2004.
[11] M. Jain and S. Upadhyaya, "Threshold *N*-policy for degraded machining system with multiple type spares and multiple vacations," *Quality Technology & Quantitative Management*, vol. 6, pp. 185-203, 2009.
[12] J.-C. Ke and C.-H. Lin, "Sensitivity analysis of machine repair problems in manufacturing systems with service interruptions," *Applied Mathematical Modelling*, vol. 32, pp. 2087-2105. 2008.
[13] J.-C. Ke and K.-H. Wang, "Vacation policies for machine repair problem with two type spares," *Applied Mathematical Modelling*, vol. 31, pp. 880-894, 2007.
[14] J.-C. Ke and C.-H. Wu, "Multi-server machine repair model with standbys and synchronous multiple vacation," *Computers & Industrial Engineering*, vol. 62, pp. 296-305. 2012.
[15] S.-L. Lv, J.-B. Li and D.-Q. Yue, "Unreliable multi-server machine repairable system with variable breakdown rates," *Journal of the Chinese Institute of Industrial Engineers*, vol. 28, pp. 400-409. 2011.
[16] K. E. Stecke and J. E. Aronson, "Review of operator/machine interference models," *International Journal of Production Research*, vol. 23, pp. 129-151, 1985.
[17] H. Takagi, *Queueing Analysis: A Foundation of Performance Evaluation. Vol. 1, Vacation and Priority Systems. Part 1.* North-Holland Publishing, Amsterdam, 1991.
[18] N. Tian and Z. G. Zhang, *Vacation Queueing Models: Theory and Applications*. Springer-Verlag, New York, 2006.
[19] K.-H. Wang, "Profit analysis of the machine-repair problem with a single service station subject to breakdowns," *Journal of the Operational Research Society*, vol. 41, pp. 1153-1160, 1990.
[20] K.-H. Wang, W.-L. Chen and D.-Y. Yang, "Optimal management of the machine repair problem with working vacation: Newton's method," *Journal of Computational and Applied Mathematics*, vol. 233, pp. 449-458. 2009.
[21] K.-H. Wang and M.-Y. Kuo, "Profit analysis of the M/E_k/1 machine repair problem with a non-reliable service station," *Computers and Industrial Engineering*, vol. 32, pp. 587-594. 1997.
[22] K.-H. Wang, Y.-J. Lai and J.-B. Ke, "Reliability and sensitivity analysis of a system with warm standbys and a repairable service station," *International Journal of Operations Research*, vol. 1, pp. 61-70. 2004.