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General Iterative Method for Convex Feasibility Problem via the Hierarchical Generalized Variational Inequality Problems

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***Abstract*—Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A_m, B_m : C \to H$ be relaxed cocoercive mappings for each $1 \le m \le r$, where $r \ge 1$ is integer. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be ξ-strongly monotone and L-Lipschitz continuous mappings. Under the assumption $\cap_{m=1}^{r} GVI(C, B_m, A_m) \neq \emptyset$, where $GVI(C, B_m, A_m)$ is the solution set of a generalized variational inequality. Consequently, we prove a strong convergence theorem for finding a point $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$ which is a unique solution of the hierarchical generalized variational inequality $\langle(\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$.**

***Index Terms*—Relaxed cocoercive mapping, convex feasibility problem, generalized variational inequality problem, hierarchical generalized variational inequality problem.**

I. Introduction

A CONVEX feasibility problem, CFP, is the problem of finding a point in the intersection of finitely many closed convex sets in a real Hilbert spaces H. That is, finding an $x \in \cap_{m=1}^{r} C_m$, where $r \ge 1$ is an integer and each C_m is a nonempty closed and convex subset of H. Many problems in mathematics, for example in physical sciences, in engineering and in real-world applications of various technological innovations can be modeled as CFP. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [1] computer tomography [2] and radiation therapy treatment planning [3].

Let H be a real Hilbert space with inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively and let C be a nonempty closed convex subset of H.
A mapping $T : C \to C$ is called $nonexpansive$ if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

We use $F(T)$ to denote the set of $fixed\ points$ of T, that is, $F(T) = \{x \in C : Tx = x\}$. It is well known that $F(T)$ is a closed convex set, if T is $nonexpansive$.

Consider the set of solutions of the following *generalized variational inequality:* given nonlinear mappings $A, B : C \to H$ find a $x \in C$ such that

$$\langle x - \hat{\lambda} Bx + \lambda Ax, x - y\rangle \ge 0, \forall y \in C, \tag{1}$$

where $\hat{\lambda}$ and λ are two positive constants. We use $GVI(C, B, A)$ to denote *the set of solutions of the generalized variational inequality* (1). It is easy to see that an element $x \in C$ is a solution to the variational inequality (1) if and only if x is a fixed point of the mapping $P_C(\hat{\lambda}B - \lambda A)$, where P_C denotes the metric projection from H onto C. That is

$$x = F(P_C(\hat{\lambda}B - \lambda A))x \Leftrightarrow x \in GVI(C, B, A). \tag{2}$$

Therefore, fixed point algorithms can be applied to solve $GVI(C, B, A)$. Next, we consider a special case of (1). If $B = I$, the identity mapping and $\hat{\lambda} = 1$, then the generalized variational inequality (1.1) is reduced to *the variational inequality* as follow: find $x \in C$ such that

$$\langle Ax, x - y\rangle \ge 0, \forall y \in C. \tag{3}$$

We use $VI(C, A)$ to denote the set of solutions of the variational inequality (3). It is well known that the variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems; which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] and the references therein.

A lot of times, we may need to find a point $\overline{x} \in GVI(C, B, A)$ with the property that $\langle F\overline{x}, x - \overline{x}\rangle \le 0, \forall x \in GVI(C, B, A)$ where $GVI(C, B, A)$ is the solution set of the generalized variational inequality. We will describe this situation by the term *hierachical generalized variational inequality problems (HGVIP).* If the set $GVI(C, B, A)$ is replaced by the set $VI(C, A)$, the solution set of the variational inequality, then the HGVIP is called a hierarchical variational inequality problems (HVIP). Many problems in mathematics, for example the signal recovery[16], the power control problem[17] and the beamforming problem[18] can be modeled as HGVIP.

In 2011, Yu and Liang [15] proved the following theorem for finding solutions to the HGVIP for a cocoercive mapping.

Theorem I.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H, $A_m : C \to H$be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping, $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \le m \le r$. Assume that $\cap_{m=1}^{r} GVI(C, B_m, A_m) \neq \emptyset$. Given $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \Sigma_{m=1}^{r} \delta_{(m,n)} P_C(T_m x_n), \ \forall n \ge 1,$$

where $T_m = \hat{\lambda}_m B_m - \lambda_m A_m, u$ is fixed element in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_{(1,n)}\}, \{\delta_{(2,n)}\}, ..., \{\delta_{(r,n)}\}$ are sequences in $(0,1)$, satisfying the following conditions:

$(C1)$ $\alpha_n + \beta_n + \gamma_n = 1 = \Sigma_{m=1}^{r} \delta_{(m,n)}, \forall n \ge 1;$
$(C2)$ $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty} \alpha_n = \infty;$

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$(C3)$ $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$(C4)$ $\lim_{n\to\infty} \delta_{(m,n)} = \delta_m \in (0,1), \forall 1 \leq m \leq r$,

and $\{\lambda_m\}_{m=1}^r, \{\hat{\lambda}_m\}_{m=1}^r$ *are two positive sequences such that for each* $1 \leq m \leq r$

$$1 \leq \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2} + \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}.$$

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in \cap_{m=1}^r GVI(C, B_m, A_m)$*, which is the unique solution of the following:*

$$\langle u - \tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in \cap_{m=1}^r GVI(C, B_m, A_m). \tag{4}$$

On the other hand, the hierarchical fixed point problems, i.e, find $x^* \in F(T)$ such that $\langle Ax^*, x - x^*\rangle \geq 0, \forall x \in F(T)$, have attracted many authors attention due to their link with some convex programming problems. See [19], [20], [21], [22], [23], [24], [25], [26]. In 2010, Tian [27] introduced a general iterative method for nonexpansive mappings and proved the following theorem.

Theorem I.2. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*,* $f : C \to C$ *be a contraction with coefficient* $k \in (0,1)$*,* $G : C \to C$ *be* ξ*-strongly monotone and* L*-Lipschitz continuous mapping, Let* $S : C \to C$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$*,* $\xi > 0, L > 0$*,* $0 < \mu < 2\xi/L^2$ *and* $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$*. Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)Sx_n, \ \forall n \geq 1, \tag{5}$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$*, satisfying the following conditions:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$;
$(C2)$ $\Sigma_{n=1}^\infty \alpha_n = \infty$;
$(C3)$ $\Sigma_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in F(S)$*, which is the unique solution of the hierarchical fixed point problem:*

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \forall x \in F(S). \tag{6}$$

Motivated and inspired by Yu and Liang's results and Tian's results, we consider and study the CFP in the case that each C_m is a solution set of generalized variational inequality $GVI(C, B_m, A_m)$ and are devoted to solve the following the HGVIP: find $\tilde{x} \in \cap_{m=1}^r GVI(C, B_m, A_m)$ such that

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \leq 0, \forall x \in \cap_{m=1}^r GVI(C, B_m, A_m). \tag{7}$$

Which is the problem (7) is general than the problem (4) and (6). Consequently, we prove a strong convergence theorem for finding a point $\tilde{x}$ which is a unique solution of the HGVIP (7).

II. Preliminaries

This section collects some definitions and lemma which be use in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Let $A : C \to H$ and $G : C \to C$ be a nonlinear mappings. Recall the following definitions: for all $x, y \in C$

(a) A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

(b) A is said to be ρ-strongly monotone if there exists a positive real number $\rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho\|x - y\|^2.$$

(c) A is said to be η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|Ax - Ay\|^2.$$

(d) A is said to be relaxed η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)\|Ax - Ay\|^2.$$

(e) A is said to be relaxed (η, ρ)-cocoercive if there exists a positive real number $\eta, \rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)\|Ax - Ay\|^2 + \rho\|x - y\|^2.$$

(f) G is said to be L-Lipschitzian on C if there exists a positive real number $L > 0$ such that

$$\|G(x) - G(y)\| \leq L\|x - y\|.$$

(g) G is said to be k-contraction if there exists a positive real number $k \in (0,1)$ such that

$$\|G(x) - G(y)\| \leq k\|x - y\|.$$

Lemma II.1. [30] *Let* H *be a Hilbert space,* C *a closed convex subset of* H *and* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* C *weakly converging to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* y*, then* $(I - T)x = y$*; in particular, if* $y = 0$ *then* $x \in F(T)$*.*

Lemma II.2. [28] *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*. Let* $S_1 : C \to C$ *and* $S_2 : C \to C$ *be nonexpansive mappings on* C*. Suppose that* $F(S_1) \cap F(S_2)$ *is nonempty. Define a mapping* $S : C \to C$ *by*

$$Sx = aS_1 + (1 - a)S_2, \quad \forall x \in C,$$

where a *is a constant in* $(0,1)$*. Then* S *is nonexpansive with* $F(S) = F(S_1) \cap F(S_2)$*.*

Lemma II.3. [27]) *Let* $F : C \to C$ *be a* η*-strongly monotone and* L*-Lipschitzian operator with* $L > 0, \eta > 0$*. Assume that* $0 < \mu < 2\eta/L^2, \tau = \mu(\eta - \mu L^2/2)$ *and* $0 < t < 1$*. Then* $\|(I - \mu tF)x - (I - \mu tF)y\| \leq (1 - t\tau)\|x - y\|$*.*

Lemma II.4. *In a real Hilbert space* H*, we have the equations hold:*

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in H$;
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle, \ \forall x, y \in H$.

Lemma II.5. [29] *Assume that* $\{a_n\}$ *is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

1) $\sum_{n=1}^\infty \gamma_n = \infty$,
2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma II.6. [27] *Let* H *be a real Hilbert space,* $f : H \to H$ *be a contraction with coefficient* $0 < k < 1$*, and* $G : H \to H$ *be a* L*-Lipschitzian continuous operator and* ξ*-strongly monotone operator with* $L > 0$*,* $\xi > 0$*. Then for* $0 < \gamma < \mu\xi/k$ *and for all* $x, y \in H$*,*

$$\langle x - y, (\mu G - \gamma f)x - (\mu G - \gamma f)y\rangle \geq (\mu\xi - \gamma k)\|x - y\|^2.$$

That is, $\mu G - \gamma f$ *is* $(\mu\xi - \gamma k)$ *- strongly monotone.*

Lemma II.7. [31] *Let* C *be a closed convex subset of* H*. Let* $\{x_n\}$ *be a bounded sequence in* H*. Assume that*

(1) *The weak* ω*-limit set* $\omega_w(x_n) \subset C$*,*
(2) *For each* $z \in C$*,* $\lim_{n\to\infty} \|x_n - z\|$ *exists.*

Then $\{x_n\}$ *is weakly convergent to a point in* C*.*

Notation. We use $\to$ for strong convergence and $\rightharpoonup$ for weak convergence.

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III. MAIN RESULT

Theorem III.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $f : C \to C$ be a contraction with coefficient $k \in (0,1)$. Let $G : C \to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping and $B_m : C \to H$ be a relaxed $(\hat{\eta}_m, \hat{\rho}_m)$-cocoercive and $\hat{\nu}_m$-Lipschitz continuous mapping for each $1 \le m \le r$. Let $p_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2 + 2\lambda_m\eta_m\nu_m^2}$ and $q_m = \sqrt{1 - 2\hat{\lambda}_m\hat{\rho}_m + \hat{\lambda}_m^2\hat{\nu}_m^2 + 2\hat{\lambda}_m\hat{\eta}_m\hat{\nu}_m^2}$, where $\{\lambda_m\}$ and $\{\hat{\lambda}_m\}$ are two positive sequences for each $1 \le m \le r$. Assume that $\cap_{m=1}^r GVI(C, B_m, A_m) \ne \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ and $p_m, q_m \in [0, \frac{1}{2})$, for each $1 \le m \le r$. Given the initial guess $x_1 \in C$ and $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)\Sigma_{m=1}^r\beta_{(m,n)}P_C(T_m x_n), \tag{8}$$

where $T_m = P_C(\hat{\lambda}_m B_m - \lambda_m A_m), \forall 1 \le m \le r$ and $\{\alpha_n\}, \{\beta_{(1,n)}\}, \{\beta_{(2,n)}\}, ..., \{\beta_{(r,n)}\}$ are sequences in $(0,1)$, satisfying the following conditions:

$(C1)$ $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^\infty\alpha_n = \infty, \Sigma_{n=1}^\infty|\alpha_{n+1} - \alpha_n| < \infty;$
$(C2)$ $\Sigma_{m=1}^r\beta_{(m,n)} = 1, \forall n \ge 1, \Sigma_{n=1}^\infty|\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$, $\lim_{n\to\infty}\beta_{(m,n)} = \beta_m \in (0,1), \forall 1 \le m \le r.$

Then the sequence $\{x_n\}$ converges strongly to a common element $\tilde{x} \in \cap_{m=1}^r GVI(C, B_m, A_m)$, which is the unique solution of the HGVIP:

$$\langle(\gamma f - \mu G)\tilde{x}, x - \tilde{x}\rangle \le 0, \quad \forall x \in \cap_{m=1}^r GVI(C, B_m, A_m). \tag{9}$$

Proof: For each $x, y \in C$ and for each $m \ge 1$, we have

$$\begin{aligned} \|T_m x - T_m y\| &\le \|(\hat{\lambda}_m B_m - \lambda_m A_m)x - (\hat{\lambda}_m B_m - \lambda_m A_m)y\| \\ &\le \|(x-y) - \lambda_m(A_m x - A_m y)\| \\ &\quad + \|(x-y) - \hat{\lambda}_m(B_m x - B_m y)\|. \end{aligned} \tag{10}$$

It follows from the assumption that each A_m is relaxed (η_m, ρ_m)-cocoercive and ν_m-Lipschitz continuous that

$$\begin{aligned} &\|(x-y) - \lambda_m(A_m x - A_m y)\|^2 \\ &= \|x-y\|^2 + \lambda_m^2\|A_m x - A_m y\|^2 \\ &\quad -2\lambda_m\langle A_m x - A_m y, x - y\rangle \\ &\le \|x-y\|^2 - 2\lambda_m\big[(-\eta_m)\|A_m x - A_m y\|^2 \\ &\quad +\rho_m\|x-y\|^2\big] + \lambda_m^2\nu_m^2\|x-y\|^2 \\ &\le (1 - 2\lambda_m\rho_m + \lambda_m^2\nu_m^2)\|x-y\|^2 \\ &\quad +2\lambda_m\eta_m\nu_m^2\|x-y\|^2 \\ &= p_m^2\|x-y\|^2. \end{aligned}$$

This shows that

$$\|(x-y) - \lambda_m(A_m x - A_m y)\| \le p_m\|x-y\|. \tag{11}$$

In a similar way, we can obtain that

$$\|(x-y) - \hat{\lambda}_m(B_m x - B_m y)\| \le q_m\|x-y\|. \tag{12}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$\begin{aligned} \|T_m x - T_m y\| &\le (p_m + q_m)\|x-y\| \\ &\le \|x-y\|. \end{aligned}$$

Hence T_m is a nonexpansive mapping and $F(T_m) = F(P_C(\hat{\lambda}_m B_m - \lambda_m A_m)) = GVI(C, B_m, A_m)$ for each $1 \le m \le r$. Put $S_n = \Sigma_{m=1}^r\beta_{(m,n)}T_m$. By Lemma II.2, we conclude that S_n is a nonexpansive mapping and $F(S_n) = \cap_{m=1}^r GVI(C, B_m, A_m), \forall n \ge 1$. We can rewrite the algorithm (8) as

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)S_n x_n. \tag{13}$$

Step 1: We will show that $\{x_n\}$ is bounded.
Take $v \in F(S_n) = \cap_{m=1}^r GVI(C, B_m, A_m)$, from (13) and lemma II.3, we have

$$\begin{aligned} \|x_{n+1} - v\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)S_n x_n - v\| \\ &= \|\alpha_n(\gamma f(x_n) - \mu Gv) + (I - \alpha_n\mu G)S_n x_n \\ &\quad -(I - \alpha_n\mu G)v\| \\ &\le \alpha_n\|\gamma(f(x_n) - f(v)) + \gamma f(v) - \mu Gv\| \\ &\quad +(1 - \alpha_n\pi)\|x_n - v\| \\ &\le \alpha_n\gamma k\|x_n - v\| + \alpha_n\|\gamma f(v) - \mu Gv\| \\ &\quad +(1 - \alpha_n\pi)\|x_n - v\| \\ &= (1 - \alpha_n(\pi - \gamma k))\|x_n - v\| + \alpha_n\|\gamma f(v) - \mu Gv\| \\ &\le max\left\{\|x_n - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}. \end{aligned}$$

By induction, we obtain

$$\|x_n - v\| \le \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}.$$

Hence $\{x_n\}$ is bounded.
Since S_n is nonexpansive mappings for $n \ge 1$, we see that

$$\begin{aligned} \|S_n x_n - v\| &= \|S_n x_n - S_n v\| \\ &\le \|x_n - v\| \\ &\le \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}. \end{aligned}$$

Therefore, $\{S_n x_n\}$ is bounded. Since$G$ is a L-Lipschitz continuous mapping, we have

$$\begin{aligned} \|GS_n x_n - Gv\| &= \|GS_n x_n - GS_n v\| \\ &\le L\|S_n x_n - S_n v\| \\ &\le L\|x_n - v\| \\ &\le \max\left\{L\|x_1 - v\|, L\frac{\|\gamma f(v) - \mu Gv\|}{\pi - \gamma k}\right\}. \end{aligned}$$

Hence $\{GS_n x_n\}$ is bounded. Since f is contraction, so $f(x_n)$ is bounded.

Step 2: We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
From (13), we consider

$$\begin{aligned} &x_{n+1} - x_n \\ &= [\alpha_n\gamma f(x_n) + (I - \alpha_n\mu G)S_n x_n] \\ &\quad -[\alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}\mu G)S_{n-1}x_{n-1}] \\ &= \alpha_n\gamma(f(x_n) - f(x_{n-1})) + [(I - \alpha_n\mu G)S_n x_n \\ &\quad -(I - \alpha_n\mu G)S_{n-1}x_{n-1}] + (\alpha_n - \alpha_{n-1})\gamma f(x_{n-1}) \\ &\quad +(\alpha_{n-1} - \alpha_n)\mu GS_{n-1}x_{n-1}, \end{aligned}$$

it follows that

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\le \alpha_n\gamma k\|x_n - x_{n-1}\| + (1 - \alpha_n\pi)\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\quad +|\alpha_n - \alpha_{n-1}|(\gamma\|f(x_{n-1})\| + \mu\|GS_{n-1}x_{n-1}\|) \\ &\le \alpha_n\gamma k\|x_n - x_{n-1}\| + (1 - \alpha_n\pi)\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\quad +|\alpha_n - \alpha_{n-1}|M_1, \end{aligned} \tag{14}$$

where $M_1 = \sup_{n\ge1}\{\gamma\|f(x_n)\| + \mu\|GS_n x_n\|\}$. On the other hand, we note that

$$\begin{aligned} &\|S_n x_n - S_{n-1}x_{n-1}\| \\ &\le \|S_n x_n - S_n x_{n-1}\| + \|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \|\Sigma_{m=1}^r\beta_{(m,n)}T_m x_{n-1} \\ &\quad -\Sigma_{m=1}^r\beta_{(m,n-1)}T_m x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + M_2\Sigma_{m=1}^r|\beta_{(m,n)} - \beta_{(m,n-1)}|, \end{aligned} \tag{15}$$

where M_2 is appropriate constant such that $M_2 = \max\{\sup_{n\ge1}\|T_m x_n\|, \forall 1 \le m \le r\}$.
Substituting (15) into (14) yields

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad\le \alpha_n\gamma k\|x_n-x_{n-1}\|+(1-\alpha_n\pi)\|x_n-x_{n-1}\| \\
&\qquad +M_1|\alpha_n-\alpha_{n-1}| \\
&\qquad +M_2\Sigma_{m=1}^r|\beta_{(m,n)}-\beta_{(m,n-1)}| \\
&\quad\le \alpha_n\gamma k\|x_n-x_{n-1}\|+(1-\alpha_n\pi)\|x_n-x_{n-1}\| \\
&\qquad +M_3(|\alpha_n-\alpha_{n-1}|+\Sigma_{m=1}^r|\beta_{(m,n)}-\beta_{(m,n-1)}|),
\end{aligned}$$

where M_3 is an appropriate constant such that $M_3 \ge \max\{M_1, M_2\}$.
By conditions $(C1)$ and $(C2)$ and Lemma II.5, we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{16}$$

Step 3: We will show that $\lim_{n\to\infty}\|Sx_n-x_n\|=0$.
Define a mapping $S: C\to C$ by

$$Sx=\Sigma_{m=1}^r\beta_m T_m x, \forall x\in C,$$

where $\beta_m=\lim_{n\to\infty}\beta_{(m,n)}$. From Lemma II.2, we see that S is a nonexpansive mapping and

$$F(S)=\cap_{m=1}^r F(T_m)=\cap_{m=1}^r GVI(C,B_m,A_m), \forall n\ge 1.$$

From (13), we observe that

$$\begin{aligned}
\|x_{n+1}-S_nx_n\| &= \alpha_n\|\gamma f(x_n)+\mu GS_nx_n\| \\
&\le \alpha_n(\gamma\|f(x_n)-f(v)\|+\|\gamma f(v)+\mu GS_nv\| \\
&\quad +\mu\|GS_nx_n-GS_nv\|).
\end{aligned}$$

It follows from the condition $(C1)$ and the boundedness of $\{f(x_n)\}$ and $\{GS_nx_n\}$, we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-S_nx_n\|=0. \tag{17}$$

We observe that

$$\begin{aligned}
\|x_n-S_nx_n\| &= \|x_n-x_{n+1}+x_{n+1}-S_nx_n\| \\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-S_nx_n\|.
\end{aligned}$$

From (16) and (17), we obtain

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0. \tag{18}$$

Now, we show that $Sx_n-x_n\to 0$ as $n\to\infty$. Note that

$$\begin{aligned}
\|Sx_n-x_n\| &= \|Sx_n-S_nx_n+S_nx_n-x_n\| \\
&\le \|\Sigma_{m=1}^r\beta_mT_mx_n-\Sigma_{m=1}^r\beta_{(m,n)}T_mx_n\| \\
&\quad +\|S_nx_n-x_n\| \\
&\le M_2(\Sigma_{m=1}^r|\beta_m-\beta_{(m,n)}|)+\|S_nx_n-x_n\|.
\end{aligned}$$

By the condition $(C2)$ and (18), we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{19}$$

From the boundedness of x_n, we deduced that x_n converges weakly in $F(S)$, say $x_n\rightharpoonup p$, by Lemma II.1 and (19), we obtain $p=Sp$. So, we have

$$\omega_w(x_n)\subset F(S). \tag{20}$$

By Lemma II.6, $\mu G-\gamma f$ is strongly monotone, so the variational inequality (9) has a unique solution $\tilde{x}\in F(S)=\cap_{m=1}^r GVI(C,B_m,A_m)$.

Step 4: We show that $\limsup_{n\to\infty}\langle(\gamma f-\mu G)\tilde{x},x_n-\tilde{x}\rangle\le 0$.
Indeed, since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-\mu G)\tilde{x},x_n-\tilde{x}\rangle=\lim_{i\to\infty}\langle(\gamma f-\mu G)\tilde{x},x_{n_i}-\tilde{x}\rangle.$$

Without loss of generality, we may further assume that $x_{n_i}\rightharpoonup p$. It follows from (20) that $p\in F(S)$. Since $\tilde{x}$ is the unique solution of (9), we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f-\mu G)\tilde{x},x_n-\tilde{x}\rangle &= \lim_{i\to\infty}\langle(\gamma f-\mu G)\tilde{x},x_{n_i}-\tilde{x}\rangle \\
&= \langle(\gamma f-\mu G)\tilde{x},p-\tilde{x}\rangle\le 0. \quad (21)
\end{aligned}$$

Step 5: Finally, we will show that $x_n\to\tilde{x}$ as $n\to\infty$.
From Lemma II.4, we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \|\alpha_n(\gamma f(x_n)-\mu G\tilde{x})+(I-\alpha_n\mu G)S_nx_n \\
&\quad -\mu G(I-\alpha_n\mu G)\tilde{x}\|^2 \\
&\le (1-\alpha_n\pi)^2\|x_n-\tilde{x}\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\alpha_n\pi)^2\|x_n-\tilde{x}\|^2 \\
&\quad +2\alpha_n\gamma\langle f(x_n)-f(\tilde{x}),x_{n+1}-\tilde{x}\rangle \\
&\quad +2\alpha_n\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\alpha_n\pi)^2\|x_n-\tilde{x}\|^2 \\
&\quad +2\alpha_n\gamma k\|x_n-\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&\quad +2\alpha_n\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le (1-\alpha_n\pi)^2\|x_n-\tilde{x}\|^2 \\
&\quad +\alpha_n\gamma k(\|x_n-\tilde{x}\|^2+\|x_{n+1}-\tilde{x}\|^2) \\
&\quad +2\alpha_n\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&\le \frac{1-2\alpha_n\pi+(\alpha_n\pi)^2+\alpha_n\gamma k}{1-\alpha_n\gamma k}\|x_n-\tilde{x}\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma k}\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&= [1-\frac{2\alpha_n(\pi-\gamma k)}{1-\alpha_n\gamma k}]\|x_n-\tilde{x}\|^2 \\
&\quad +\frac{(\alpha_n\pi)^2}{1-\alpha_n\gamma k}\|x_n-\tilde{x}\|^2 \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma k}\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle \\
&= (1-\theta_n)\|x_n-\tilde{x}\|^2+\delta_n,
\end{aligned}$$

where $\theta_n := \frac{2\alpha_n(\pi-\gamma k)}{1-\alpha_n\gamma k}$ and $\delta_n := \frac{\alpha_n}{1-\alpha_n\gamma k}[\alpha_n\pi^2\|x_n-\tilde{x}\|^2+2\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle]$.
Note that,

$$\theta_n := \frac{2\alpha_n(\pi-\gamma k)}{1-\alpha_n\gamma k}\le\frac{2(\pi-\gamma k)}{1-\gamma k}\alpha_n.$$

By the condition $(C1)$, we obtain that

$$\lim_{n\to\infty}\theta_n=0. \tag{22}$$

On the other hand, we have

$$\theta_n := \frac{2\alpha_n(\pi-\gamma k)}{1-\alpha_n\gamma k}\ge 2\alpha_n(\pi-\gamma k).$$

From the condition $(C1)$, we have

$$\sum_{n=1}^{\infty}\theta_n=\infty. \tag{23}$$

Put $M=\sup_{n\in\mathbb{N}}\{\|x_n-\tilde{x}\|\}$, we have

$$\frac{\delta_n}{\theta_n}=\frac{1}{2(\pi-\gamma k)}[\alpha_n\pi^2M+2\langle\gamma f(\tilde{x})-\mu G\tilde{x},x_{n+1}-\tilde{x}\rangle].$$

From the condition $(C1)$ and (21), we have

$$\limsup_{n\to\infty}\frac{\delta_n}{\theta_n}\le 0. \tag{24}$$

Hence, by Lemma II.5, (22), (23) and (24), we conclude that

$$\lim_{n\to\infty}\|x_n-\tilde{x}\|=0.$$

This completes the proof. ■

If $B_m=I$, the identity mapping and $\hat{\lambda_m}=1$, then Theorem III.1 is reduced to the following result on the classical variational inequality (3).

Corollary III.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C\pm C\subset C$. Let $f: C\to C$ be a contraction with coefficient $k\in(0,1)$. Let $G: C\to C$ be a ξ-strongly monotone and L-Lipschitz continuous mapping. Let $A_m: C\to H$ be a relaxed (η_m,ρ_m)-cocoercive and ν_m-Lipschitz continuous mapping, for each $1\le m\le r$. Let $p_m=\sqrt{1-2\lambda_m\rho_m+\lambda_m^2\nu_m^2+2\lambda_m\eta_m\nu_m^2}$, where $\{\lambda_m\}$*

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is a positive sequence, for each $1 \leq m \leq r$. *Assume that* $\cap_{m=1}^{r} VI(C, A_m) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ *and* $p_m \in [0, 1)$], *for each* $1 \leq m \leq r$. *Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)\Sigma_{m=1}^{r} \beta_{(m,n)} P_C(x_n - \lambda_m A_m x_n),$$

where $\{\alpha_n\}, \{\beta_{(1,n)}\}, \{\beta_{(2,n)}\}, ..., \{\beta_{(r,n)}\}$ *are sequences in* $(0, 1)$, *satisfying the following conditions:*

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty} \alpha_n = \infty, \Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
$(C2)$ $\Sigma_{m=1}^{r} \beta_{(m,n)} = 1, \forall n \geq 1, \Sigma_{n=1}^{\infty} |\beta_{(m,n+1)} - \beta_{(m,n)}| < \infty$, $\lim_{n\to\infty} \beta_{(m,n)} = \beta_m \in (0, 1), \forall 1 \leq m \leq r$,

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in \cap_{m=1}^{r} VI(C, A_m)$, *which is the unique solution of the HVIP:*

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \cap_{m=1}^{r} VI(C, A_m).$$

If $r = 1$, then Theorem III.1 is reduced to the following Corollary.

Corollary III.3. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H *such that* $C \pm C \subset C$. *Let* $f : C \to C$ *be a contraction with coefficient* $k \in (0, 1)$. *Let* $G : C \to C$ *be a* ξ*-strongly monotone and* L*-Lipschitz continuous mapping. Let* $A : C \to H$ *be a relaxed* (η, ρ)*-cocoercive and* ν*-Lipschitz continuous mapping. Let* $B : C \to H$ *be a relaxed* $(\hat{\eta}, \hat{\rho})$*-cocoercive and* $\hat{\nu}$*-Lipschitz continuous mapping. Let* $p = \sqrt{1 - 2\lambda\rho + \lambda^2\nu^2 + 2\lambda\eta\nu^2}$ *and* $q = \sqrt{1 - 2\hat{\lambda}\hat{\rho} + \hat{\lambda}^2\hat{\nu}^2 + 2\hat{\lambda}\hat{\eta}\hat{\nu}^2}$, *where* λ *and* $\hat{\lambda}$ *are two positive real numbers. Assume that* $GVI(C, B, A) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ *and* $p, q \in [0, \frac{1}{2})$. *Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) P_C(\hat{\lambda} B x_n - \lambda A x_n),$$

where $\{\alpha_n\}$ *is a sequences in* $(0, 1)$, *satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \Sigma_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in GVI(C, B, A)$, *which is the unique solution of the HGVIP:*

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in GVI(C, B, A).$$

For the variational inequality (3), we can obtain from Corollary III.3 the following immediately.

Corollary III.4. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H *such that* $C \pm C \subset C$. *Let* $f : C \to C$ *be a contraction with coefficient* $k \in (0, 1)$. *Let* $G : C \to C$ *be a* ξ*-strongly monotone and* L*-Lipschitz continuous mapping. Let* $A : C \to H$ *be a relaxed* (η, ρ)*-cocoercive and* ν*-Lipschitz continuous mapping. Let* $p = \sqrt{1 - 2\lambda\rho + \lambda^2\nu^2 + 2\lambda\eta\nu^2}$, *where* λ *is a positive real number. Assume that* $VI(C, A) \neq \emptyset$, $\xi > 0, L > 0$, $0 < \mu < 2\xi/L^2$, $0 < \gamma < \mu(\xi - \mu L^2/2)/k = \pi/k$ *and* $p \in [0, 1)$. *Given the initial guess* $x_1 \in C$ *and* $\{x_n\}$ *is a sequence generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) P_C(x_n - \lambda A x_n),$$

where $\{\alpha_n\}$ *is a sequences in* $(0, 1)$, *satisfying the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \Sigma_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\tilde{x} \in VI(C, A)$, *which is the unique solution of the HVIP:*

$$\langle (\gamma f - \mu G)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in VI(C, A).$$

Remark III.5. (1) If we take $G = A$ and $\mu = 1$, where A is a strongly positive linear bounded operator on C in Theorem III.1, then our iterative algorithm define by (8) converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle (\gamma f - A)\tilde{x}, x - \tilde{x} \rangle \leq 0, \ \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, Equivalently, $\tilde{x}$ is the unique solution to the minimization problem:

$$\min_{x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

(2) If we taking $G = I$ and $\gamma = \mu = 1$, where I is a identity mapping in Theorem III.1, then our iterative algorithm define by (8) converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle (f - I)\tilde{x}, x - \tilde{x} \rangle \leq 0, \ \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$.
In case, $f = 0$, our iterative algorithm define by (8) converges strongly to $\tilde{x}$ which is the unique solution to the quadratic minimiztion problem:

$$z = \operatorname*{arg\,min}_{x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)} \|x\|^2. \tag{25}$$

In case, $f = u$, where u is fixed element in C, our iterative algorithm define by (8) converges strongly to a common element $\tilde{x} \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$, such that $\langle u - \tilde{x}, x - \tilde{x} \rangle \leq 0, \ \forall x \in \cap_{m=1}^{r} GVI(C, B_m, A_m)$.

(3) Note that, our iterative algorithm define by (8) are more flexible in solving the HGVIP than the one introduced by Yu and Liang.

IV. Conclusion

We studied the convex feasibility problem (CFP) in the case that each closed convex set is a solution set of generalized variational inequality and exhibits an algorithm for finding solution of the hierarchical generalized variational inequality problem (HGVIP). The result of this paper extends and generalizes the corresponding results given by Yu and Liang [15] and some authors in the literature.

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