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Random Fuzzy Multiobjective Linear Programming Through Probability Maximization

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***Abstract*—In this paper, we propose an interactive decision making method for random fuzzy multiobjective linear programming problems (RFMOLP) through a probability maximization model. In the proposed method, it is assumed that the decision maker has fuzzy goals for not only permissible objective levels of a probability maximization model but also the corresponding distribution function values. Using the fuzzy decision, such two kinds of membership functions are integrated. In the integrated membership space, a satisfactory solution is obtained from among a Pareto optimal solution set through the interaction with the decision maker.**

***Index Terms*—random fuzzy variable, a probability maximization model, satisfactory solution, interactive decision making.**

I. Introduction

In the real world decision making situations, we often have to make a decision under uncertainty. In order to deal with decision problems involving uncertainty, stochastic programming approaches [1], [2], [3], [7] and fuzzy programming approaches [13], [17], [18] have been developed. Recently, in order to deal with mathematical programming problems involving the randomness and the fuzziness, random fuzzy programming has been developed [8], in which the coefficients of the objective functions and/or the constraints are represented with random fuzzy variables [14], [15]. As a natural extension, a random fuzzy multiobjective programming problem (RFMOLP) was formulated and the interactive decision making methods were proposed to obtain the satisfactory solution of the decision maker from among the Pareto optimal solution set [9], [10], [11], [12]. Moreover, in order to show the efficiency of random fuzzy programming techniques, real-world decision making problems under random fuzzy environments were formulated as random fuzzy programming problems, and the corresponding algorithms to obtain the optimal solutions were proposed [5], [6], [16].

Under these circumstances, we focus on the interactive decision making method [8], [9], [10] for RFMOLP to obtain a satisfactory solution, in which a probability maximization model or a fractile optimization model is adopted in order to deal with RFMOLP. In their proposed methods, it seems to be very difficult for the decision maker to specify permissible objective levels or permissible probability levels appropriately. From such a point of view, in this paper, under the assumption that the decision maker has fuzzy goals for permissible objective levels of a probability maximization model, we propose an interactive decision making method for RFMOLP to obtain a satisfactory solution of the decision maker.

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II. Problem Formulation

In this section, we focus on RFMOLP in which random variable coefficients are involved in objective functions.

[RFMOLP]

$$\min_{\boldsymbol{x}\in X} \bar{\tilde{\boldsymbol{C}}}\boldsymbol{x} = (\bar{\tilde{\boldsymbol{c}}}_1\boldsymbol{x}, \cdots, \bar{\tilde{\boldsymbol{c}}}_k\boldsymbol{x})$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is an n dimensional decision variable column vector, $\bar{\tilde{\boldsymbol{c}}}_i = (\bar{\tilde{c}}_{i1}, \cdots, \bar{\tilde{c}}_{in}), i = 1, \cdots, k$, are coefficient vectors of objective function $\bar{\tilde{\boldsymbol{c}}}_i\boldsymbol{x}$, whose elements are random fuzzy variables [14], and the symbols "-" and "~" mean randomness and fuzziness respectively.

In this paper, according to Katagiri et al. [8], [9], [10], we assume that a random fuzzy variable $\bar{\tilde{c}}_{ij}$ is normally distributed with the fuzzy number $\tilde{M}_{ij}$ as mean and σ_{ij}^2 as variance. As a result, we assume that a probability density function $f_{ij}(y)$ for a random fuzzy variable $\bar{\tilde{c}}_{ij}$ is formally represented with the following form.

$$f_{ij}(y) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} e^{-\frac{(y-\tilde{M}_{ij})^2}{2\sigma_{ij}^2}}, 1 \le i \le k, 1 \le j \le n \quad (1)$$

where $\tilde{M}_{ij}$ is an L-R fuzzy number characterized by the following membership function.

$$\mu_{\tilde{M}_{ij}}(t) = \begin{cases} L\left(\frac{m_{ij}-t}{\alpha_{ij}}\right) & \text{if} \quad m_{ij} \ge t \\ R\left(\frac{t-m_{ij}}{\beta_{ij}}\right) & \text{if} \quad m_{ij} \le t \end{cases} \quad (2)$$

L and R are called reference functions, m_{ij} is the mean value, and α_{ij}, β_{ij} are spread parameters [4].

Then, a random fuzzy variable $\bar{\tilde{c}}_{ij}$ can be characterized by the following membership function [8], [9], [10].

$$\mu_{\bar{\tilde{c}}_{ij}}(\bar{\gamma}_{ij}) = \sup_{s_{ij}}\{\mu_{\tilde{M}_{ij}}(s_{ij}) | \bar{\gamma}_{ij} \sim N(s_{ij}, \sigma_{ij}^2)\} \quad (3)$$

where $N(s_{ij}, \sigma_{ij}^2)$ means a normal distribution with mean s_{ij} and standard deviation σ_{ij}. Moreover, using Zadeh's extension principle [17], [18], the objective function $\bar{\tilde{c}}_i\boldsymbol{x}$ become a random fuzzy variable characterized by the following membership function [8], [9], [10].

$$\mu_{\bar{\tilde{\boldsymbol{c}}}_i\boldsymbol{x}}(\bar{u}_i) = \sup_{(s_{i1},\cdots,s_{in})\in \mathrm{R}^n} \left\{ \min_{1\le j\le n} \mu_{\tilde{M}_{ij}}(s_{ij}) \,\middle|\, \bar{u}_i \sim N\left(\sum_{j=1}^n s_{ij}x_j, \sum_{j=1}^n \sigma_{ij}^2 x_j^2\right)\right\} \quad (4)$$

Unfortunately, we can not treat RFMOLP directly because it is ill-defined. Katagiri et al. [8], [10] formulated RFMOLP using permissible objective levels of a probability maximization model and the possibility measure. For permissible

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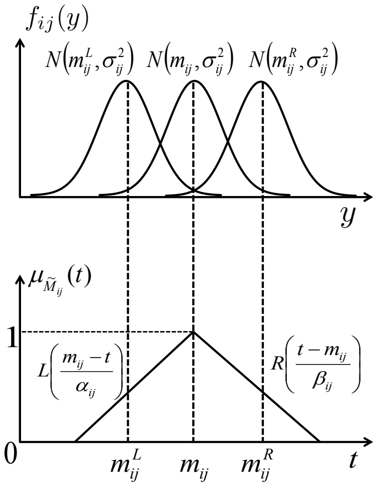

Fig. 1. An Image of a random fuzzy variable $\bar{\tilde{c}}_{ij}$

objective levels $f_i, i = 1, \cdots, k$ specified by the decision maker, a probability maximization model for RFMOLP can be formulated as follows.

[MOP1($\boldsymbol{f}$)]

$$\max_{\boldsymbol{x}\in X}(\Pr(\omega|\bar{\tilde{\boldsymbol{c}}}_1(\omega)\boldsymbol{x} \leq f_1), \cdots, \Pr(\omega|\bar{\tilde{\boldsymbol{c}}}_k(\omega)\boldsymbol{x} \leq f_k))$$

It should be noted here that the each objective function :

$$\tilde{P}_i(\boldsymbol{x}, f_i) \stackrel{\text{def}}{=} \Pr(\omega|\bar{\tilde{\boldsymbol{c}}}_i(\omega)\boldsymbol{x} \leq f_i), \tag{5}$$

becomes a fuzzy set and the corresponding membership function is defined as follows[8], [9], [10].

$$\mu_{\tilde{P}_i(\boldsymbol{x},f_i)}(p_i) \stackrel{\text{def}}{=} \sup_{\bar{u}_i}\left\{\mu_{\bar{\tilde{\boldsymbol{c}}}_i\boldsymbol{x}}(\bar{u}_i)\middle| p_i = \Pr(\omega|\bar{u}_i(\omega) \leq f_i),\ \bar{u}_i \sim N\left(\sum_{j=1}^n s_{ij}x_j, \sum_{j=1}^n \sigma_{ij}^2 x_j^2\right)\right\} \tag{6}$$

Katagiri et al. [8], [9], [10] showed that, from (4), the membership function (6) can be transformed as follows.

$$\mu_{\tilde{P}_i(\boldsymbol{x},f_i)}(p_i) = \sup_{(s_{i1},\cdots,s_{in})\in\mathrm{R}^n} \min_{1\leq j\leq n}\left\{\mu_{\tilde{M}_{ij}}(s_{ij})\middle| p_i = \Pr(\omega|\bar{u}_i(\omega) \leq f_i),\ \bar{u}_i \sim N\left(\sum_{j=1}^n s_{ij}x_j, \sum_{j=1}^n \sigma_{ij}^2 x_j^2\right)\right\} \tag{7}$$

Using (5), MOP1($\boldsymbol{f}$) can be transformed as follows.

[MOP2($\boldsymbol{f}$)]

$$\max_{\boldsymbol{x}\in X}(\tilde{P}_1(\boldsymbol{x}, f_1), \cdots, \tilde{P}_k(\boldsymbol{x}, f_k))$$

MOP2($\boldsymbol{f}$) is ill-defined yet, because objective functions of MOP2($\boldsymbol{f}$) are fuzzy sets depending on permissible objective level $f_i, i = 1, \cdots, k$. In order to deal with MOP2($\boldsymbol{f}$), let us assume that the decision maker has a fuzzy goal $\tilde{G}_i$ for each objective function $\tilde{P}_i(\boldsymbol{x}, f_i)$, which is expressed in words such as "$\tilde{P}_i(\boldsymbol{x}, f_i)$ should be substantially less than p_i". For the corresponding membership function $\mu_{\tilde{G}_i}(p_i)$, we make the following assumption.

Assumption 1.

$\mu_{\tilde{G}_i}(p_i), i = 1, \cdots, k$ are strictly increasing and continuous with respect to $p_i \in [p_{i\min}, p_{i\max}]$, and $\mu_{p_i}(p_{i\min}) = 0$, $\mu_{p_i}(p_{i\max}) = 1$, where $0.5 < p_{i\min}$ is a maximum value of an unacceptable levels and $p_{i\max} < 1$ is a minimum value of a sufficiently satisfactory levels.

Using possibility measure [4],

$$\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i) \stackrel{\text{def}}{=} \sup_{p_i} \min\{\mu_{\tilde{P}_i(\boldsymbol{x},f_i)}(p_i), \mu_{\tilde{G}_i}(p_i)\}, \tag{8}$$

Katagiri et al. [8], [9], [10] transformed MOP2($\boldsymbol{f}$) into the following well-defined multiobjective programming problem.

[MOP3($\boldsymbol{f}$)]

$$\max_{\boldsymbol{x}\in X}(\Pi_{\tilde{P}_1(\boldsymbol{x},f_1)}(\tilde{G}_1), \cdots, \Pi_{\tilde{P}_k((\boldsymbol{x},f_k))}(\tilde{G}_k))$$

Unfortunately, in MOP3($\boldsymbol{f}$), the decision maker must specify permissible objective levels in advance. However, it seems very difficult to specify such values because $\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i)$ depends on a permissible objective levels f_i. From such a point of view, in this paper, instead of MOP3($\boldsymbol{f}$), we consider the following extended problem where $f_i, i = 1, \cdots, k$ are not constants but decision variables.

[MOP4]

$$\max_{\boldsymbol{x}\in X, f_i\in\mathrm{R}^1, i=1,\cdots,k} (\Pi_{\tilde{P}_1(\boldsymbol{x},f_1)}(\tilde{G}_1), \cdots, \Pi_{\tilde{P}_k((\boldsymbol{x},f_k))}(\tilde{G}_k), -f_1, \cdots, -f_k)$$

Considering the imprecise nature of the decision maker's judgment, we assume that the decision maker has a fuzzy goal for each permissible objective level. Such a fuzzy goal can be quantified by eliciting the corresponding membership function. Let us denote a membership function of a permissible objective level f_i as $\mu_{\tilde{F}_i}(f_i)$. For the membership function $\mu_{\tilde{F}_i}(f_i)$, we make the following assumption.

Assumption 2.

$\mu_{\tilde{F}_i}(f_i), i = 1, \cdots, k$ are strictly decreasing and continuous with respect to $f_i \in [f_{i\min}, f_{i\max}]$, and $\mu_{f_i}(f_{i\min}) = 1$, $\mu_{f_i}(f_{i\max}) = 0$, where $f_{i\min}$ is a maximum value of a sufficiently satisfactory levels. and $f_{i\max}$ is a minimum value of an unacceptable levels.

Then, MOP4 can be transformed as the following multiobjective programming problem.

[MOP5]

$$\max_{\boldsymbol{x}\in X, f_i\in\mathrm{R}^1, i=1,\cdots,k} (\Pi_{\tilde{P}_1(\boldsymbol{x},f_1)}(\tilde{G}_1), \cdots, \Pi_{\tilde{P}_k((\boldsymbol{x},f_k))}(\tilde{G}_k), \mu_{\tilde{F}_1}(f_1), \cdots, \mu_{\tilde{F}_k}(f_k))$$

It should be noted here that, $\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i)$ is strictly increasing with respect to f_i. If the decision maker adopts the fuzzy decision [17], [18] to integrate $\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i)$ and $\mu_{\tilde{F}_i}(f_i)$, MOP5 can be transformed into the following form.

[MOP6]

$$\max_{\boldsymbol{x}\in X, f_i\in\mathrm{R}^1, i=1,\cdots,k} (\mu_{D_1}(\boldsymbol{x}, f_1), \cdots, \mu_{D_k}(\boldsymbol{x}, f_k))$$

where

$$\mu_{D_i}(\boldsymbol{x}, f_i) \stackrel{\text{def}}{=} \min\{\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i), \mu_{\tilde{F}_i}(f_i)\}$$

In order to deal with MOP6, we introduce a D-Pareto optimal solution concept.

Definition 1.

$\boldsymbol{x}^* \in X, f_i^* \in \mathrm{R}^1, i = 1, \cdots, k$ is said to be a D-Pareto optimal solution to MOP6, if and only if there does not exist another $\boldsymbol{x} \in X, f_i \in \mathrm{R}^1, i = 1, \cdots, k$ such that $\mu_{D_i}(\boldsymbol{x}, f_i) \geq \mu_{D_i}(\boldsymbol{x}^*, f_i^*)$ $i = 1, \cdots, k$ with strict inequality holding for at least one i.

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For generating a candidate of a satisfactory solution which is also D-Pareto optimal, the decision maker is asked to specify the reference membership values [17]. Once the reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$ are specified, the corresponding D-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX1($\hat{\boldsymbol{\mu}}$)]

$$\min_{\boldsymbol{x}\in X, f_i\in \mathrm{R}^1, i=1,\cdots,k, \lambda\in[0,1]} \lambda \tag{9}$$

subject to

$$\hat{\mu}_i - \Pi_{\tilde{P}_i(\boldsymbol{x}, f_i)}(\tilde{G}_i) \leq \lambda, i = 1, \cdots, k \tag{10}$$
$$\hat{\mu}_i - \mu_{\tilde{F}_i}(f_i) \leq \lambda, i = 1, \cdots, k \tag{11}$$

From [8], [9], [10], each constraint of (10) can be equivalently transformed into the following form.

$$\begin{aligned} & \hat{\mu}_i - \Pi_{\tilde{P}_i(\boldsymbol{x}, f_i)}(\tilde{G}_i) \leq \lambda \\ \Leftrightarrow \ & \sum_{j=1}^{n} \{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda)\alpha_{ij}\} x_j \\ & + \Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda)) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \leq f_i \end{aligned} \tag{12}$$

where $\Phi(\cdot)$ is a distribution function of the standard Gaussian random variable, $\Phi^{-1}(\cdot)$ is a corresponding inverse function, and $L^{-1}(\cdot), \mu_{\tilde{G}_l}^{-1}(\cdot)$ are pseudo-inverse functions of $L(\cdot), \mu_{\tilde{G}_i}(\cdot)$ respectively. Moreover, since the inequalities (11) can be transformed into $f_i \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda)$, MINMAX1($\hat{\boldsymbol{\mu}}$) can be reduced to the following problem.

[MINMAX2($\hat{\boldsymbol{\mu}}$)]

$$\min_{\boldsymbol{x}\in X, \lambda\in\Lambda} \lambda \tag{13}$$

subject to

$$\begin{aligned} & \sum_{j=1}^{n} \{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda)\alpha_{ij}\} x_j \\ & + \Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda)) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda), \\ & i = 1, \cdots, k \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Lambda & \stackrel{\text{def}}{=} [\lambda_{\min}, \lambda_{\max}] \\ & = [\max_{i=1,\cdots,k} \hat{\mu}_i - 1, \min_{i=1,\cdots,k} \hat{\mu}_i]. \end{aligned} \tag{15}$$

The relationships between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$) and D-Pareto optimal solutions can be characterized by the following theorem.

Theorem 1.

(1) If $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution of MINMAX2($\hat{\boldsymbol{\mu}}$), then $\boldsymbol{x}^* \in X, \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*) \in \mathrm{R}^1, i = 1, \cdots, k$ is a D-Pareto optimal solution.

(2) If $\boldsymbol{x}^* \in X, f_i^* \in \mathrm{R}^1, i = 1, \cdots, k$ is a D-Pareto optimal solution, then $\boldsymbol{x}^* \in X$, $\lambda^* = \hat{\mu}_i - \Pi_{\tilde{P}_i(\boldsymbol{x}^*, f_i^*)}(\tilde{G}_i) = \hat{\mu}_i - \mu_{\tilde{F}_i}(f_i^*), i = 1, \cdots, k$ is an optimal solution of MINMAX2($\hat{\boldsymbol{\mu}}$) for some reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$.

(Proof)

(1) Assume that $\boldsymbol{x}^* \in X, f_i^* \stackrel{\text{def}}{=} \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*), i = 1, \cdots, k$ is not a D-Pareto optimal solution. Then, from (10), there exist $\boldsymbol{x} \in X, f_i \in \mathrm{R}^1, i = 1, \cdots, k$ such that

$$\begin{aligned} \mu_{D_i}(\boldsymbol{x}, f_i) & = \min\{\Pi_{\tilde{P}_i(\boldsymbol{x}, f_i)}(\tilde{G}_i), \mu_{\tilde{F}_i}(f_i)\} \\ & \geq \mu_{D_i}(\boldsymbol{x}^*, f_i^*) \\ & = \min\{\Pi_{\tilde{P}_i(\boldsymbol{x}^*, f_i^*)}(\tilde{G}_i), \mu_{\tilde{F}_i}(f_i^*)\} \\ & = \hat{\mu}_i - \lambda^*, i = 1, \cdots, k, \end{aligned}$$

with strict inequality holding for at least one i. Then it holds that

$$\Pi_{\tilde{P}_i(\boldsymbol{x}, f_i)}(\tilde{G}_i) \geq \hat{\mu}_i - \lambda^*, i = 1, \cdots, k, \tag{16}$$
$$\mu_{\tilde{F}_i}(f_i) \geq \hat{\mu}_i - \lambda^*, i = 1, \cdots, k. \tag{17}$$

From (12), (16) can be transformed as follows.

$$\begin{aligned} & \sum_{j=1}^{n} \{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_{ij}\} x_j \\ & + \Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \leq f_i \end{aligned} \tag{18}$$

From (17), it holds that $f_i \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*)$. As a result, there exists $\boldsymbol{x} \in X$ such that

$$\begin{aligned} & \sum_{j=1}^{n} \{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_{ij}\} x_j \\ & + \Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*)) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*) \\ & i = 1, \cdots, k \end{aligned} \tag{19}$$

which contradicts the fact that $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX2($\hat{\boldsymbol{\mu}}$).

(2) Assume that $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is not an optimal solution to MINMAX2($\hat{\boldsymbol{\mu}}$) for any reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \cdots, \hat{\mu}_k)$, which satisfy the equalities :

$$\hat{\mu}_i - \lambda^* = \Pi_{\tilde{P}_i(\boldsymbol{x}^*, f_i^*)}(\tilde{G}_i) = \mu_{\tilde{F}_i}(f_i^*), i = 1, \cdots, k. \tag{20}$$

Then, there exists some $\boldsymbol{x} \in X, \lambda < \lambda^*$ such that

$$\begin{aligned} & \sum_{j=1}^{n} \{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda)\alpha_{ij}\} x_j \\ & + \Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda)) \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2 x_j^2} \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda), \\ & i = 1, \cdots, k. \end{aligned} \tag{21}$$

This means that

$$\begin{aligned} \Pi_{\tilde{P}_i(\boldsymbol{x}, f_i)}(\tilde{G}_i) & \geq \hat{\mu}_i - \lambda > \hat{\mu}_i - \lambda^*, i = 1, \cdots, k \\ \mu_{\tilde{F}_i}(f_i) & = \hat{\mu}_i - \lambda > \hat{\mu}_i - \lambda^*, i = 1, \cdots, k, \end{aligned}$$

where $f_i \stackrel{\text{def}}{=} \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda)$. From (20), there exists $\boldsymbol{x} \in X, f_i \in \mathrm{R}^1, i = 1, \cdots, k$ such that

$$\mu_{D_i}(\boldsymbol{x}, f_i) > \mu_{D_i}(\boldsymbol{x}^*, f_i^*), i = 1, \cdots, k.$$

This contradicts the fact that $\boldsymbol{x}^* \in X, f_i^* \in \mathrm{R}^1, i = 1, \cdots, k$ is a D-Pareto optimal solution. □

Since the constraints (14) are nonlinear, it is not easy to solve MINMAX2($\hat{\boldsymbol{\mu}}$) directly. Before considering the algorithm to solve MINMAX2($\hat{\boldsymbol{\mu}}$), we first define the following functions corresponding to (14).

$$\begin{aligned} g_i(\boldsymbol{x},\lambda) &\stackrel{\text{def}}{=} \mu^{-1}_{\tilde{F}_i}(\hat{\mu}_i-\lambda)-\sum_{j=1}^{n}\{m_{ij}-L^{-1}(\hat{\mu}_i-\lambda)\alpha_{ij}\}x_j \\ &\quad -\Phi^{-1}(\mu^{-1}_{\tilde{G}_i}(\hat{\mu}_i-\lambda))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2x_j^2}, \\ &\quad i=1,\cdots,k \end{aligned} \tag{22}$$

Because of $\mu^{-1}_{\tilde{G}_i}(\hat{\mu}_i-\lambda)>0.5$ for any $\lambda\in\Lambda$, it holds that $\Phi^{-1}(\mu^{-1}_{p_i}(\hat{\mu}_i-\lambda))>0$. This means that $g_i(\boldsymbol{x},\lambda), i=1,\cdots,k$ are concave with respect to $\boldsymbol{x}\in X$ for any fixed value $\lambda\in\Lambda$. Let us define the following feasible set $X(\lambda)$ of MINMAX2($\hat{\boldsymbol{\mu}}$) for some fixed value $\lambda\in\Lambda$.

$$X(\lambda)\stackrel{\text{def}}{=}\{\boldsymbol{x}\in X\mid g_i(\boldsymbol{x},\lambda)\geq 0, i=1,\cdots,k\} \tag{23}$$

Then, it is clear that $X(\lambda)$ is a convex set. $X(\lambda)$ satisfies the following property.

Property 1.

If $\lambda_{\min}\leq\lambda_1\leq\lambda_2\leq\lambda_{\max}$, then it holds that $X(\lambda_1)\subset X(\lambda_2)$.

In the following, it is assumed that $X(\lambda_{\min})=\phi, X(\lambda_{\max})\neq\phi$. From Property 1, we can obtain the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$) using the following simple algorithm which is based on the bisection method and the convex programming technique.

[**Algorithm 1.**]

Step 1: Set $\lambda_0=\lambda_{\min},\lambda_1=\lambda_{\max}$, $\lambda\leftarrow(\lambda_0+\lambda_1)/2$.

Step 2: Solve the following convex programming problem for the fixed value λ, and denote the optimal solution as $\boldsymbol{x}(\lambda)$.

$$\max_{\boldsymbol{x}\in X} g_j(\boldsymbol{x},\lambda) \tag{24}$$

subject to

$$g_i(\boldsymbol{x},\lambda)\geq 0, i=1,\cdots,k, i\neq j \tag{25}$$

Step 3: If $g_\ell(\boldsymbol{x}(\lambda),\lambda)\geq 0,\ell=1,\cdots,k$ then set $\lambda_1\leftarrow\lambda,\lambda\leftarrow(\lambda_0+\lambda_1)/2$. Otherwise, set $\lambda_0\leftarrow\lambda,\lambda\leftarrow(\lambda_0+\lambda_1)/2$. If $\mid\lambda_1-\lambda_0\mid<\epsilon$ then go to Step 4, where ϵ is a sufficiently small positive constant. Otherwise, go to Step 2.

Step 4: Set $\lambda^*\leftarrow\lambda$ and $\boldsymbol{x}^*\leftarrow\boldsymbol{x}(\lambda)$. The optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$) is obtained.

III. An Interactive Algorithm

In this section, we propose an interactive algorithm to obtain a satisfactory solution from among a D-Pareto optimal solution set. From Theorem 1, it is not guaranteed that the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$) is D-Pareto optimal, if it is not unique. In order to guarantee the D-Pareto optimality, we first assume that k constraints (14) are active at the optimal solution $(\boldsymbol{x}^*,\lambda^*)$, $i.e.$,

$$\begin{aligned} &\sum_{j=1}^{n}\{m_{ij}-L^{-1}(\hat{\mu}_i-\lambda^*)\alpha_{ij}\}x_j^* \\ &+\Phi^{-1}(\mu^{-1}_{\tilde{G}_i}(\hat{\mu}_i-\lambda^*))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2x_j^{*2}}=\mu^{-1}_{\tilde{F}_i}(\hat{\mu}_i-\lambda^*), \\ &i=1,\cdots,k. \end{aligned} \tag{26}$$

If the ℓ-th constraint of (14) is inactive, $i.e.$,

$$\begin{aligned} &\sum_{j=1}^{n}\{m_{\ell j}-L^{-1}(\hat{\mu}_\ell-\lambda^*)\alpha_{\ell j}\}x_j^* \\ &+\Phi^{-1}(\mu^{-1}_{\tilde{G}_\ell}(\hat{\mu}_\ell-\lambda^*))\sqrt{\sum_{j=1}^{n}\sigma_{\ell j}^2x_j^{*2}}<\mu^{-1}_{\tilde{F}_\ell}(\hat{\mu}_\ell-\lambda^*), \end{aligned} \tag{27}$$

we can convert the inactive constraint (27) into the active one by applying the bisection method for the reference membership value $\hat{\mu}_\ell\in[\lambda^*,\lambda^*+1]$.

For the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$), where the active conditions (26) are satisfied, we solve the D-Pareto optimality test problem defined as follows.

[D-Pareto optimality test problem]

$$\max_{\boldsymbol{x}\in X,\epsilon_i\geq 0,i=1,\cdots,k} w\stackrel{\text{def}}{=}\sum_{i=1}^{k}\epsilon_i \tag{28}$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n}\{m_{ij}-L^{-1}(\hat{\mu}_i-\lambda^*)\alpha_{ij}\}x_j \\ &+\Phi^{-1}(\mu^{-1}_{\tilde{G}_i}(\hat{\mu}_i-\lambda^*))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2x_j^2}+\epsilon_i \\ \leq\ &\sum_{j=1}^{n}\{m_{ij}-L^{-1}(\hat{\mu}_i-\lambda^*)\alpha_{ij}\}x_j^* \\ &+\Phi^{-1}(\mu^{-1}_{\tilde{G}_i}(\hat{\mu}_i-\lambda^*))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2x_j^{*2}}, \\ &i=1,\cdots,k \end{aligned} \tag{29}$$

For the optimal solution of the above test problem, the following theorem holds.

Theorem 2.

For the optimal solution $\check{\boldsymbol{x}},\check{\epsilon}_i,i=1,\cdots,k$ of the test problem (28)-(29), if $w=0$ (equivalently, $\check{\epsilon}_i=0,i=1,\cdots,k$), $\boldsymbol{x}^*\in X, f_i^*\stackrel{\text{def}}{=}\mu^{-1}_{\tilde{F}_i}(\hat{\mu}_i-\lambda^*)\in\mathrm{R}^1,i=1,\cdots,k$ is a D-Pareto optimal solution.

(Proof)

From the active condition (26) at the optimal solution $(\boldsymbol{x}^*,\lambda^*)$ of MINMAX2($\hat{\boldsymbol{\mu}}$), it holds that

$$\begin{aligned} \hat{\mu}_i-\lambda^* &= \Pi_{\tilde{P}_i(\boldsymbol{x}^*,f_i^*)}(\tilde{G}_i), i=1,\cdots,k, \\ \hat{\mu}_i-\lambda^* &= \mu_{\tilde{F}_i}(f_i^*), i=1,\cdots,k. \end{aligned}$$

Assume that $\boldsymbol{x}^*\in X,\mu^{-1}_{\tilde{F}_i}(\hat{\mu}_i-\lambda^*),i=1,\cdots,k$ is not a D-Pareto optimal solution. Then, there exist $\boldsymbol{x}\in X, f_i\in\mathrm{R}^1, i=1,\cdots,k$ such that

$$\begin{aligned} \mu_{D_i}(\boldsymbol{x},f_i) &= \min\{\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i),\mu_{\tilde{F}_i}(f_i)\} \\ &\geq \mu_{D_i}(\boldsymbol{x}^*,f_i^*) \\ &= \hat{\mu}_i-\lambda^*, i=1,\cdots,k, \end{aligned}$$

with strict inequality holding for at least one i. This means that

$$\Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i) \;\geq\; \hat{\mu}_i-\lambda^*, i=1,\cdots,k, \tag{30}$$

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$$\mu_{\tilde{F}_i}(f_i) \quad \geq \quad \hat{\mu}_i - \lambda^*, i = 1, \cdots, k. \qquad (31)$$

From (12), (30) and (31), the following inequalities hold,

$$\sum_{j=1}^{n}\{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda^*)\alpha_{ij}\}x_j$$
$$+\Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda^*))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2 x_j^2} \leq \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*)$$
$$i = 1, \cdots, k \qquad (32)$$

with strict inequality holding for at least one i. This contradicts the fact that $\check{\epsilon}_i = 0, i = 1, \cdots, k$. □

Now, following the above discussions, we can present the interactive algorithm in order to derive a satisfactory solution from among a D-Pareto optimal solution set.

[Algorithm 2]

Step 1: The decision maker sets each of the membership functions $\mu_{\tilde{F}_i}(f_i), i = 1, \cdots, k$ of the fuzzy goal $\tilde{F}_i$ for permissible objective level f_i according to Assumption 2.

Step 2: Corresponding to the fuzzy goal $\tilde{G}_i$ for the probability that the objective function $\bar{\tilde{c}}_i \boldsymbol{x}$ is less than f_i, the decision maker sets each of the membership functions $\mu_{\tilde{G}_i}(p_i), i = 1, \cdots, k$ according to Assumption 1.

Step 3: Set the initial reference membership values as $\hat{\mu}_i = 1, i = 1, \cdots, k$.

Step 4: Solve MINMAX2($\hat{\boldsymbol{\mu}}$) by applying Algorithm 1, and obtain the optimal solution $(\boldsymbol{x}^*, \lambda^*)$. For the optimal solution $(\boldsymbol{x}^*, \lambda^*)$, The corresponding D-Pareto optimality test problem (28)-(29) is formulated and solved.

Step 5: If the decision maker is satisfied with the current values of the D-Pareto optimal solution $\mu_{D_i}(\boldsymbol{x}^*, f_i^*), i = 1, \cdots, k$ where $f_i^* = \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*)$, then stop. Otherwise, the decision maker updates his/her reference membership values $\hat{\mu}_i, i = 1, \cdots, k$, and return to Step 4.

IV. A Numerical Example

We consider the following two-objective random fuzzy linear programming problem to demonstrate the feasibility of the proposed method under the hypothetical decision maker.

[RFMOLP]

$$\min_{\boldsymbol{x} \geq \boldsymbol{0}} \bar{\tilde{\boldsymbol{c}}}_1 \boldsymbol{x} = \sum_{j=1}^{5} \bar{\tilde{c}}_{1j} x_j$$
$$\min_{\boldsymbol{x} \geq \boldsymbol{0}} \bar{\tilde{\boldsymbol{c}}}_2 \boldsymbol{x} = \sum_{j=1}^{5} \bar{\tilde{c}}_{2j} x_j$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 6x_4 + 4x_5 &\leq 240 \\ 3x_1 + 3x_2 + 2x_3 + 5x_4 + 3x_5 &\leq 230 \\ 4x_1 + 2x_2 + 3x_3 + 7x_4 + 6x_5 &\leq 250 \end{aligned}$$

where random fuzzy variables $\bar{\tilde{c}}_{ij}, i = 1, 2, j = 1, \cdots, 5$ are normally distributed with the fuzzy number $\tilde{M}_{ij}$ as mean and σ_{ij}^2 as variance, and a probability density function $f_{ij}(y)$ for a random fuzzy variable $\bar{\tilde{c}}_{ij}$ is formally represented with the following form.

$$f_{ij}(y) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} e^{-\frac{(y-\tilde{M}_{ij})^2}{2\sigma_{ij}^2}}, 1 \leq i \leq k, 1 \leq j \leq n \qquad (33)$$

TABLE I
THE PARAMETERS OF THE FUZZY NUMBER $\tilde{M}_{ij}$ AND THE VARIANCE σ_{ij}^2 OF THE RANDOM FUZZY VARIABLE $\bar{\tilde{c}}_{ij}$

$\tilde{M}_{ij}$	m_{ij}	$\alpha_{ij}(=\beta_{ij})$	σ_{ij}^2 (the variance of $\bar{\tilde{c}}_{ij}$)
$\tilde{M}_{11}$	5	0.5	1
$\tilde{M}_{12}$	7	0.4	2
$\tilde{M}_{13}$	6	0.3	2
$\tilde{M}_{14}$	10	0.2	3
$\tilde{M}_{15}$	8	0.3	1
$\tilde{M}_{21}$	-10	0.3	2
$\tilde{M}_{22}$	-5	0.4	3
$\tilde{M}_{23}$	-5	0.4	1
$\tilde{M}_{24}$	-9	0.2	2
$\tilde{M}_{25}$	-5	0.1	1

$\tilde{M}_{ij}$ is an L-R fuzzy number characterized by the following membership function.

$$\mu_{\tilde{M}_{ij}}(t) = \begin{cases} L\left(\frac{m_{ij}-t}{\alpha_{ij}},\right) & m_{ij} \geq t \\ R\left(\frac{t-m_{ij}}{\beta_{ij}},\right) & m_{ij} \leq t \end{cases} \qquad (34)$$

where $L(t) = R(t) = \max\{0, 1-t\}$, and m_{ij},$\alpha_{ij}(= \beta_{ij})$, σ_{ij}^2 are given in Table I.

In RFMOLP, let us assume that the hypothetical decision maker sets the membership functions $\mu_{\tilde{F}_i}(\cdot)$, $\mu_{\tilde{G}_i}(\cdot)$,$i = 1, 2$ as follows (Step 1, 2).

$$\begin{aligned} \mu_{\tilde{F}_1}(f_1) &= \frac{f_1 - 500}{100 - 500} \\ \mu_{\tilde{F}_2}(f_2) &= \frac{f_2 - (-400)}{(-30) - (-400)} \\ \mu_{\tilde{G}_1}(p_1) &= \frac{p_1 - 0.7}{0.85 - 0.7} \\ \mu_{\tilde{G}_2}(p_2) &= \frac{p_2 - 0.8}{0.9 - 0.8} \end{aligned}$$

Set the initial reference membership values as $(\hat{\mu}_1, \hat{\mu}_2) = (1, 1)$ (Step 3), and solve MINMAX2($\hat{\boldsymbol{\mu}}$) to obtain the corresponding D-Pareto optimal solution $(\boldsymbol{x}^*, \lambda^*)$ (Step 4).

$$(\mu_{D_1}(\boldsymbol{x}^*, f_1^*), \mu_{D_2}(\boldsymbol{x}^*, f_2^*)) = (0.7363, 0.7363)$$

where $f_i^* = \mu_{\tilde{F}_i}^{-1}(\hat{\mu}_i - \lambda^*)$, $i = 1, 2$. The hypothetical decision maker is not satisfied with the current value of the D-Pareto optimal solution $(\boldsymbol{x}^*, f_i^*)$, and, in order to improve $\mu_{D_1}(\cdot)$ at the expense of $\mu_{D_2}(\cdot)$, he/she updates his/her reference membership values as $(\hat{\mu}_1, \hat{\mu}_2) = (1, 0.7)$ (Step 5). Then, the corresponding D-Pareto optimal solution is obtained by solving MINMAX2($\hat{\boldsymbol{\mu}}$) (Step 4). The interactive processes under the hypothetical decision maker are summarized in Table II.

In order to compare our proposed approach with the previous ones, let us consider the following multiobjective programming problem based on a probability maximization model for RFMOLP.

[MOP2'($\boldsymbol{f}$)]

$$\max_{\boldsymbol{x} \in X}(\Pi_{\tilde{P}_1(\boldsymbol{x}, f_1)}(\tilde{G}_1), \Pi_{\tilde{P}_2((\boldsymbol{x}, f_2))}(\tilde{G}_2))$$

where f_1 and f_2 are permissible objective levels specified by the decision maker in his/her subjective manner. Once the reference membership values $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \hat{\mu}_2)$ are specified

TABLE II
INTERACTIVE PROCESSES

Iteration	1	2	3
$\hat{\mu}_1$	1	1	0.86
$\hat{\mu}_2$	1	0.7	0.7
f_1^*	205.4521	159.2246	180.7777
f_2^*	-302.457	-234.217	-266.081
p_1^*	0.8104	0.8277	0.8197
p_2^*	0.8736	0.8551	0.8638
$\mu_{D_1}(\boldsymbol{x}^*, f_1^*)$	0.7363	0.8519	0.7980
$\mu_{D_2}(\boldsymbol{x}^*, f_2^*)$	0.7363	0.5519	0.6380

TABLE III
COMPARISON BETWEEN THE PROPOSED METHOD AND THE PROBABILITY MAXIMIZATION METHOD

	proposed method	probability max.
f_1^*	205.4521	200
f_2^*	-302.457	-310
p_1^*	0.8104	0.7831
p_2^*	0.8736	0.8544
$\mu_{f_1}(f_1^*)$	0.7363	0.75
$\mu_{f_2}(f_2^*)$	0.7363	0.7567
$\mu_{\tilde{G}_1}(p_1^*)$	0.7363	0.5546
$\mu_{\tilde{G}_2}(p_2^*)$	0.7363	0.5546

by the decision maker, the corresponding Pareto optimal solution is obtained by the following minmax problem.

[MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{f}$)]

$$\min_{\boldsymbol{x}\in X,\lambda\in\Lambda} \lambda \tag{35}$$

subject to

$$\hat{\mu}_i - \Pi_{\tilde{P}_i(\boldsymbol{x},f_i)}(\tilde{G}_i) \le \lambda, i = 1,2$$

From (12), MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{f}$) can be equivalently transformed to the following form.

[MINMAX4($\hat{\boldsymbol{\mu}}, \boldsymbol{f}$)]

$$\min_{\boldsymbol{x}\in X,\lambda\in\Lambda} \lambda$$

subject to

$$\sum_{j=1}^{n}\{m_{ij} - L^{-1}(\hat{\mu}_i - \lambda)\alpha_{ij}\}x_j$$

$$+\Phi^{-1}(\mu_{\tilde{G}_i}^{-1}(\hat{\mu}_i - \lambda))\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2 x_j^2} \le f_i, i = 1,2 \tag{36}$$

We can easily solve MINMAX4($\hat{\boldsymbol{\mu}}, \boldsymbol{f}$) by applying Algorithm 1, because the constraint set (36) are convex for any fixed $\lambda \in \Lambda$. Let us assume that the decision maker sets his/her reference membership values as $(\hat{\mu}_1, \hat{\mu}_2) = (1, 1)$ and permissible objective levels as $(f_1, f_2) = (200, -310)$. Then, the corresponding Pareto optimal solution can be obtained as shown in Table III, where the left side shows the D-Pareto optimal solution of the proposed method with reference membership values $(\hat{\mu}_1, \hat{\mu}_2) = (1, 1)$ (see the first iteration in Table II). In Table III, it is clear that, in the proposed method, a proper balance between permissible probability levels and the corresponding objective functions in a probability maximization model is attained in membership space. On the other hand, In a probability maximization model based method, although permissible objective levels are improved in comparison with the proposed method, the corresponding probability function values was changed for the worse.

V. CONCLUSION

In this paper, we have proposed an interactive decision making method for RFMOLP based on a probability maximization model to obtain a satisfactory solution from among a Pareto optimal solution set. In the proposed method, the decision maker is required to specify the membership functions for the fuzzy goals of not only the permissible objective levels in a probability maximization model but also the corresponding distribution function. Such two kinds of membership functions are integrated and, in the integrated membership space, a D-Pareto optimal solution concept is introduced. The satisfactory solution can be obtained by updating the reference membership values and solving the corresponding minmax problem by applying the convex programming technique. At any D-Pareto optimal solution, it is guaranteed that a proper balance between permissible objective levels and the corresponding distribution function values in a probability maximization model is attained.

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