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Fast Partial Subset Convolution for Steiner Tree Problem

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***Abstract*—We present a new algorithm that solves the minimum Steiner tree problem on an undirected graph with n nodes, k terminals, and m edges with bounded integer weights in time $\tilde{O}((2-\varepsilon)^k n^2 + nm)$. Our algorithm beats the aesthetically appealing and seemingly inherent 2^k bound of the current best algorithm for Steiner tree problem from Björklund, Husfeldt, Kaski, and Koivisto. Our algorithm is based on fast subset convolution and a variant of the classical tree-separator theorem.**

***Index Terms*—Steiner tree; Fast subset convolution; Mobius transform; Tree separator theorem;**

I. Introduction

A. Steiner Tree Problem

IN the standard Steiner tree problem, given an n nodes undirected graph $G(V,E)$, a subset $R \subset V$ of $k = |R|$ *terminals* and a length function $c : E \to \mathbb{R}$ on the edges of G, then the Steiner tree problem asks for a shortest network connecting the vertices of R. The nodes in subset R are called terminal vertices, and the nodes in the subset V/R are called Steiner vertices. The Steiner tree problem appears in many different kinds of applications.

This problem is well known to be NP-hard[2] and therefore we cannot expect to find polynomial time algorithms for solving it exactly. This motives the search for good approximation algorithms for the Steiner tree problem in graphs(i.e., algorithms that have polynomial running time and return solutions that are not far from an optimum solution), or faster exact algorithms that can give a lower time complexity. The best approximation algorithm for Steiner tree problem comes from Byrka et al.[3], where they give a 1.39-approximation ratio based on iterated randomized rounding.

For more than 30 years the fastest parameterized algorithm for the Steiner tree problem was the classical $O^*(3^k)$ dynamic programming algorithm(The O^* notation omits polynomial factors, and k denotes the number of terminals in R.) by Dreyfus and Wagner[4]. Dreyfus-Wagner's algorithm is still probably the most popular algorithm used for solving different variants of the Steiner tree problem in practice[5][6]. This algorithm and its variations are also used as a subroutine in many other algorithms. Recent progress in parameterized complexity and exact algorithms led to new insights on the Steiner tree problem. Mölle, Richter, and Rossmanith[7] improved the running time to $O^*((2+\epsilon)^k)$, for any constant $\epsilon > 0$. More recently, Björklund, Husfeldt, Kaski, and Koivisto[1] obtained an $O^*(2^k)$ time algorithm for the version of the problem where edges have bounded integer weights.

B. Subset Convolution

Many hard computational problems admit a recursive solution via a convolution-like recursion step over the subsets of an n-element ground set N. More precisely, for every $S \subset N$, one computes the "solution" h(S) defined by

$$h(S) = \sum_{X \subset S} f(X) g(S \backslash X) \tag{1}$$

where $f(X)$ and $g(S\backslash X)$ are previously computed solutions for the subproblems specified by X and $S\backslash X$, and the arithmetic is carried out in an appropriate semiring; the most common examples in applications being perhaps the integer sum-product ring and the integer max-sum semiring. Given f and g, a direct evaluation of h for all $S \subset N$ requires $O(3^n)$ semiring operations. For a long time, this is the fastest known evaluation. In 2007, Björklund, Husfeldt et al.[1] introduced a fast subset convolution algorithm that improved substantially upon the straightforward $O(3^n)$ algorithm. Their algorithm achieved a time complexity of $O(n^2 2^n)$ via a product ("convolution over rank") of "ranked" extensions of the classical Möbius transforms of f and g on the subset lattice, followed by a "ranked" Möbius inversion.

The key of Björklund's algorithm[1] is the use of Yates's fast Fourier transform[9] in combination with Möbius inversion, which will be introduced in the following paragraph. From the way it is normally stated, Yates's algorithm[9] seems to face an inherent 2^n lower bound, up to a polynomial factor, and it also seems to be oblivious to the structural properties of the transform it computes.

The motivation of the present investigation is to expedite the running time of Yates's algorithm[9] for certain structures so as to get running times with a dominating factor of the form $(2-\varepsilon)^n$. From the perspective of running times alone, our improvements are modest at best, but apart from providing evidence that the aesthetically appealing 2^n bound from[1] can be beaten, the combinatorial framework we present seems to be new and may present a fruitful direction for exact exponential time algorithms.

II. Fast Subset Convolution Over A Ring

The key factor of Björklund, Husfeldt et al.[1]'s algorithm is the use of a product ("convolution over rank") of "ranked" extensions of the classical Möbius transforms of f and g on the subset lattice, followed by a "ranked" Möbius inversion.

The *ranked Möbius transform* of f is the function $\hat{f}_r$ that associates with every $k = 0, 1, \ldots, n$ and $S \subset N$ the ring

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element

$$\hat{f}_r(k,S) = \sum_{\substack{X\subset S\\ |X|=k}} f(X) \quad (2)$$

While the *classical Möbius transform* of f is the function $\hat{f}$ that associates with every $S \subset N$ the ring element

$$\hat{f}(S) = \sum_{X\subset S} f(X) \quad (3)$$

In particular, the classical Möbius transform of f is obtained in terms of the ranked transform by taking the sum over k, that is, $\hat{f}(S) = \sum_{k=0}^{|S|} \hat{f}_r(k,S)$.

Given the Möbius transform $\hat{f}$, the original function f may be recovered via the *Möbius inversion* formula

$$f(S) = \sum_{X\subset S} (-1)^{|S\backslash X|}\hat{f}(X) \quad (4)$$

For the ranked transform, inversion is achieved simply by $f(S) = \hat{f}_r(|S|,S)$, or in a somewhat more redundant form,

$$f(S) = \sum_{X\subset S} (-1)^{|S\backslash X|}\hat{f}_r(|S|,X) \quad (5)$$

Throughout this section we assume that R is an arbitrary (possibly noncommutative) ring and that N is a set of n elements, $n \geq 0$. Let f (respectively, g) be a function that associates with every subset $S \subset N$ an element $f(S)$ (respectively, $g(S)$) of the ring R. Define the *convolution* $f * g$ for all $S \subset N$ by

$$(f*g)(S) = \sum_{X\subset S} f(X)g(S\backslash X) \quad (6)$$

For two ranked Möbius transforms, $\hat{f}_r$ and $\hat{g}_r$, define the convolution $\hat{f}_r \circledast \hat{g}_r$ for all $k = 0,1,\ldots,n$ and $S \subset N$ by

$$(\hat{f}_r \circledast \hat{g}_r)(k,S) = \sum_{j=0}^{k} \hat{f}_r(j,S)\hat{g}_r(k-j,S) \quad (7)$$

The *fast Möbius transform*[8][9] is the following algorithm for computing the Möbius transform (3) in $O(n2^n)$ ring operations. By relabeling if necessary, we may assume that $N = 1,2,\ldots,n$. To compute $\hat{f}$ given f, let initially

$$\hat{f}_0(S) = f(S) \quad (8)$$

for all $S \subset N$, and then iterate for all $j = 1,2,\ldots,n$ and $S \subset N$ as follows:

$$\hat{f}_j(S) = \begin{cases} \hat{f}_{j-1}(S) & \text{if } j \notin S, \\ \hat{f}_{j-1}(S\backslash\{j\}) + \hat{f}_{j-1}(S) & \text{if } j \in S. \end{cases} \quad (9)$$

It is easy to verify by induction on j that this recurrence gives $\hat{f}_n(S) = \hat{f}(S)$ for all $S \subset N$ in $O(n2^n)$ ring operations. The inversion operation (4) can be implemented in a similar fashion.

Expression (5) provides the key to fast evaluation of the subset convolution (6). Namely, we will "invert" a function that, in general, cannot be represented via ranked Möobius transform but via a convolution (over rank) of two such transforms. To set the stage, it is immediate that the ranked transform (2) can be computed in $O(n^2 2^n)$ ring operations by carrying out the fast transform (9) independently for each $k = 0,1,\ldots,n$. Similarly, the ranked inversion (5) can be computed in $O(n^2 2^n)$ ring operations by carrying out the fast inversion independently for each $k = 0,1,\ldots,n$. Note that this convolution operation is over the rank parameter rather than over the subset parameter. In Björklund, Husfeldt, Kaski, and Koivisto[1] paper, their main result is

Theorem 1: The subset convolution over an arbitrary ring can be evaluated in $O(n^2 2^n)$ ring operations.

Another important result in their paper that relates to our algorithm is

Theorem 2: The subset convolution over the integer max-sum (min-sum) semiring can be computed in $\tilde{O}(2^n M)$ time, provided that the range of the input functions is $\{-M, -M+1, \ldots, M\}$.

Readers who are interested in the proofs of the above two theorems are referred to the article of Björklund er al.[1].

III. Algorithms

A. Dreyfus-Wagner Algorithm

Dreyfus and Wagner[4]'s algorithm is based on the following observation. For a given instance of the Steiner tree problem G(V, A), with a minimum Steiner tree T on the terminal set R, and $k = |R| \geq 3$, there must be an internal node $p \in T$, terminal vertex or steiner vertex, that can separate T into two forests $\mathcal{R}_1$ and $\mathcal{R}_2$, each containing at least one terminal vertex. Let R_i be the terminal vertices in $\mathcal{R}_i$, $i \in 1,2$. If we compute minimum Steiner trees on the terminal vertices subset $R_i \cup p$, $i \in 1,2$, and merge them, we obtain a minimum Steiner tree for R. We need to enumerate all the possible separations since we do not know p nor (R_1, R_2) a prior. Let $MST(R)$ be the minimum Steiner tree on terminal set R in graph $G(V,A)$, and $mst(R)$ be the responding cost. The following equation holds for Dreyfus and Wagner[4]'s algorithm.

$$mst(R) = \min_{p\in V} \min_{(R_1,R_2)\in\mathcal{P}(p,R)} \{mst(R_1\cup\{p\}) + mst(R_2\cup\{p\})\} \quad (10)$$

where $\mathcal{P}(p,R)$ is the set of possible partitions R_1, R_2 of $R\backslash\{p\}$ in two nonempty subsets. Dreyfus and Wagner[4]'s algorithm simply applies (10) to any subset of R in a bottom-up fashion, storing each partial solution computed for the later computations.

B. Björklund-Husfeldt Algorithm

For a given vertex subset $Y \subset R$, denote by $W(Y)$ the total weight of a Steiner tree connecting Y in G. Equation(10) can be deposed into two parts.

$$W(\{q\} \cup X) = min\{W(\{p,q\}) + g_p(X) : p \in V\} \quad (11)$$

$$g_p(X) = min\{W(\{p\}\cup D) + W(\{p\}\cup(X\backslash D)) : 0 \subset D \subset X\} \quad (12)$$

for all $q \in Y$ and $X = Y\backslash\{q\}$.

In [1], Björklund et al. apply the fast subset convolution over the min-sum semiring to expedite the evaluation of the Dreyfus-Wagner recursion in (12). They define the function f_p for all $X \subset R$ by

$$f_p(X) = W(\{p\} \cup X) \quad (13)$$

Applying the subset convolution over the min-sum semiring, it is immediate from (12) and (13) that $g_p(X) = (f_p * f_p)(X)$ holds for all $X \subset R$. Thus by Theorem 2, they can computer $g_p(X)$ for all $p \in V$ and $X \subset R$ using n evaluations of the subset convolution with integers bounded by nM, which leads to $\hat{O}(2^k n^2)$ total time.

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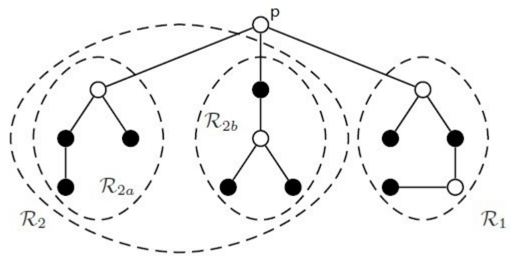

Fig. 1. Tight example for Lemma 1 (black nodes are terminals): a Steiner separator p, and the corresponding forests $\mathcal{R}_1$ and $\mathcal{R}_2$ with $|R_1| = k/3$ and $|R_2| = 2k/3$ terminals, respectively. Note that in $\mathcal{R}_2 \cup \{s\}$, node s separates two perfectly balanced forests $\mathcal{R}_{2a}$ and $\mathcal{R}_{2b}$.

C. Our Improvement

In this paper, we exploite a variant of the classical tree-separator theorem to reduce the time complexity of Björklund et al.[1] algorithm. It is well known that any n-node tree contains a node s (separator) whose removal divides the tree in two forests, each one containing at most $2n/3$ nodes. The same basic result holds if we put weights on the nodes [10]. In particular, the following lemma holds (see Fig. 1[11] for a tight example).

Lemma 1: [10]Consider any Steiner tree T on the set of terminals R, $|R| = k \geq 3$. Then there exists an internal node $p \in T$ (Steiner-separator), not necessarily a terminal, whose removal divides the tree in two forests, each one containing at most $2k/3$ terminals.

As a consequence of Lemma 1, when applying Equation (10), we do not really need to consider all the partitions in $P(p, R)$. It is sufficient to consider only the subset $B(p, R) \subset P(p, R)$ of ("almost balanced") partitions (R_1, R_2) where $|R_1| \leq |R_2| \leq 2k/3$:

$$mst(R) = \min_{p \in V} \min_{(R_1, R_2) \in \mathcal{B}(p,R)} \{mst(R_1 \cup \{p\}) + mst(R_2 \cup \{p\})\} \tag{14}$$

Equation (14) shows that when we are counting the subset of R, we can save some cases. It is clear that in the case of Equation (10), the total number of subsets of R that needs to be considered is

$$\sum_{i=0}^{k} \binom{k}{i} = \binom{k}{0} + \cdots + \binom{k}{i} + \cdots + \binom{k}{k} = 2^k \tag{15}$$

But as Equation (14) shows, the total number of subsets of R that needs to be considered in this problem is

$$\sum_{i=0}^{2/3k} \binom{k}{i} = \binom{k}{0} + \cdots + \binom{k}{i} + \cdots + \binom{k}{2/3k} \leq 2^k \tag{16}$$

In R.L. Graham, D.E. Knuth, O. Patashnik's book *Concrete Mathematics*[12], they give a asymptotic formula to calculate partial sum of binomial coefficients.

$$\sum_{i=0}^{\lambda k} \binom{k}{i} = 2^{kH(\lambda) - lg(k)/2 + O(1)} \tag{17}$$

where $0 < \lambda < 1/2$, $H(\lambda) = \lambda lg(1/\lambda) + (1-\lambda) lg(1/(1-\lambda))$ is the *binary entropy* of λ and lg is the binary logarithm. Here we have $\lambda = \frac{1}{3}$, which means

$$\sum_{i=0}^{k/3} \binom{k}{i} = 2^{kH(1/3) - lg(k)/2 + O(1)} \tag{18}$$

where $H(1/3) = 0.9183$. If we put Equation (18) into Equation (16), we get

$$\sum_{i=0}^{2/3k} \binom{k}{i} = 2^k - 2^{kH(1/3) - lg(k)/2 + O(1)} = (2 - \varepsilon)^k \tag{19}$$

where ε is a number that related to k. The second e-quality is because that function $f(x) = x^k$ with $x > 1$ and $k > 1$ is a continuous monotonic function and that $\frac{2^{kH(1/3) - lg(k)/2 + O(1)}}{2^k} \to 0$ as $k \to \infty$. The following table gives a insight into the relationship between k and $2 - \varepsilon$.

TABLE I
NUMERICAL RELATIONSHIP BETWEEN k AND $2 - \varepsilon$

k	5	10	15	20	25
ε	0.0814	0.0374	0.0081	0.0059	0.0044
$2 - \varepsilon$	1.9186	1.9626	1.9919	1.9941	1.9956
$\sum_{i=0}^{2/3k} \binom{k}{i} / \sum_{i=0}^{k} \binom{k}{i}$	0.8125	0.8281	0.9408	0.9423	0.9461

In Parallel with Equation (12), we can decompose Equation (14) into two parts and define

$$g'_p(X) = min\{W(\{p\} \cup D) + W(\{p\} \cup (X \backslash D)) : \\ 0 \subset D \subset X, 1/3|X| \leq |D| \leq 2/3|X|\} \tag{20}$$

and at the same time, we can define *partial subset convolution* for all $S \subset N$ by

$$\begin{aligned}(f \odot g)(S) &= \sum_{\substack{X \subset S \\ 1/3|S| \leq |X| \leq 2/3|S|}} f(X) g(S \backslash X) \\ &= \sum_{\substack{X \subset S \\ 1/3|S| \leq |X| \leq 2/3|S|}} (-1)^{S \backslash X} (\hat{f} \circledast \hat{g})(|S|, X)\end{aligned}$$

When we want to calculate $(\hat{f} \circledast \hat{g})(|S|, S)$ and further to calculate $(f \odot g)(S)$, we first need to calculate all the necessary *ranked Möbius transform* Equation (2) and stored them as what the proof of Theorem 1 indicates. Björklund-Husfeldt's algorithm need to calculate *ranked Möbius transform* Equation (2) for all $S \subset N$ and $i = 0, 1, \ldots, n$. Lemma 1 and Equation (14) gives us two chance to reduce the number of *ranked Möbius transform* Equation (2) that need to be calculated and stored.

On one hand, we only need to calculate the subset S of N with $|S| \leq 2/3|N|$. On the other hand, we only need to calculate Equation (2) for $0 < i < 2/3n$. As we have discussed above, we can compute all the necessary Equation (2) in $O(2/3n * n * (2 - \varepsilon)^n) = O(n^2 (2 - \varepsilon)^n)$.
Now we can apply the fast partial subset convolution over the min-sum semiring to expedite the evaluation of the recursion in (20). However, we cannot simply replace (20) by fast subset convolution as each $g'_p(X)$ is defined in terms of other values $g'_r(Z)$, for $Z \subset X$ and $r \in V$, which need to be precomputed. To this end, we carry out the computations in a level-wise manner.

Algorithm based on Partial subset convolution

A_1 : *Initialization* Calculate all-pairs shortest paths and

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stored as $W(p,q)$ for $\forall p \in V, \forall q \in V$

A_2 : For each level $i = 2, 3 \ldots, 2/3k$

A_{21} : for $\forall p \in V, \forall q \in V, X \subset R$ with $|X| = i$ define f_p for all $X \subset R$ by

$$f_p(X) = \begin{cases} w(\{p\} \cup X) & \text{if } 1 \leq |X| \leq i-1, \\ \infty & \text{otherwise.} \end{cases} \quad (21)$$

A_{22} : compute $g'_p(X) = (f_p \odot f_p)(X)$ using *partial subset convolution* and stored;

A_{23} : compute $W(\{q\} \cup X) = min\{W(\{p,q\}) + g'_p(X) : p \in V\}$ and stored;

A_3 : Construct a optimal Steiner tree by tracing backwards a path of the above optimal calculated choices.

Theorem 3: The steiner tree problem with edge weights in $\{1, 2, \ldots, M\}$ can be solved in $\tilde{O}((2-\varepsilon)^k n^2 M + nm)$.

Proof: A_1 takes $\tilde{O}(n^2 + nm)$ to compute all-pairs shortest paths using Johnson's algorithm[13]. A_{22} needs to compute $g'_p(X)$ for all $p \in V$ and $X \subset R$ using n evaluations of the partial subset convolution with integers bounded by nM, which leads to $\tilde{O}((2-\varepsilon)^k n^2 M)$ total time. A_{23} needs $\tilde{O}(\binom{k}{i})n^2$ to compute $W(\{q\} \cup X)$ for all $X \subset R$ and $q \in V$ with $|X| = i$, and $\tilde{O}((2-\varepsilon)^k n^2)$ for all $i = 2, \ldots, 2/3k$. A_3 can be solved in the same time bound.

IV. Conclusion

This paper presents a splitting technique based on a variant of the classical tree-separator theorem to speed up the fast subset convolution to minimum Steiner tree computation. It's interesting to note that if k is small relative to n (e.g., $k = \omega(log(n))$, which is a realistic situation), the factor n^2 seems to be more time-consuming than 2^k, so we think that we may pay more attention to reduce the factor corresponding to n instead of k. The ε in our algorithm is related to k, which makes the reduction in time-complexity seems not substantial as $k \to \infty$. But if we consider the (realistic) assumption that $k << n$, this reduction is in fact quite attracting. However, we still hope that we can get an fixed ε in our future work.

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