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Convergence Theorem for Class T Mappings in Hilbert Spaces

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***Abstract*—In this study, we would like to propose the iterative algorithm for solving the equilibrium problem and fixed point problem for the class $\mathfrak{T}$ mappings which contain various kind of operators that are found in many problems in applied mathematics and other fields. We prove that the sequence x_n which is generated by the proposed iterative algorithm converges strongly to a common element of these two sets. Furthermore, we give a numerical example which supports our main theorem in the last part. Our result extended and improve the existing result of Qiao-Li Dong and Songnian He and many others.**

***Index Terms*—Class $\mathfrak{T}$- mapping, Equilibrium problem, Fixed Point Problems.**

I. Introduction

THROUGHOUT this paper, we focus on the framework of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. First, we recall the basic concept of mappings as shown in the following:

- $T : H \to H$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in H$. We denote the set of fixed point of T by $F(T)$.
- $T : H \to H$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and $\|Tx-p\| \leq \|x-p\|$ for all $x \in H$ and $p \in F(T)$.
- $T : H \to H$ is said to be the class $\mathfrak{T}$ if $T \in \mathfrak{T} = \{T : H \to H | dom(T) = H \text{ and } F(T) \subset H(x, Tx)\}$ for all $x \in H$ and $H(x, y) := \{z \in H : \langle z-y, x-y\rangle \leq 0\}$ for all $x, y \in H$. We called $H(x, y)$ a half-space generated by (x, y).
- $f : H \to H$ is said to be an α-contraction if there exists a constant $\alpha \in [0, 1)$ which satisfy the following statement:

$$\|f(x) - f(y)\| \leq \alpha\|x-y\|, \ \forall x, y \in H.$$

Since 1967, Halpern introduced an explicit iterative scheme as shown in the following:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \ \forall n \geq 0,$$

where $\{\alpha_n\} \subset [0, 1]$. He proposed the convergence theorem which states that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2004, Xu studied the iteration process $\{x_n\}$ called viscosity approximation method as shown in the following:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \ \text{ for } n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $f : C \to C$ is a contraction. He also proved the strong convergence theorem of the sequence $\{x_n\}$ which generated by the above scheme under the appropriate conditions.

In 2010, Maingè, proposed the viscosity approximation scheme for quasi-nonexpansive mappings as shown in the following:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_w x_n,$$

where $\{\alpha_n\} \subset (0, 1)$ and T_w was generated by $T_w = (1-w)I + wT, \ w \in (0, 1)$. He also proved the convergence theorem under the suitable conditions.

Bauschke and Combettes proposed the research about the properties of a class $\mathfrak{T}$ mappings. Their method was shown in the following: for $x_0 \in H$

$$x_{n+1} = P_{H(x_0,x_n)\cap H(x_n,T_n x_n)} x_0.$$

Recently, Dong and He studied the shrinking projection method for the class of $\mathfrak{T}$ mappings and proved a strong convergence theorem.

Motivated and inspired by the previous mentioned researches, we combined all the ideas and then proposed the iterative scheme for finding the common solution of fixed point problems for class $\mathfrak{T}$ mappings and equilibrium problem as shown in the following:

$$\begin{cases} x_0 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n P_{H(x_n,S_n u_n)} f(x_n) + (1-\alpha_n)S_n u_n, \end{cases} \tag{1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy the appropriate conditions.

II. Preliminaries

Definition II.1. A sequence of mappings $\{T_n\}$ having a common fixed point is said to satisfy the condition (Z) if every bounded sequence $\{x_n\}$ with $\|x_n - T_n x_n\| \to 0$ satisfies $\omega_w(x_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

Definition II.2. A mapping T is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H, \ x_n \rightharpoonup x$ and $Tx_n \to y$.

Lemma II.3. *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

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(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma II.4. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

Lemma II.5. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in H$, defined a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\} \quad (2)$$

for all $z \in H$. Then, the following conclusions hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*
$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Lemma II.6. *Let $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, 1)$ be sequences of real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n, \text{ for all } n \in \mathbb{N},$$
$$\sum_{n=1}^{\infty} c_n = \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma II.7. *Let $C = \{z \in H : \langle x - u, z - u\rangle \leq 0\}$. Assume $x \neq u$ and $x_0 \notin C$. Then*

$$P_C x_0 = x_0 - \frac{\langle x - u, x_0 - u\rangle}{\|x - u\|^2}(x - u).$$

III. Main result

Lemma III.1. *Assume a sequence of mappings $S_n \in \mathfrak{T} : H \to H$ satisfies condition (Z). If x^* is a solution of $\mathcal{F}$ where $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_n) \cap EP(F) \neq \emptyset$ and $\{x_n\}$ is a bounded sequence such that $\|S_n x_n - x_n\| \to 0$, then*

$$\lim_{n\to\infty} \sup \langle P_{H(x_n, S_n u_n)} f(x^*) - x^*, x_n - x^*\rangle \leq 0. \quad (3)$$

Proof: *Since $\{x_n\}$ is a bounded sequence and $\{S_n\}$ satisfies condition (Z) which means that every bounded sequence $\{x_n\}$ satisfies $\omega_w(x_n) \subset \mathcal{F}$. Hence, there exists $\bar{x}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$ and such that*

$$\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{n\to\infty} \langle f(x^*) - x^*, x_{n_k} - x^*\rangle.$$

Since $\forall v \in \bigcap_{n=1}^{\infty} F(S_n)$, then $\langle f(x^*) - x^*, v - x^*\rangle \leq 0$ obviously leads to

$$\begin{aligned}\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{n\to\infty} \langle f(x^*) - x^*, x_{n_k} - x^*\rangle \\ &= \lim_{n\to\infty} \langle f(x^*) - x^*, \bar{x} - x^*\rangle \\ &= \langle f(x^*) - x^*, \bar{x} - x^*\rangle \leq 0.\end{aligned}$$

Next, we consider in two cases, first if $f(x^*) \in H(x_n, S_n u_n)$, then $P_{H(x_n,S_n u_n)} f(x^*) = f(x^*)$ and it is obvious that $\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \leq 0$ can imply $\limsup_{n\to\infty} \langle P_{H(x_n,S_n u_n)} f(x^*) - x^*, x_n - x^*\rangle \leq 0$. The second case is $f(x^*) \notin H(x_n, S_n u_n)$. Then by the definition of $H(x_n, S_n u_n)$, we have

$$\langle x_n - S_n u_n, f(x^*) - S_n u_n\rangle > 0. \quad (4)$$

From $x^* \in \mathcal{F} \subset H(x_n, S_n u_n)$, we get

$$\begin{aligned}\langle x_n - S_n u_n, x_n - x^*\rangle &= \|x_n - S_n u_n\|^2 \\ &\quad + \langle x_n - S_n u_n, S_n u_n - x^*\rangle \\ &> 0. \quad (5)\end{aligned}$$

From Lemma (II.7), it follows

$$\begin{aligned}P_{H(x_n,S_n u_n)} f(x^*) &= f(x^*) \\ &\quad - \frac{\langle x_n - S_n u_n, f(x^*) - S_n u_n\rangle}{\|x_n - S_n u_n\|^2} \\ &\quad \times (x_n - S_n u_n). \quad (6)\end{aligned}$$

Combining (4), (5) and (6) together, we obtain

$$\begin{aligned}&\langle P_{H(x_n,S_n u_n)} f(x^*) - x^*, x_n - x^*\rangle \\ = &\ \langle f(x^*) - x^*, x_n - x^*\rangle \\ &- \frac{\langle x_n - S_n u_n, f(x^*) - S_n u_n\rangle}{\|x_n - S_n u_n\|^2}\langle x_n - S_n u_n, x_n - x^*\rangle \\ < &\ \langle f(x^*) - x^*, x_n - x^*\rangle. \quad (7)\end{aligned}$$

We can conclude that

$$\lim_{n\to\infty} \sup \langle P_{H(x_n,S_n u_n)} f(x^*) - x^*, x_n - x^*\rangle \leq 0.$$

■

Theorem III.2. *Let H be a real Hilbert space and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $S_n : H \to H$ be a mapping in class $\mathfrak{T}$ and satisfies the condition (Z) such that $\mathfrak{F} := \bigcap_{n=1}^{\infty} F(S_n) \cap EP(F) \neq \emptyset$. Let f be an α-contraction mapping of H into itself and $\{x_n\}$ be a sequence generated by the following:*

$$\begin{cases} x_0 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n P_{H(x_n,S_n u_n)} f(x_n) + (1 - \alpha_n) S_n u_n, \end{cases} \quad (8)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

1. *$\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*
2. *$\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,*
3. *$\liminf_{n\to\infty} r_n > 0$.*

Then, $\{x_n\}$ converges strongly to $z \in \mathfrak{F} := \bigcap_{n=1}^{\infty} F(S_n) \cap EP(F)$, where $z = P_{\mathfrak{F}} f(z)$.

Proof: **Step 1.** From $S_n \in \mathfrak{T}$ and $F(S_n) \subset H(x_n, S_n u_n)$ for all $x \in H$. Therefore, we have $P_{H(x_n,S_n u_n)} p = p$ for all $p \in \mathfrak{F}$.
Since $u_n = T_{r_n} x_n$, we consider

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|$$

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and

$$\begin{aligned}
& \|x_{n+1}-p\| \\
= \ & \|\alpha_n P_{H(x_n,S_nu_n)}f(x_n) \\
& +(1-\alpha_n)S_nu_n-p\| \\
\leq \ & \alpha_n\|P_{H(x_n,S_nu_n)}f(x_n)-P_{H(x_n,S_nu_n)}p\| \\
& +(1-\alpha_n)\|S_nu_n-p\| \\
\leq \ & \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\| \\
& +(1-\alpha_n)\|S_nu_n-p\| \\
\leq \ & \alpha\alpha_n\|x_n-p\|+\alpha_n\|f(p)-p\| \\
& +(1-\alpha_n)\|x_n-p\| \\
= \ & \alpha_(1-\alpha)\frac{\|f(p)-p\|}{1-\alpha} \\
& +[1-\alpha_n(1-\alpha)]\|x_n-p\|.
\end{aligned}$$

By the induction, we have

$$\|x_n-p\| \leq \max\{\frac{\|f(p)-p\|}{1-\alpha}, \|x_0-p\|\}.$$

Hence, $\{x_n\}$ is bounded and $\{P_{H(x_n,S_nu_n)}f(x_n)\}$ is also bounded.

Step 2. We will show that $lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Consider,

$$\begin{aligned}
& \|x_{n+1}-x_n\| && (9) \\
= \ & \|\alpha_nP_{H(x_n,S_nu_n)}f(x_n)+(1-\alpha_n)S_nu_n && (10) \\
& -\alpha_{n-1}P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1}) \\
& -(1-\alpha_{n-1})S_{n-1}u_{n-1}\| \\
\leq \ & \alpha_n\|P_{H(x_n,S_nu_n)}f(x_n) \\
& -P_{H(x_n,S_nu_n)}f(x_{n-1})\| \\
& +\alpha_n\|P_{H(x_n,S_nu_n)}f(x_{n-1}) \\
& -P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +|\alpha_n-\alpha_{n-1}|\|P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +(1-\alpha_n)S_nu_n-(1-\alpha_{n-1})S_{n-1}u_{n-1} \\
\leq \ & \alpha_n\|P_{H(x_n,S_nu_n)}f(x_n)-P_{H(x_n,S_nu_n)}f(x_{n-1})\| \\
& +\alpha_n\|P_{H(x_n,S_nu_n)}f(x_{n-1}) \\
& -P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +|\alpha_n-\alpha_{n-1}|\|P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +(1-\alpha-n)\|u_n-u_{n-1}\| \\
& +(1-\alpha_n)\|S_nu_{n-1}-S_{n-1}u_{n-1}\| \\
& +|\alpha_{n-1}-\alpha_n|\|S_{n-1}u_{n-1}\|. && (11)
\end{aligned}$$

From $u_n=T_{r_n}x_n$ and $u_{n-1}=T_{r_n}x_{n-1}$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0 \quad (12)$$

and

$$F(u_{n-1},y)+\frac{1}{r_{n-1}}\langle y-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0. \quad (13)$$

Let $y=u_{n-1}$ in (12) and $y=u_n$ in (13), we have

$$F(u_n,u_{n-1})+\frac{1}{r_n}\langle u_{n-1}-u_n,u_n-x_n\rangle \geq 0, \text{ and}$$
$$F(u_{n-1},u_n)+\frac{1}{r_{n-1}}\langle u_n-u_{n-1},u_{n-1}-x_{n-1}\rangle \geq 0.$$

Since F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$, we have

$$\langle u_n-u_{n-1},\frac{u_{n-1}-x_{n-1}}{r_{n-1}}-\frac{u_n-x_n}{r_n}\rangle \geq 0.$$

Hence,

$$\langle u_n-u_{n-1},u_{n-1}-u_n+u_n-x_n-\frac{r_{n-1}}{r_n}(u_n-x_n)\rangle \geq 0.$$

Without loss of generality, we can assume that there exists a real number b such that $r_n>b>0$ for all $n\in\mathbb{N}$. Then we have

$$\begin{aligned}
\|u_n-u_{n-1}\|^2 \ \leq \ & \langle u_n-u_{n-1},x_n-x_{n-1} \\
& +(1-\frac{r_{n-1}}{r_n})(u_n-x_n)\rangle \\
\leq \ & \|u_n-u_{n-1}\|\{\|x_n-x_{n-1}\| \\
& +|1-\frac{r_{n-1}}{r_n}|\|u_n-x_n\|\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_n-u_{n-1}\| \ \leq \ & \|x_n-x_{n-1}\|+|1-\frac{r_{n-1}}{r_n}|\|u_n-x_n\| \\
= \ & \|x_n-x_{n-1}\|+|\frac{r_n-r_{n-1}}{r_n}|\|u_n-x_n\| \\
\leq \ & \|x_n-x_{n-1}\|+\frac{1}{b}|r_n-r_{n-1}|L \qquad (14)
\end{aligned}$$

where $L=\sup\{\|u_n-x_n\|\},\ n\in\mathbb{N}$.
By substituting (14) into (9), we have

$$\begin{aligned}
& \|x_{n+1}-x_n\| && (15) \\
& \leq \alpha_n\|P_{H(x_n,S_nu_n)}f(x_n)-P_{H(x_n,S_nu_n)}f(x_{n-1})\| \\
& +\alpha_n\|P_{H(x_n,S_nu_n)}f(x_{n-1}) \\
& -P_{H(x-n-1,S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +|\alpha_n-\alpha_{n-1}|\|P_{H(x_{n-1},S_{n-1}x_{n-1})}f(x_{n-1})\| \\
& +(1-\alpha-n)[\|x_n-x_{n-1}\|+\frac{1}{b}|r_n-r_{n-1}|L] \\
& +(1-\alpha_n)\|S_nu_{n-1}-S_{n-1}u_{n-1}\| \\
& +|\alpha_{n-1}-\alpha_n|\|S_{n-1}u_{n-1}\| \\
\leq \ & (1-\alpha_n(1-\alpha))\|x_n-x_{n-1}\|+b_n. && (16)
\end{aligned}$$

From Lemma (II.6), we can conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

From (14) and $|r_n-r_{n-1}|\to 0$, we have

$$\lim_{n\to\infty}\|u_n-u_{n-1}\|=0$$

which also implies that $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$.
Next, we consider

$$\begin{aligned}
\|x_n-S_nu_n\| \ = \ & \|x_n-x_{n+1}+x_{n+1}-S_nu_n\| \\
\leq \ & \|x_n-x_{n+1}\| \\
& +\alpha_n\|P_{H(x_n,S_nu_n)}f(x_n)-S_nu_n\|.
\end{aligned}$$

Since $\|x_{n+1}-x_n\|\to 0$ and $\lim_{n\to\infty}\alpha_n=0$, then

$$\lim_{n\to\infty}\|x_n-S_nu_n\|=0.$$

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For $p \in \mathcal{F}$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\ &\leq \langle T_{r_n x_n} - T_{r_n}p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

therefore,

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Consider the following from the convexity of $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n_1} - p\|^2 &= \|\alpha_n P_{H(x_n,S_nu_n)}f(x_n) \\ &\quad +(1-\alpha_n)S_nu_n - p\|^2 \\ &\leq \alpha_n\|P_{H(x_n,S_nu_n)}f(x_n)\|^2 \\ &\quad +(1-\alpha_n)\|S_nu_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 \\ &\quad -(1-\alpha_n)\|x_n - u_n\|^2,\end{aligned}$$

then

$$\begin{aligned}(1-\alpha_n)\|x_n - u_n\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 \\ &\quad +\|x_n - x_{n+1}\|(\|x_n - p\| \\ &\quad +\|x_{n+1} - p\|).\end{aligned}$$

It is obvious that $\|x_n - u_n\| \to 0$.
Next, we consider

$$\|S_nu_n - u_n\| \leq \|S_nu_n - x_n\| + \|x_n - u_n\|,$$

then we have $\|S_nu_n - u_n\| \to 0$. Similarly, we also obtain

$$\|x_n - S_nx_n\| \leq \|x_n - S_nu_n\| + \|u_n - x_n\|.$$

So, $\|x_n - S_nx_n\| \to 0$.
Step 3. We will show that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_{\bigcap_{n=1}^{\infty} F(S_n)\cap EP(F)}f(z)$.
We choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{ij}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. With out loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|S_nu_n - u_n\| \to 0$, we can obtain $S_nu_{n_i} \rightharpoonup w$. Next we will show that $w \in EP(F)$. By $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \ \forall y \in C.$$

By the monotonicity of F, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A4) we have

$$0 \geq F(y, w) \text{ for all } y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$\begin{aligned}0 &= F(y_t, y_t) \\ &\leq tF(y_t, y_t) + (1-t)F(y_t, w) \\ &\leq tF(y_t, y)\end{aligned}$$

and hence $0 \leq F(y_t, y_t)$. From (A3), we have $0 \leq F(w, y)$ for all $y \in C$. Therefore, we have $w \in EP(F)$. Next we shall show that $w \in \bigcap_{n=1}^{\infty} F(S_n)$. Assume that $w \notin \bigcap_{n=1}^{\infty} F(S_n)$ and since $u_{n_i} \rightharpoonup w$ and $w \neq S_nw$, from the Opial's theorem, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\liminf_{i\to\infty}\|u_{n_i} - w\| &< \liminf_{i\to\infty}\|u_{n_i} - S_nw\| \\ &\leq \liminf_{i\to\infty}\{\|u_{n_i} - S_nu_{n_i}\| \\ &\quad +\|S_nu_{n_i} - S_nw\|\} \\ &\leq \liminf_{i\to\infty}\|u_{n_i} - w\|.\end{aligned}$$

This is the contradiction. So, $w \in \bigcap_{n=1}^{\infty} F(S_n)$. Therefore, $w \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(F)$.
Since $z = P_{\bigcap_{n=1}^{\infty} F(S_n)\cap EP(F)}f(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \\ &= \langle f(z) - z, w - z\rangle \leq 0.\end{aligned}$$

By Lemma III.1, we have

$$\lim_{n\to\infty}\sup\langle P_{H(x_n,S_nu_n)}f(x^*) - x^*, x_n - x^*\rangle \leq 0.$$

Step 4. We will show that $\{x_n\}$ converges to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(F)$.
From

$$x_{n+1} - z = \alpha_n(P_{H(x_n,S_nu_n)}f(x_n) - z) + (1-\alpha_n)(S_nu_n - z),$$

we have

$$\begin{aligned}&(1-\alpha_n)^2\|S_nu_n - z\|^2 \\ &\geq \|x_{n+1} - z\|^2 \\ &-2\alpha_n\langle P_{H(x_n,S_nu_n)}f(z) - z, x_{n+1} - z\rangle.\end{aligned}$$

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So, we have

$$\begin{aligned}
& \|x_{n+1}-z\|^2 \\
\leq\ & (1-\alpha_n)^2\|S_nu_n-z\|^2 \\
& +2\alpha_n\langle P_{H(x_n,S_nu_n)}f(x_n)-z,x_{n+1}-z\rangle \\
\leq\ & (1-\alpha_n)^2\|u_n-z\|^2 \\
& +2\alpha_n\langle P_{H(x_n,S_nu_n)}f(x_n) \\
& -P_{H(x_n,S_nu_n)}f(z),x_{n+1}-z\rangle \\
& +2\alpha_n\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
\leq\ & (1-\alpha_n)^2\|x_n-z\|^2+2\alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| \\
& +2\alpha_n\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
\leq\ & (1-\alpha_n)^2\|x_n-z\|^2 \\
& +\alpha_n\alpha\{\|x_n-z\|^2+\|x_{n+1}-z\|^2\} \\
& +2\alpha_n\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
\leq\ & \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z\|^2 \\
& +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
=\ & \frac{1-2\alpha_n+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-z\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z\|^2 \\
& +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
\leq\ & (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha})\|x_n-z\|^2+\frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z\|^2 \\
& +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle \\
=\ & (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha})\|x_n-z\|^2+b_n
\end{aligned}$$

where

$$\begin{aligned}
b_n\ =\ & \frac{\alpha_n^2}{1-\alpha_n\alpha}\|x_n-z\|^2 \\
& +\frac{2\alpha_n}{1-\alpha_n\alpha}\langle P_{H(x_n,S_nu_n)}f(z)-z,x_{n+1}-z\rangle
\end{aligned}$$

and $\sum_{n=1}^{\infty}<\infty$. It is obvious that $c_n=\dfrac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}$ and $\sum_{n=1}^{\infty}c_n=\infty$.

By Lemma II.6, Lemma III.1, we have that $\{x_n\}$ converges strongly to z. This completes the proof. ■

Based on Theorem III.2, we can deduce our result to the following corollary.

Corollary III.3. *Let H be a real Hilbert space and F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $S_n: H\to H$ be a mapping in class $\mathfrak{T}$ and satisfies the condition (Z) such that $\mathfrak{F}:=\bigcap_{n=1}^{\infty}F(S_n)\cap EP(F)\neq\emptyset$. Let f be an α-contraction mapping of H into itself and $\{x_n\}$ be a sequence generated by the following:*

$$\begin{cases} x_0=x\in C, \\ F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0,\ \ \forall y\in C, \\ x_{n+1}=\alpha_nP_{H(x_n,S_nu_n)}f(x_n)+(1-\alpha_n)Su_n, \end{cases} \tag{17}$$

for all $n\in\mathbb{N}$, where $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(2) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$,
(3) $\liminf_{n\to\infty}r_n>0$.

Then, $\{x_n\}$ converges strongly to $z\in\mathfrak{F}:=\bigcap_{n=1}^{\infty}F(S_n)\cap EP(F)$, where $z=P_{\mathfrak{F}}f(z)$.

IV. Numerical test

In this section, we give a numerical example of the main result as follow:
Example For simplicity, we assume $H=\mathbb{R}$ and $C=[-1,1]$. Let $F(x,y)=-2x^2+xy+y^2$, $f(x)=\frac{x}{2}$, $T_1(x)=sin(x)$. Furthermore, let $r=0.5$ and $\alpha_n=\frac{1}{10n}$.
By using MATLAB 7.11.0, we can give the result that support our Main Theorem as shown by the following:

n	x_n
1	1.000000000000000
2	0.389418342308651
3	0.155138191495153
4	0.062015456596188
5	0.024803638649656
6	0.009921292690458
⋮	⋮
38	0.000000000000002
39	0.000000000000001
40	0.000000000000000

Fig. 1. This table shows the value of sequence $\{x_n\}$ on each iteration steps.

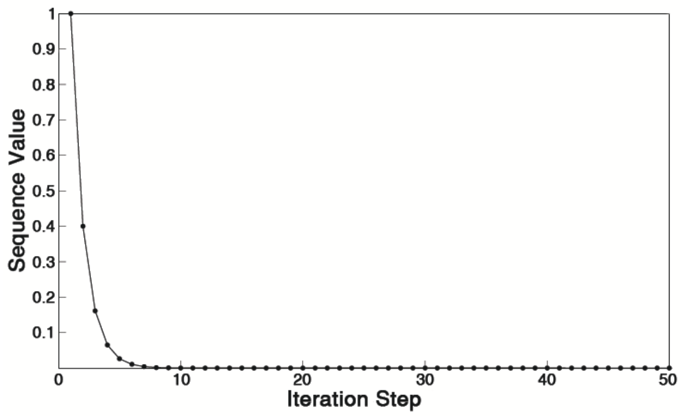

Fig. 2. This figure shows the graph of the above table, we can see that x_n converges to zero.

V. Conclusion

We focused on an iterative scheme for the class of $\mathfrak{T}$ mappings in Hilbert spaces. We established the strong convergence theorem and gave a numerical test to illustrate our main theorem. Our scheme can be used for determining common solutions of fixed point problems and equilibrium problems which will lead to solve variational inequality problems or optimization problems as its advanced applications. The result of this paper extended and improved the corresponding result given by Qiao-Li Dong and Songnian He [2] and some authors in the literature.

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REFERENCES

[1] P.E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, *Computers and Mathematics with Applications*, 59(2010), 74-79.

[2] Q.L. Dong, S. He and F. Su, Strong Convergence Theorems by Shrinking Projection Methods for Class $\mathfrak{T}$ Mappings, *Fixed Point Theory and Applications*, Volume 2011, Article ID 681214, 7 pages.

[3] H. H. Bauschke and P. L. Combettes, A Weak-to-Strong Convergence Principle for Fejér-Monotone Methods in Hilbert Spaces, *Mathematics of Operations Research*, Volume 26, No. 2, May 2001, pp.248-264.

[4] , H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *Journal of Mathematical Analysis and Applications*, 298(2004), 279–291.

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