Numerical Method for Solving Wave Equation with Non Local Boundary Conditions

A. Cheniguel

Abstract— **The hyperbolic partial differential equations with nonlocal boundary conditions arise in many branches of science and engineering. In this paper a numerical new technique (ADM) is presented and used for solving wave equation with nonlocal boundary conditions. Numerical experiments show that the series form of approximate solution converges rapidly, and the obtained results are in very good agreement with the exact ones.**

Index Terms—**Adomian decomposition method, wave equation, non local problem, numerical solutions for partial differential equations.**

I. Introduction

IN this paper, we deal with non classical initial boundary value problems that is, the solution of hyperbolic differential equations with non local boundary specifications. These non local conditions arise mainly when the data on the boundary cannot be measured directly. Many physical phenomena are modeled by hyperbolic initial boundary value problems with non local boundary conditions. Hyperbolic equationswith non local integral conditions are widely used in chemistry, plasma physics, thermoelasticity, engineering and so forth. The solutions of hyperbolic and parabolic equations with integral conditions were studied by several authors [1-10]. Numerical solution of hyperbolic partial differential equations with integral conditions are still a major research area with widespread applications in engineering, physics and technology.

We consider the following one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = q(x,t)\ 0 < x < 1, 0 < t \le T \qquad (1)$$

Subject to the initial condition:

$$u(x,0) = r(x), 0 \le x \le 1 \qquad (2)$$

$$u_t(x,0) = s(x) \qquad (3)$$

And the non local boundary conditions

$$u(0,t) = p(t),\ \ 0 < t \le T \qquad (4)$$

$$\int_0^1 u(x,t)dx = q\ (t),\ \ 0 < t \le T \qquad (5)$$

Where r, s, p and q are known functions, we suppose that f is sufficiently smooth to produce a smooth classical solution.

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II. Adomian Decomposition Method

A. Operator for:

The Adomian decomposition method has been applied [11-14] for solving a large classe of linear and non linear ordinary and partial differential equations with approximate solutions which converges rapidly to accurate solutions. In recent years, many papers were devoted to the problem of approximate of one-dimensional wave equation with non local boundary conditions. The motivation of this work is to apply the decomposition method for solving the one-dimensional wave equation with an integral boundary condition. It is well known in the literature that this algorithm provides solution in rapidly convergent series. The implementation of theAdomian method has shown reliable results in that few terms only are needed to obtain accurate solutions.

Consider equations (1)-(5) written in the form

$$L_{tt}(u) = L_{xx}(u) + q(x,t) \qquad (6)$$

Where the differential operators are givenas :

$L_{tt}(.) = \frac{\partial^2}{\partial t^2}(.)andL_{xx} = \frac{\partial^2}{\partial x^2}$

The inverse operator $L_{tt}^{-1}$is therefore considered a two-fold integral operator defined by :

$$L_{tt}^{-1} = \int_0^t \int_0^t (.)\ dtdt$$

Operating with L_{tt}^{-1} on equation (6), it then follows that:

$$L_{tt}^{-1}\left(L_{tt}(u)\right) = L_{tt}^{-1}\left(L_{xx}(u)\right) + L_{tt}^{-1}\left(q(x,t)\right) \qquad (7)$$

And specified initial conditions yield:

$$u(x,t) = r(x) + ts(x) + L_{tt}^{-1}\left(L_{xx}\left(u(x,t)\right)\right) + L_{tt}^{-1}\left(q(x,t)\right) \qquad (8)$$

B. Application to the solution of the problem

The decomposition method assumes an infinite series solution for unknown function $u(x,t)$ given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k \qquad (9)$$

Where the components u_k $(k = 0,1,2,3, \ldots)$ are determined recursively by using the relation:

$$u_0 = r(x) + ts(x) + L_{tt}^{-1}(q(x,t) \qquad (10)$$

And

$$u_{k+1} = L_{tt}^{-1}\left(L_{xx}(u_k)\right), k \ge 0\ (11)$$

If the series converges in a suitable way then, the general solution is obtained as:

$$u(x,t) = lim_{n\to\infty} \sum_{k=0}^{\infty} u_k\ (x,t)$$

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III. EXAMPLES

A. Example 1

We consider the following wave equation:

$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, 0 < t < 0.5$_(12)

With the initial conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1$$

$u_t(x,0) = \pi\cos(\pi x)\,,\ 0 \le x \le 1$(13)

And the boundary conditions:

$$u(0,t) = p(t) = \sin(\pi t)$$

$\int_0^t u(x,t)dt = q(t) = 0$ (14)

Substituting in equations (10) and (11) we obtain the following:

$u_0 = t(\pi\cos(\pi x))$ (15)

$u_{k+1} = L_{tt}^{-1}(L_{xx}(u_k), k \ge 0$ (16)

We can then, proceed to compute the first few terms of the series:

$u_0 = t(\cos(\pi x))$ (17)

$u_1 = \pi\cos(\pi x)\int_0^t dt\int_0^t t dt = \cos(\pi x)\,(-\pi^3\frac{t^3}{3!})$ (18)

$u_2 = L_{tt}^{-1}(L_{xx}(u_1)) = \cos(\pi x)\int_0^t dt\int_0^t \pi^5\left(\frac{t^3}{3!}\right)dt = \cos(\pi x)(\pi^5\frac{t^5}{5!})$

$u_3 = L_{tt}^{-1}(L_{xx}(u_2)) = \cos(\pi x)\int_0^t dt\int_0^t(-\pi^7\frac{t^5}{5!})dt = \cos(\pi x)(-\pi^7\frac{t^7}{7!})$

$u_4 = L_{tt}^{-1}(L_{xx}(u_3)) = \cos(\pi x)\int_0^t dt\int_0^t \pi^9\frac{t^7}{7!}dt = \cos(\pi x)\,(\frac{t^9}{9!})$ (19)

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.. And so on:

$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots +$ (20)

Hence:

$u(x,t) = \cos(\pi x)\,(\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \frac{(\pi t)^7}{7!} + \frac{(\pi t)^9}{9!} -. + \cdots)$

Or

$u(x,t) = \cos(\pi x)\sin(\pi t)$ (21)

The result shows that the method provides an excellent approximation.

B. Example 2

Consider the wave equation :

$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < 1, t > 0$ (22)

With the initial conditions:

$u(x,0) = x, 0 < x < 1$

$u_t(x,0) = e^x$, $0<x<1$ (23)

And the boundary conditions:

$u_x(0,t) = 2\sinh(\frac{t}{2})$, t>0

$u_x(1,t) = 2e^x \times (\sinh\left(\frac{t}{2}\right) + 1)\,, t > 0$ (24)

Writing equation (22) in an operator form yields:

$L_{tt}(u(x,t)) - \frac{1}{4}L_{xx}(u(x,t)) = 0$ (25)

Operating with the inverse operator L_{tt}^{-1} on both sides of equation (25) and imposing the corresponding initial conditions we obtain:

$u(x,t) = u(x,0) + tu_t(x,0) + L_{tt}^{-1}(L_{xx}(u(x,t)))$ (26)

Starting with:

$u_0 = u(x,0) + tu_t(x,0) = x + te^x$ (27)

And using:

$u_{k+1} = L_{tt}^{-1}(L_{xx}(u_k)), k \ge 0$ (28)

We can obtain:

$u_1 = L_{tt}^{-1}(L_{xx}(u_0)) = 2 \times e^x\int_0^t dt\int_0^t(\frac{t}{2})dt = 2e^x\frac{(\frac{t}{2})^3}{3!}$ (29)

$u_2 = L_{tt}^{-1}(L_{xx}(u_1)) = 2e^x\int_0^t dt\int_0^t\frac{(\frac{t}{2})^3}{3!}dt = 2e^x\frac{(\frac{t}{2})^5}{5!}$ (30)

$u_3 = L_{tt}^{-1}(L_{xx}(u_2)) = 2e^x\int_0^t dt\int_0^t\frac{\left(\frac{t}{2}\right)^5}{5!}dt = 2e^x\frac{(\frac{t}{2})^7}{7!}$(31)

By continuing the iteration, we find that:

$u_k = 2e^x\frac{(\frac{t}{2})^{2k+1}}{(2k+1)!}$ (32)

Which implies that :

$u(x,t) = x + 2e^x\sum_{k=0}^{\infty}u_k(x,t) = x + 2e^x\sinh(\frac{t}{2})$ (33)

Which converges to the exact solution.

C. Example 3

Now we consider, the following problem:

$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, 0 < x < 1, 0 < t \le 0.5$ (34)

Subject to the initial conditions:

$u(x,0) = \cos(\pi x)\,, 0 < x < 1$ (35)

$u_t(x,0) = 0, 0 < x < 1$ (36)

And the boundary conditions:

$u_x(0,t) = 0$ (37)

$\int_0^1 u(x,t)dx = 0$ (38)

Which is easily seen to have the exact solution $u(x,t) = \cos(\pi x)\cos(\pi t)$. Rewriting equation (34) in an operator form:

$L_{tt}(u(x,t)) = L_{xx}(u(x,t))$ (39)

We operate with the inverse operator L_{tt}^{-1}

On both sides of equation (39), we get the following equations:

$L_{tt}^{-1}(L_{tt}(u(x,t)) = L_{tt}^{-1}(L_{xx}(u(x,t)))$ (40)

$$L_{tt}^{-1}(u(x,t)) = \int_0^t dt\int_0^t\frac{d^2u}{dt^2}dt$$
$$= u(x,t) - u(x,0) - tu_t(x,0)$$

Then :

$u(x,t) - u(x,0) - tu_t(x,0) = L_{tt}^{-1}(L_{xx}(u(x,t)))$ (41)

Or:

$u(x,t) = u(x,0) + tu_t(x,0) + L_{tt}^{-1}(L_{xx}(u(x,t)))$ (42)

From the above equations we can get the different terms of the approximate series solution as:

$u_0 = u(x,0) + tu_t(x,0) = \cos(\pi x)$ (43)

$u_1 = L_{tt}^{-1}(L_{xx}(u_0)) = -\pi^2 cos\pi x\int_0^t dt\int_0^t dt = \cos(\pi x)\,(-\frac{(\pi t)^2)}{2!})$ (44)

$u_2 = L_{tt}^{-1}(L_{xx}(u_1)) = \cos\pi x\int_0^t dt\int_0^t\frac{(\pi t)^2}{2!}dt = \cos\pi x(\frac{(\pi t)^4}{4!})$ (45

..

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$u_k = L_{tt}^{-1}(L_{xx}(u_{k-1})) = cos\pi x \times (-1)^k\frac{(\pi t)^{2k}}{(2k)!}$ (46)

Hence, the approximate series solution is given by:

$$u(x,t) = u_0 + u_1 + u_2 + \cdots .. + u_k + \cdots.$$

Or:

$u(x,t) = cos\pi x(1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + \cdots + (-1)^k\frac{(\pi t)^{2k}}{(2k)!} + \cdots)$ (47)

Which converges to the exact solution:

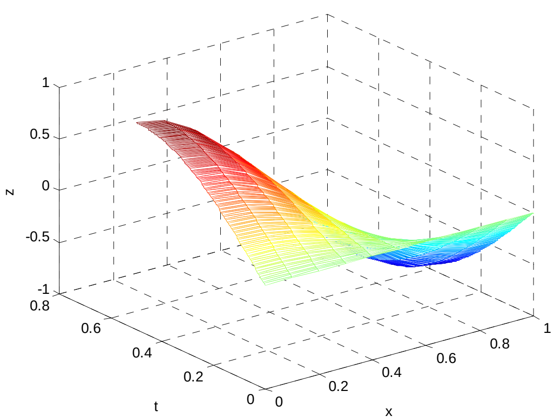
$$u(x,t) = cos\pi x \times cos\pi t \quad (48)$$

IV. Conclusion

In this work we have applied the Adomian decomposition method for the solution of the wave equation with non local boundary conditions. This algorithm is simple and easy to implement. The obtained results confirmed a good accuracy of the method.On the other hand, thecalculations are simpler and faster than in traditional techniques.

Table 1

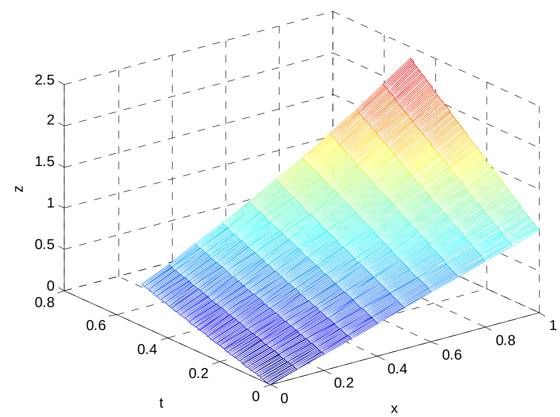
Example 1			Absolute Error
x_i	u_{ex}	u_{Ad} 5-Iterates	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	$1.2566*10^{-2}$	$1.2566*10^{-2}$	0.0
0.1	1.1951×10^{-2}	1.1951×10^{-2}	0.0
0.2	1.0166×10^{-2}	1.0166×10^{-2}	0.0
0.3	7.3851×10^{-3}	7.3851×10^{-3}	0.0
0.4	3.8831×10^{-3}	3.8831×10^{-3}	0.0
0.5	0.0	0.0	0.0
0.6	-3.8831×10^{-3}	-3.8831×10^{-3}	0.0
0.7	-7.3851×10^{-3}	-7.3851×10^{-3}	0.0
0.8	-1.0166×10^{-2}	-1.0166×10^{-2}	0.0
0.9	-1.1951×10^{-2}	-1.1951×10^{-2}	0.0
1.0	-1.2566×10^{-2}	-1.2566×10^{-2}	0.0



Variation of the approximate solution for differentvaluesofxand

$Table\ 2$

$Example\ 2$			$Absolute\ Error$
x_i	u_{ex}	u_{Ad} $5 - Iterates$	$\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.004	0.004	0.0
0.1	0.10442	0.10442	0.0
0.2	0.20489	0.20489	0.0
0.3	0.30540	0.30540	0.0
0.4	0.40592	0.40592	0.0
0.5	0.50659	0.50659	0.0
0.6	0.60729	0.60729	0.0
0.7	0.70806	0.70806	0.0
0.8	0.8089	0.8089	0.0
0.9	0.90984	0.90984	0.0
1.0	0.0109	0.0109	0.0



$\cos(\pi x) \sin(\pi t)$

Variation of the approximate solution for different values of x and t

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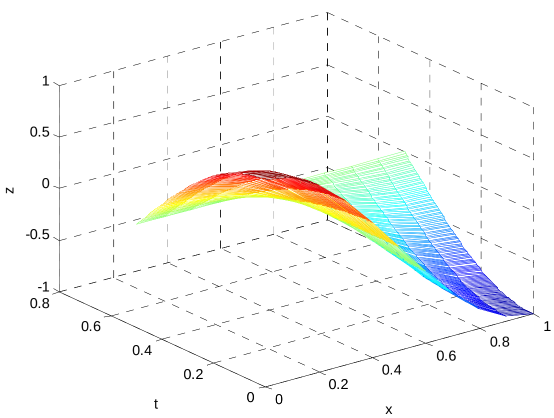
Table 3

Example 3

$h_x = \frac{1}{10}$ $h_t = \frac{1}{250}$

x_i	u_{ex}	u_{Ad} 5 − *Iterates*	*Absolute Error* $\lvert u_{ex} - u_{Ad} \rvert$
0.0	0.99992	0.99992	0.0
0.1	0.95098	0.95098	0.0
0.2	0.80895	0.80895	0.0
0.3	0.58774	0.58774	0.0
0.4	0.30899	0.30899	0.0
0.5	0.0.0	0.0	0.0
0.6	-0.30899	-0.30899	0.0
0.7	-0.58774	-0.58774	0.0
0.8	-0.80895	-0.80895	0.0
0.9	-0.95098	-0.95098	0.0
1.0	-0.99992	-0.99992	0.0



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