Strong Convergence Theorems of Multivalued Nonexpansive Mappings and Maximal Monotone Operators in Banach Spaces

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***Abstract*—In this paper, we prove a strong convergence theorem for fixed points of sequence for multivalued nonexpansive mappings and a zero of maximal monotone operator in Banach spaces by using the hybrid projection method. Our results modify and improve the recent results in the literatures.**

***Index Terms*—Fixed Point, multivalued nonexpansive mapping, maximal monotone operator.**

I. Introduction

THROUGHOUT this paper, we let C be a nonempty closed convex subset of a real Banach space E. A mapping $t : C \to C$ is said to be *nonexpansive* if $\|tx-ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(t)$ the set of fixed points of t, that is $F(t) = \{x \in C : x = tx\}$. A mapping t is said to be an asymptotic fixed point of t (see [11]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - tx_n\| = 0$. The set of asymptotic fixed points of t will be denoted by $\widehat{F}(t)$. A mapping t from C into itself is said to be *relatively nonexpansive* [9], [12], [16] if $\widehat{F}(t) = F(t)$ and $\phi(p, tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(t)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [2], [3].

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. Let $H : CB(C) \times CB(C) \to \mathbb{R}^+$ be the Hausdorff distance on $CB(C)$, that is

$$H(A < B) = max\{\sup_{a\in A} dist(a, A), \sup_{b\in B} dist(b, B)\},$$

for every $A, B \in CB(C)$, where $dist(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B of C.

A multi-valued mapping $T : E \to CB(C)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|,$$

for all $x, y \in C$. An element $p \in C$ is called a fixed point of $T : C \to CB(C)$, if $p \in Tp$. The set of fixed point T is denoted by $F(T)$.

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A point $p \in C$ is said to be an asymptotic fixed point of $T : C \to CB(C)$, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x \in E$ and $d(x_nTx_n) \to 0$. Denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. T is said to be relatively nonexpansive, if $F(T) \neq \emptyset$, $\widehat{F}(T) = F(T)$ and $\phi(p, z) \leq \phi(p, x)$, $\forall x \in C$, $p \in F(T)$, $z \in Tz$. A mapping T is said to be closed, if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $d(y, Tx_n) \to 0$ then $d(y, Tx) \to 0$. T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, z_n) \leq \phi(p, x), \forall x \in C, p \in F(T), z \in T^n(x)$. T is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exists a real sequence $k_n \subset [1, +\infty), k_n \to 1$ such that

$$\phi(x, z_n) \leq k_n\phi(p, x), \forall x \in C, p \in F(T), z_n \in T^nx. \quad (1)$$

A mapping T is said to be total quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$ and $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$\begin{aligned}\phi(x, z_n) \quad &\leq \quad k_n\phi(p, x) + \nu_n\zeta[\phi(p, x)] + \mu_n, \forall x \in C,\\ &\qquad p \in F(T), z_n \in T^nx. \end{aligned} \quad (2)$$

A mapping T is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that $\|x_n - y_n\| \leq L\|x - y\|$, where $x, y \in C, x_n \in T^nx, y_n \in T^ny$.

Let E be a real Banach space with dual E^*. Denote by $\langle\cdot,\cdot\rangle$ the duality product. The *normalized duality mapping* J from E to 2^{E^*} is defined by $Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, for all $x \in E$. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \qquad \text{for all } x, y \in E. \quad (3)$$

A mapping T is said to be *hemi-relatively nonexpansive* (see [12]) if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \qquad \text{for all } \ x \in C \quad \text{and} \quad p \in F(T).$$

A point p in C is said to be an *asymptotic* fixed point of T [2] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A hemi-relatively nonexpansive mapping T from C into itself is called *relatively nonexpansive* if $\hat{F}(T) = F(T)$, see [8], [10], [14]) for more details.

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Matsushita and Takahashi [8] introduced the iteration in a Banach space E:

$$\begin{cases} x_0 = x \in C, \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n = 0, 1, 2, \dots, \end{cases} \tag{4}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$.

Qin and Su [10] showed that the sequence $\{x_n\}$, which is generated by a relatively nonexpansive mappings T in a Banach space E, as following

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + \\ \qquad (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{5}$$

converges strongly to $\Pi_{F(T)}x_0$.
Moreover, they also showed that the the sequence $\{x_n\}$, which is generated by

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{6}$$

converges strongly to $\Pi_{F(T)}x_0$.

In 2012, Chang et al. [4] modified the Halpern-type iteration algorithm for total quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition in Banach space. Recently, Tang and Chang [15] introduce the concept of total quasi-ϕ-asymptotically nonexpansive multi-value mapping in Banach space, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in C, \quad \text{is arbitrary}, \\ C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTw_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{7}$$

$\forall n \ge 0$, where $w_n \in T^n x_n$, $\xi_n = \nu_n \sup_{p \in F} \zeta(\phi(p, x_n)) + \mu_n$ and showed that the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

In this paper, motivated by Tang and Chang [15], we prove strong convergence theorems for fixed points of sequence for multicalued nonexpansive mapping and a zero of maximal monotone operator in Banach space by using the hybrid projection methods. Our results extend and improve the recent results by Tang and Chang [15] and many others.

II. Preliminaries

In this section, we will recall some basic concepts and useful well known results.

A Banach space E is said to be *strictly convex* if

$$\left\| \frac{x+y}{2} \right\| < 1, \tag{8}$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. It is said to be *uniformly convex* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and

$$\lim_{n\to\infty} \|x_n + y_n\| = 2, \tag{9}$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{10}$$

exists for each $x, y \in U$. It is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (10) is attained uniformly for $y \in U$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be uniformly smooth) if the limit (10) is attained uniformly for $x, y \in U$. .

In our work, the concept duality mapping is very important. Here, we list some known facts, related to the duality mapping J, as following:

(a) E (E^*, resp) is uniformly convex if and only if E^* (E, resp.) is uniformly smooth.
(b) $J(x) \ne \emptyset$ for each $x \in E$.
(c) If E is reflexive, then J is a mapping of E onto E^*.
(d) If E is strictly convex, then $J(x) \cap J(y) \ne \emptyset$ for all $x \ne y$.
(e) If E is smooth, then J is single valued.
(f) If E has a Fréchet differentiable norm, then J is norm to norm continuous.
(g) If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of E.
(h) If E is a Hilbert space, then J is the identity operator.

For more information, the readers may consult [5], [13].

If C is a nonempty closed convex subset of real a Hilbert space H and $P_C : H \to C$ is the *metric projection*, then P_C is nonexpansive. Alber [1] has recently introduced a *generalized projection* operator Π_C in a Banach space E which is an analogue representation of the metric projection in Hilbert spaces.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x).$$

Notice that the existence and uniqueness of the operator Π_C is followed from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J, and moreover, in the Hilbert spaces setting we have $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\| y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \text{for all } x, y \in E. \tag{11}$$

Remark II.1. If E is a strictly convex and a smooth Banach space, then for all $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$, see Matsushita and Takahashi [8].

To obtain our results, following lemmas is very important.

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Lemma II.2. (*Kamimura and Takahashi* [6]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma II.3. (*Kamimura and Takahashi* [6]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a strictly convex, smooth and reflexive Banach space, let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V : E \times E^* \to \mathbb{R}$ as follows (see [7]):

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{12}$$

for all $x \in Ex \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

Lemma II.4. (*Kohsaka and Takahashi* [7]). *Let E ba a strictly convex, smooth and reflexive Banach space, and let V be as in* (12)*. Then*

$$V(x, x^*) + 2\langle J^{-1}(x^* - x), y^*\rangle \leq V(x, x^* + y^*) \tag{13}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma II.5. (*Alber* [1]). *Let E be a reflexive, strict convex and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{14}$$

A set-value mapping $A : E \to E^*$ with domain $D(A) = \{x \in E : A(x)\} \neq \emptyset$ and range $R(A) = \{x^* \in E^* : x^* \in A(x), x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x^* \in A(x), y^* \in A(y)$. We denote the set $\{s \in E : 0 \in Ax\}$ by $A^{-1}0$. A is maximal monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set A^{-1} is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is known that A is a maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.

Define the resolvent of A by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(A)$. Also, $A^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set os all fixed points of J_r. Define, for $r > 0$, the Yosida approximation of A by $A_r = (J - JJ_r)/r$. We know that $A_r x \in A(J_r x)$ for all $R > 0$ and $x \in E$.

Lemma II.6. (*Kohsaka and Takahashi* [7]). *Let E ba a strictly convex, smooth and reflexive Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1} \neq \emptyset$, let $r > 0$ and let $J_r = (J + rT)^{-1}J$. Then*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y) \tag{15}$$

for all $x \in A^{-1}(0)$ and $y \in E$.

III. Main result

Theorem III.1. *Let E be a real uniformly smooth and uniformly convex Banach space with Kadec-Klee property and let C be a nonempty closed convex subset of E. Let $T : C \to CB(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multivalued mapping with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and $\zeta(0) = 0$, let $t : C \to C$ be a relatively nonexpansive mapping, let $A : E \to E^*$ be a maximal monotone operator satisfying $D(A) \subset C$. Let $J_r = (J + rA)^{-1}J$ for $r > 0$ such that $F := A^{-1}(0) \cap F(T) \cap F(t)$ and let a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C, \ \ \text{chosen arbitrarity} \ \ \text{and} \ \ C_0 = C, \\ w_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J(J_{r_n} z_n)), z_n \in T^n x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jtw_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_n) \\ \qquad +(1 - \alpha_n)phi(z, w_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{16}$$

for $n \in \mathbb{N} \cup \{0\}$, where J is the single-valued duality mapping on E and $\xi_n = \nu_n \sup_{u^* \in F} \zeta(\phi(u^*, x_n)) + \mu_n$. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfying

(i) $0 < \beta_1 \leq \beta_n \leq \beta_2 < 1$,
(ii) $0 \leq \alpha_n \leq \alpha < 1$,
(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Proof. We first show that C_{n+1} is closed and convex for each $n \geq 0$. Obviously, from the definition of C_{n+1}, we see that C_{n+1} is closed for each $n \geq 0$. Now we show that C_{n+1} is convex for any $n \geq 0$. Since

$$\begin{aligned} \phi(z, y_n) \ \ &\leq \ \ \phi(z, x_n) + \xi_n \iff 2\langle v, Jx_n - Jy_n\rangle + \|y_n\|^2 \\ &\quad -\|x_n\|^2 - \xi_n \leq 0, \end{aligned} \tag{17}$$

this implies that C_{n+1} is a convex set.

Next, we show that $\{x_n\}$ is bounded and $\{\phi(x_n, x_0)\}$ is convergent sequence. Put $u_n = J_{r_n} z_n, \forall n \geq 0$, let $p \in F := A^{-1}(0) \cap F(T) \cap F(t)$. By (16), we obtain

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jtw_n)) \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1 - \alpha_n)\langle p, Jtw_n\rangle \\ &\quad +\alpha_n\|x_n\|^2 + (1 - \alpha_n)\|tw_n\|^2 \\ &\quad -\alpha_n(1 - \alpha_n)g\|Jx_n - Jtw_n\|^2 \\ &= \alpha_n\phi(p, x_n) + (1 - \alpha_n)\phi(p, tw_n) \\ &\quad -\alpha_n(1 - \alpha_n)g\|Jx_n - Jtw_n\|^2 \\ &\leq \alpha_n\phi(p, x_n) + (1 - \alpha_n)\phi(p, w_n) \\ &\quad -\alpha_n(1 - \alpha_n)g\|Jx_n - Jtw_n\|^2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} \phi(p, w_n) &= \phi(p, J^{-1}(\beta_n Jx_n + (1 - \beta_n)Ju_n)) \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1 - \beta_n)\langle p, Ju_n\rangle \\ &\quad +\beta_n\|x_n\|^2 + (1 - \beta_n)\|u_n\|^2 \\ &\quad -\beta_n(1 - \beta_n)g\|Jx_n - Ju_n\|^2 \\ &= \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, u_n) - \beta_n(1 - \beta_n) \\ &\quad g\|Jx_n - Ju_n\|^2 \end{aligned}$$

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$$\begin{aligned}
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\phi(p,z_n)-\phi(u_n,z_n)] \\
&\quad -\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\phi(p,T^nx_n) \\
&\quad -\phi(u_n,z_n)]-\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&\leq \beta_n\phi(p,x_n)+(1-\beta_n)[\phi(p,x_n) \\
&\quad +\nu_n\zeta(\phi(p,x_n))+\mu_n-\phi(u_n,z_n)] \\
&\quad -\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&= \phi(p,x_n)+\nu_n\sup_{u^*\in F}\zeta(\phi(u^*,x_n))+\mu_n \\
&\quad -(1-\beta_n)\phi(u_n,z_n) \\
&\quad -\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&= \phi(p,x_n)+\xi_n-(1-\beta_n)\phi(u_n,z_n) \\
&\quad -\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&= \phi(p,x_n)+\xi_n, \qquad (19)
\end{aligned}$$

where $\xi_n=\nu_n\sup_{u^*\in F}\zeta(\phi(u^*,x_n))+\mu_n$.
By the assumptions of $\{\nu_n\}$, $\{\mu_n\}$, we obtain

$$\xi_n=\nu_n\sup_{u^*\in F}\zeta(\phi(u^*,x_n))+\mu_n\to 0,\quad \text{as}\quad n\to\infty. \qquad (20)$$

Substituting (19) into (18), we have

$$\begin{aligned}
\phi(p,y_n) &\leq \alpha_n\phi(p,x_n)+(1-\alpha_n)[\phi(p,x_n)+\xi_n \\
&\quad -\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2] \\
&\leq \phi(p,x_n)+\xi_n- \\
&\quad (1-\alpha_n)\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2 \\
&\leq \phi(p,x_n)+\xi_n, \forall p\in F. \qquad (21)
\end{aligned}$$

This means that, $p\in C_{n+1}$ for all $n\geq 0$. As consequently, the sequence $\{x_n\}$ is well defined. Moreover, since $x_n=\Pi_{C_n}x_0$ and $x_{n+1}\in C_{n+1}\subset C_n$, we get

$$\phi(x_n,x_0)\leq\phi(x_{n+1},x_0),$$

for all $n\geq 0$. Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing.
By definition of x_n and Lemma II.6, we have

$$\begin{aligned}
\phi(x_n,x_0)=\phi(\Pi_{C_n}x_0,x_0) &\leq \phi(p,x_0)-\phi(p,\Pi_{C_n}x_0) \\
&\leq \phi(p,x_0),
\end{aligned}$$

for all $p\in\bigcap_{n=0}^{\infty}F(T_n)\subset C_n$. Thus, $\{\phi(x_n,x_0)\}$ is a bounded sequence. Moreover, by (11), we know that $\{x_n\}$ is bounded. So, $\lim_{n\to\infty}\phi(x_n,x_0)$ exists. Again, by Lemma II.6, we have

$$\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_0) \\
&\leq \phi(x_{n+1},x_0)-\phi(\Pi_{C_n}x_0,x_0) \\
&= \phi(x_{n+1},x_0)-\phi(x_n,x_0),
\end{aligned}$$

for all $n\geq 0$. Thus, $\phi(x_{n+1},x_n)\to 0$ as $n\to\infty$.

Since $\{x_n\}$ is bounded and x is reflexive, there exists a subsequence $\{x_{n_i}\}\subset\{x_n\}$ such that $x_{n_i}\rightharpoonup q\in C$.

From C_n is closed and convex and $C_{n+1}\subset C_n$, this implies that C_n is weakly closed and $q\in C_n$, for each $n\geq 0$.

In view of $x_{n_i}=\Pi_{C_{n_i}}x_0$, we have

$$\phi(x_{n_i},x_0)\leq\phi(q,x_0),\forall n_i\geq 0.$$

Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
\liminf_{n_t\to\infty}\phi(x_{n_i},x_0) &= \liminf_{n_i\to\infty}\|x_{n_i}\|^2-2\langle x_{n_i},Jx_0\rangle \\
&\quad +\|x_0\|^2| \\
&\geq \|q\|^2-2\langle q,Jx_0\rangle+\|x_0\|^2| \\
&= \phi(q,x_0).
\end{aligned}$$

So,

$$\phi(q,x_0)\leq\liminf_{n_i\to\infty}\phi(x_{n_i},x_0)\leq\limsup_{n_i\to\infty}\phi(x_{n_i},x_0)\leq\phi(q,x_0).$$

this implies that $\lim_{n_i\to\infty}\phi(x_{n_i},x_0)=\phi(q,x_0)$. By Kadec-Klee property of E, we have

$$\lim_{n_i\to\infty}x_{n_i}=q,\quad \text{as}\quad n_i\to\infty.$$

Since the sequence $\{\phi(x_n,x_0)\}$ is convergent and $\lim_{n_i\to\infty}\phi(x_{n_i},x_0)=\phi(q,x_0)$ which implies that $\lim_{n\to\infty}\phi(x_n,x_0)=\phi(q,x_0)$. If there exists some subsequence $\{x_{n_j}\}\subset\{x_n\}$ such that $x_{n_j}\rightharpoonup q^*$, then from Lemma II.6

$$\begin{aligned}
\phi(q,q^*) &= \lim_{n_i,n_j\to\infty}\phi(x_{n_i},x_{n_j}) \\
&= \lim_{n_i,n_j\to\infty}\phi(x_{n_i},\Pi_{C_{n_j}}x_0) \\
&\leq \lim_{n_i,n_j\to\infty}[\phi(x_{n_i},x_0)-\phi(\Pi_{C_{n_j}}x_0,x_0)] \\
&= \lim_{n_i,n_j\to\infty}[\phi(x_{n_i},x_0)-\phi(x_{n_j}x_0,x_0)] \\
&= \phi(q,x_0)-\phi(q,x_0)=0.
\end{aligned}$$

This implies that $q=q^*$ and so

$$\lim_{n\to\infty}x_n=q. \qquad (22)$$

By definition of $\Pi_{C_n}x_0$, we have

$$\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_0) \\
&\leq \phi(x_{n+1},x_0)-\phi(\Pi_{C_n}x_0,x_0) \\
&= \phi(x_{n+1},x_0)-\phi(x_n,x_0). \qquad (23)
\end{aligned}$$

From $\lim_{n\to\infty}\phi(x_n,x_0)$ exists, we have

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \qquad (24)$$

It follows from Lemma II.2, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (25)$$

By definition of C_n and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}$, we have

$$\phi(x_{n+1},y_n)\leq\phi(x_{n+1},x_n)+\xi_n. \qquad (26)$$

It follows from (24) and $\xi_n\to 0$ as $n\to\infty$ that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=0. \qquad (27)$$

Again from Lemma II.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \qquad (28)$$

By using the triangle inequality, we get

$$\|y_n-x_n\|\leq\|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|, \qquad (29)$$

again by (25) and (28), we also have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \qquad (30)$$

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Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0. \tag{31}$$

From (21), for $u^* \in F(T)$ and $u_n = J_{r_n}z_n$, $z_n \in T^n x_n$, we have

$$\phi(p, y_n) \leq \phi(p, x_n)+\xi_n-(1-\alpha_n)\beta_n(1-\beta_n)g\|Jx_n-Ju_n\|^2,$$

and hence

$$\begin{aligned}(1-\alpha_n)\beta_n(1-\beta_n)g\|Jx_n - Ju_n\|^2 \\ \leq \phi(p, x_n) - \phi(p, y_n) + \xi_n.\end{aligned} \tag{32}$$

On the other hand, we note that

$$\begin{aligned}\phi(p, x_n) - \phi(p, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle p, Jx_n - Jy_n\rangle \\ &\leq \|x_n - y_n\|(\|x_n + y_n\|) \\ &\quad +2\|p\|\|Jx_n - Jy_n\|.\end{aligned} \tag{33}$$

It follows from $\|x_n - y_n\| \to 0$ and $\|Jx_n - Jy_n\| \to 0$, that

$$\phi(p, x_n) - \phi(p, y_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{34}$$

Since condition (i), (20) and (34), it follows from (32)

$$g\|Jx_n - Ju_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{35}$$

It follows from the property of g that

$$\|Jx_n - Ju_n\| \to 0, \quad \text{as} \quad n \to \infty, \tag{36}$$

and so

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = \lim_{n\to\infty} \|Jx_n - JJ_{r_n}z_n\| = 0. \tag{37}$$

Since J^{-1} is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - J_{r_n}z_n\| = 0. \tag{38}$$

From (18) and (19), we obtain that

$$\begin{aligned}\phi(p, y_n) &\leq \phi(p, x_n) + (1 - \alpha_n)\xi_n \\ &\quad -(1 - \alpha_n)(1 - \beta_n)\phi(u_n, z_n),\end{aligned} \tag{39}$$

and hence

$$(1-\alpha_n)(1-\beta_n)\phi(u_n, z_n) \leq \phi(p, x_n)-\phi(p, y_n)+\xi_n. \tag{40}$$

By condition (i), (ii) and (20), we have

$$\lim_{n\to\infty} \phi(u_n, z_n) = 0. \tag{41}$$

From Lemma II.2, we get

$$\lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|J_{r_n}z_n - z_n\| = 0. \tag{42}$$

Since J is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty} \|Ju_n - Jz_n\| = \lim_{n\to\infty} \|JJ_{r_n}z_n - Jz_n\| = 0. \tag{43}$$

From definition of C_n, we have

$$\alpha_n\phi(z, x_n) + (1 - \alpha_n)\phi(z, w_n) \leq \phi(z, x_n) + \xi_n$$

$$\Leftrightarrow \phi(z, w_n) \leq \phi(z, x_n) + \xi_n.$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, w_n) \leq \phi(x_{n+1}, x_n) + \xi_n. \tag{44}$$

It follows from (24) and $\xi_n \to 0$ as $n \to \infty$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0. \tag{45}$$

From Lemma II.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0. \tag{46}$$

By using the triangle inequality, we get

$$\|w_n - x_n\| \leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\|, \tag{47}$$

again by (25) and (46), we also have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{48}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jw_n - Jx_n\| = 0. \tag{49}$$

From (18) and (19), that

$$\begin{aligned}\phi(p, y_n) &\leq \phi(p, x_n) + \xi_n \\ &\quad -\alpha_n(1 - \alpha_n)g\|Jx_n - Jtw_n\|,\end{aligned} \tag{50}$$

and hence

$$\alpha_n(1-\alpha_n)g\|Jx_n-Jtw_n\| \leq \phi(p, x_n)-\phi(p, y_n)+\xi_n. \tag{51}$$

By condition (ii), (20) and (34), we obtain that

$$g\|Jx_n - Jtw_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{52}$$

It follows from the property of g that

$$\|Jx_n - Jtw_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{53}$$

Since J^{-1} is uniformly norm-to-norm continuous, we have

$$\|x_n - tw_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{54}$$

By using the triangle inequality, again

$$\|w_n - tw_n\| \leq \|w_n - x_n\| + \|x_n - tw_n\|. \tag{55}$$

By (48) and (54), we have

$$\|w_n - tw_n\| \to 0. \tag{56}$$

Since $\{x_n\}$ is bounded and $x_{n_i} \rightharpoonup q \in C$. It follows from (48), we have $w_{n_i} \rightharpoonup q$ as $i \to \infty$ and t is relatively nonexpansive. We have that $q \in \widehat{F}(t) = F(t)$.

Next, we show that $q \in A^{-1}(0)$. Indeed, since $\liminf_{n\to\infty} r_n > 0$, it follows from (42) that

$$\lim_{n\to\infty} \|B_{r_n}z_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|z_n - u_n\| = 0. \tag{57}$$

If $(z, z^*) \in A$, then it holds from the monotonicity of B that

$$\langle z - z_{n_i}, z^* - B_{r_{n_i}}z_{n_i}\rangle \geq 0,$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - q, z^*\rangle \geq 0$. Then, the maximality of A implies $q \in A^{-1}$.

By (38) and (42), we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{58}$$

From (22), we get

$$z_n \to q \quad \text{as} \quad n \to \infty. \tag{59}$$

From $u_n \in T^n x_n$ and let $\{s_n\}$ be a sequence generated by

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$$\begin{cases} s_2 \in Tz_1 \subset T^2x_1; \\ s_3 \in Tz_2 \subset T^3x_2; \\ s_4 \in Tz_3 \subset T^4x_3; \\ \vdots \\ s_n \in Tz_{n-1} \subset T^nx_{n-1}; \\ s_{n+1} \in Tz_n \subset T^{n+1}x_n; \\ \vdots \end{cases} \quad (60)$$

By the assumption that T is uniformly L-Lipschitz continuous and any $z_n \in T^nx_n$ and $a_{n+1} \in Tz_n \subset T^{n+1}x_n$, we have

$$\begin{aligned} \|s_{n+1} - z_n\| &\leq \|s_{n+1} - z_{n+1}\| + \|z_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\leq L\|x_n - x_{n+1}\| + \|z_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - z_n\| \\ &\leq (L+1)\|x_n - x_{n+1}\| + \|z_{n+1} - x_{n+1}\| \\ &\quad + \|x_n - z_n\|. \end{aligned}$$

From (22), (25) and (59) that

$$\lim_{n\to\infty} \|s_{n+1} - z_n\| = 0 \text{ and } \lim_{n\to\infty} s_{n+1} = q. \quad (61)$$

In view of closedness of T, it yields that $q \in Tq$. Therefore $q \in F(T)$.

Finally, we show that $x_n \to q = \Pi_F x_0$. Let $p^* = \Pi_F x_0$. Since $p^* \in F \subset C_n$ and $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(p^*, x_0), \forall n \geq 0.$$

This implies that

$$\phi(q, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(p^*, x_0). \quad (62)$$

In view of definition of $\Pi_F x_0$, we have $q = p^*$. Therefore, $x_n \to q = \Pi_F x_0$. This completes the proof. □

Corollary III.2. *Let E be a real uniformly smooth and uniformly convex Banach space with Kadec-Klee property and let C be a nonempty closed convex subset of E. Let $T : C \to CB(C)$ be a closed and total quasi-ϕ-asymptotically nonexpansive multivalued mapping with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu_1 = 0$, $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and $\zeta(0) = 0$ such that $F := F(T)$ and let a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarity and } C_0 = C, \\ w_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n), z_n \in T^nx_n \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jw_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \alpha_n \phi(z, x_n) \\ \qquad + (1-\alpha_n)phi(z, w_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \quad (63)$$

for $n \in \mathbb{N} \cup \{0\}$, where J is the single-valued duality mapping on E and $\xi_n = \nu_n \sup_{u^* \in F(T)} \zeta(\phi(u^*, x_n)) + \mu_n$. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfying

(i) $0 < \beta_1 \leq \beta_n \leq \beta_2 < 1$,
(ii) $0 \leq \alpha_n \leq \alpha < 1$,

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

IV. Conclusion

We established the strong convergence theorem for fixed points of sequence for multivalued nonexpansive mappings and a zero of maximal monotone operator in Banach spaces. The results of this paper extended and improved the corresponding results given by Tang and Chang [15] and some authors in the literature.

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