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Optimal Control Methods and the Variational Approach to Differential Equations

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***Abstract*—The calculus of variations is an important tool in the study of boundary value problems for differential systems. A development of this approach, called the control variational method, is based on the use of the optimal control theory, especially of the Pontryagin maximum principle. In this presentation, we review the results established in the literature on the control variational method and its applications, in the last decade.**

***Index Terms*—optimal control, variational method, boundary value problems.**

I. Introduction

THE control variational method was introduced in the papers [1], [18] in connection with new efficient methods for the solution of the biharmonic equation and the associated thickness (volume) optimization problems for plates. Several important applications, including the case of nonsmooth Kirchhoff - Love arches and related geometric optimization problems, are colected in the monograph [11], Ch. 3.4 and Ch. 6.1.

One of the simplest and most intuitive examples is the case of a simply supported plate:

$$\Delta(u^3\Delta y) = f \quad \text{in } \Omega, \tag{1.1}$$

$$y = \Delta y = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset R^d$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ ($d \in N$ and the case $d = 2$ corresponds to the plate model), $u \in L^\infty(\Omega)_+$ is the thickness of the plate and $y \in H^2(\Omega) \cap H_0^1(\Omega)$ denotes the deflection of the plate under the load $f \in L^2(\Omega)$. The existence of a unique weak solution to (1.1), (1.2) follows by the Lax - Milgram lemma.

If turns out that (1.1), (1.2) can be equivalently formulated as an unconstrained optimal control problem

$$\underset{h\in L^2(\Omega)}{\text{Min}} \left\{ \frac{1}{2} \int_\Omega l(x)h^2(x)dx \right\}, \tag{1.3}$$

$$\Delta y = lz + lh \quad \text{in } \Omega, \tag{1.4}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where $l = u^{-3} \in L^\infty(\Omega)_+$ and $z \in H^2(\Omega) \cap H_0^1(\Omega)$ is defined as the unique solution of $\Delta z = f$ in Ω.

The equivalence of (1.1), (1.2) with (1.3) - (1.5) becomes quite intuitive if one writes (1.1), (1.2) as a system

$$\Delta y = zl \quad \text{in } \Omega,$$
$$\Delta z = f \quad \text{in } \Omega,$$
$$y = z = 0 \quad \text{on } \partial\Omega,$$

which looks "almost" as the first order optimality conditions (state equation and adjoint equation) associated to (1.3) - (1.5). The equivalence is proved by writing in detail the maximum principle for (1.3) - (1.5), [11].

Although such remarks are simple, one can see already that interesting applications may arise. For instance, the numerical approximation of (1.3) - (1.5) may be performed via the simplest piecewise linear and continuous finite elements in Ω. Consequently, a similar approximation may be obtained for (1.1), (1.2). However, in the mathematical literature, higher order finite elements are generally used for the numerical approximation of (1.1), (1.2), [6], [7].

More numerical results obtained by the control variational method in various linear or nonlinear boundary value problems may be found in [1], [2], [8].

Another, rather surprising, consequence of the above equivalence appears in connection with shape optimization problems associated to (1.1), (1.2). We take the example of the volume minimization problem, under constraints on the thickness and on the deflection:

$$\text{Min} \left\{ \int_\Omega u(x)dx \right\}, \tag{1.6}$$

$$0 < m \leq u(x) \leq M \quad \text{a.e. in } \Omega, \tag{1.7}$$

$$y \in C, \tag{1.8}$$

where $C \subset L^2(\Omega)$ is nonempty and closed, $0 < m < M$ are constants and y, u satisfies (1.1), (1.2) as well.

The application of the control variational method allows to rewrite (1.6) - (1.8), (1.1), (1.2) in the equivalent form:

$$\text{Min} \left\{ \int_\Omega l^{-\frac{1}{3}}(x)dx \right\}, \tag{1.9}$$

$$\Delta y = zl \quad \text{in } \Omega, \tag{1.10}$$

$$(1.11),\qquad y = 0 \quad \text{on } \partial\Omega,$$

$$0 < M^{-3} \leq l(x) \leq m^{-3} \quad \text{a.e. in } \Omega, \tag{1.12}$$

and (1.8).

Notice that (1.6) - (1.8), (1.1), (1.2) is a nonconvex optimization problem due to the nonlinear correspondence $u \to y$ as defined by (1.1), (1.2). However, if C in (1.8) is a convex subset, the transformed problem (1.8) - (1.12) is strictly convex since the application $l \to y$ is linear in (1.10) and (1.9) is strictly convex. We get, [21]:

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Theorem 1.1 If C is convex and the admissibility hypothesis is fulfilled, then the volume optimization problem (1.6) - (1.8), (1.1), (1.2) has a unique optimal pair $[y^*, u^*] \in H^2(\Omega) \times L^\infty(\Omega)_+$.

The shape optimization problem may have infinitely many local optimal pairs, but the global optimum is unique. Such convexity and/or uniqueness properties are very rare in shape optimization [9].

Another important remark is that the substitution of lh as given by (1.4), in (1.3) and simple computations involving the definitions of l, z show that the cost functional (1.3) represents, up to a constant, the usual energy functional associated to (1.1), (1.2).

Therefore, (1.3) - (1.5) is, in fact, a reformulation of the classical Dirichlet principle associated to the biharmonic operator. Consequently, the control variational method is a modification of the classical variational approaching, via the use of optimal control theory. From this point of view, it is also important to notice that our method is essentially different from the optimal control approaches obtained via the least squares fitting to the data procedure applied in various situations. Moreover, the control variational approach allows many variants of such modifications. This flexibility may be very advantageous in certain applications. We shall exemplify it in the next section via some abstract schemes. Section 3 is devoted to applications to unilateral problems where constrained control problems have to be used. The last section briefly discusses the case of time-dependent problems, which is more difficult and still under development. The paper ends with a short conclusion.

For various applications of the control variational method, we quote [2], [8], [16], [17], [19], [20], [22].

II. ABSTRACT VARIANTS

Let $V \subset H \subset V^*$ be separable Hilbert spaces with dense and continuous embeddings, endowed with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$, the pairing $(\cdot,\cdot)_{V\times V*}$ and the norms $|\cdot|_V$, $|\cdot|_H$, $|\cdot|_{V^*}$ respectively. Let $A_1, A_2 : V \to V^*$ be linear, bounded symmetric operators with A_1 positively defined and $\varphi : V \to]-\infty, +\infty]$ be lower semicontinuous, proper, satisfying the following subquadratic growth (descent) condition at infinity:

$$(2.1) \qquad \varphi(x) \geq -c|x|_V^\alpha + \beta, \text{if } |x|_V \geq K > 0$$

where $K > 0$, $\alpha \in]0, 2[$, $c > 0$, $\beta \in R$ are some constants. In (2.1), it is possible to choose $\alpha = 2$ if $c > 0$ is dominated by the coercivity constant of A_1.

We consider first the linear equation

$$(2.2) \qquad (A_1 + A_2)y = f \in H.$$

The existence of a unique solution of (2.2) is assumed (or stronger hypotheses may be imposed on A_2 in order to prove it).

We also consider the nonlinear multivalued equation:

$$(2.3) \qquad A_1y + A_2y + Fy \ni f$$

where $F = \partial\varphi$ is the Clarke [4] subdifferential. Equations (2.2), (2.3) may be discussed in the case when A_1 is nonlinear as well (for instance, A_1 is the p - Laplacian, $p > 1$). We associate to (2.2), respectively (2.3), the following unconstrained optimal control problems:

$$(2.4) \quad \operatorname*{Min}_{u\in V^*} \{(u, y)_{V^*\times V} - 3(u, g)_{V^*\times V} + (A_2y, y)_{V^*\times V}\}$$

subject to:

$$(2.5) \qquad A_1y = u - f,$$

where $g \in V$ is the unique solution of $A_1g = f$;

The second control problem is

$$(2.6) \quad \operatorname*{Min}_{u\in U} \{< u, Gy >_{U\times U^*} -3 < u, Gg >_{U\times U^*} + +(A_2y, y)_{V^*\times V} + 2\varphi(y)\}$$

subject to

$$(2.7) \quad (A_1y, v)_{V^*\times V} =< u, Gv >_{U\times U^*} -(f, v)_{V^*\times V}, \ \forall\, v \in V,$$

where $G : V \to U^*$ is a linear continuous operator and U is another Hilbert space with dual U^* and $< \cdot,\cdot >_{U\times U^*}$ is their pairing. Obviously, the state equation (2.7) may be rewritten in the more usual form

$$(2.8) \qquad A_1y = G^*u - f$$

where $G^* : U \to V^*$ is the adjoint operator of G. However, in certain applications G^* may be difficult to compute and it is easier to use (2.7), which is the weak form of (2.8).

The coercivity of A_1 ensures the existence of a unique solution $y \in V$ to (2.5), respectively (2.7). That is, the optimal control problems (2.4), (2.5), respectively (2.6), (2.7) are well defined. They have a simple structure due to the absence of constraints.

The abstract equations (2.2), respectively (2.3) model elliptic boundary value problems or nonlinear problems like variational inequalities, quasi - variational inequalities [10], [12]. In such applications, the corresponding optimal control problems may involve state or control constraints as well [1], [14], [17] (see Section 3). The aim of introducing the optimal control problems (2.4), (2.5), respectively (2.6), (2.7) is to separate the ”good” operator A_1 from the possibly singular operator A_2 or the nonlinear operator F. One should notice that in the optimal control problems just operator A_1 has to be inverted, while operators A_2, F have simply to be computed in the corresponding cost functional.

The following results from [14], [21] show that the solution of the equations may be replaced by the solution of the corresponding optimal control problem. Its proof is based on the Pontryagin maximum principle.

Theorem 2.1 If $[y^*, u^*]$ is an optimal pair of (2.4), (2.5), respectively (2.6), (2.7), then y^* is a solution of (2.2), respectively (2.3).

Remark In the above theorem, the existence of the solutions is assumed and the uniqueness plays no role. Supplementary arguments and hypotheses are necessary in order to discuss such properties. Moreover, it is clear from the formulations of the control problems (2.4), (2.5), respectively (2.6), (2.7) that to some given linear/nonlinear equation, the associated control problem via the control variational method is not unique. In fact one may associate an infinity of such control problems to any equation of the form

$$Ay = f$$

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by choosing various (convenient) decompositions $A = A_1 + A_2$. In particular, the classical calculus of variational approach (the Dirichlet principle) is a special case of the problems discussed in this section. See [21] for a more detailed presentation.

We close this section with an example of application of the control variational method to a multiscale elliptic problem in $\Omega \in R^2$:

$$\frac{\partial^2 y}{\partial x_1} + \varepsilon \frac{\partial^2 y}{\partial x_2} = f \text{ in } \Omega, \tag{2.9}$$

$$y = 0 \text{ on } \partial\Omega, \tag{2.10}$$

where $f \in L^2(\Omega)$ and $\varepsilon > 0$ is "very small". Other boundary conditions may be considered as well in (2.4), (2.10).

The associated unconstrained optimal control problem is

(2.11)

$$\text{Min} \left\{ \frac{1}{2}|w|^2_{L^2(\Omega)^2} - \frac{1-\varepsilon}{2} \left| \frac{\partial y}{\partial x_2} \right|^2_{L^2(\Omega)} - \int_\Omega fy dx \right\},$$

$$\int_\Omega \nabla y \cdot \nabla \psi = \int_\Omega w \cdot \nabla \psi, \ \forall\, \psi \in H_0^1(\Omega). \tag{2.12}$$

The symbol "$\cdot$" is the scalar product in R^2.
Notice that the state equation (2.12) is exactly the Laplace equation in the weak form.

We denote by $[w^*, y^*]$ an optimal pair for (2.11), (2.12) assumed to exist. The Euler - Lagrange equation corresponding to (2.11), (2.12) is

$$\int_\Omega w^* v dx - (1-\varepsilon) \int_\Omega \frac{\partial y^*}{\partial x_2} \frac{\partial z}{\partial x_2} dx - \int_\Omega fz dx = 0$$

with $[v, z]$ arbitrary and satisfying the equation in variations

$$\int_\Omega \nabla z \cdot \nabla \psi = \int_\Omega v \cdot \nabla \psi, \ \forall\, \psi \in H_0^1(\Omega)$$

(which is again just the Laplace equation).

Choosing $v = \nabla z$ (which is admissible), for arbitrary $z \in H_0^1(\Omega)$ and combining with (2.12), we get

$$0 = \int_\Omega w^* \nabla z dx - (1-\varepsilon) \int_\Omega \frac{\partial y^*}{\partial x_2} \frac{\partial z}{\partial x_2} dx - \int_\Omega fz dx =$$

$$= \int_\Omega \nabla y^* \cdot \nabla z dx - (1-\varepsilon) \int_\Omega \frac{\partial y^*}{\partial x_2} \frac{\partial z}{\partial x_2} dx - \int_\Omega fz dx =$$

$$= \int_\Omega \frac{\partial y^*}{\partial x_1} \frac{\partial z}{\partial x_1} dx + \varepsilon \int_\Omega \frac{\partial y^*}{\partial x_2} \frac{\partial z}{\partial x_2} dx - \int_\Omega fz, \ \forall\, z \in H_0^1(\Omega).$$

which is exactly the weak formulation of (2.9), (2.10).

Remark It is possible to prove that the problem (2.11), (2.12) has a unique optimal pair by the direct method in the calculus of variations.

Remark The boundary value problem (2.9), (2.10) is an example of "stiff" problems, studied for instance in [13] in connection with the "locking problem" - the loss of the numerical stability due to the presence of the "very small" parameter $\varepsilon > 0$, [3].

The control variational method offers alternative approaches for the solution of this difficulty. Such multiscale problems arise frequently in the setting of thin elastic structures like beams, arches, curved rods, plates or shells. The "very small" parameter is the thickness of the structure which enters into the coefficients of the differential system modelling the behavior of the structure.

III. CONSTRAINED CONTROL PROBLEMS

We start with an example of a variational inequality with unilateral conditions in the whole domain $\Omega \subset R^d$, associated to plate models. The weak formulation is given by

(3.1)

$$\int_\Omega u^3 \Delta y (\Delta y - \Delta w) dx \le \int_\Omega f(y-w) dx, \ \forall\, w \in K \cap H_0^2(\Omega).$$

Here, K is a closed convex subset in $H^2(\Omega)$ and some examples are:

$$K = \{w \in H^2(\Omega) \cap H_0^1(\Omega);\ a \le w \le b \text{ a.e. } \Omega\}, \tag{3.2}$$

$$K = \{w \in H^2(\Omega) \cap H_0^1(\Omega);\ a \le \Delta w \le b \text{ a.e. } \Omega\}, \tag{3.3}$$

with $a, b \in R$, $a < 0 < b$. The second example was discussed in [6] by a different method. The case of plate models corresponds to $d = 2$ and the physical interpretation of $u \in L^\infty(\Omega)_+$, $y \in K \cap H_0^2(\Omega)$, $f \in L^2(\Omega)$ is as in Section 1. The existence of a unique weak solution $w \in K$ for (3.1), with K given by (3.2) or (3.3) is obtained immediately by studying the minimization of the associated energy:

$$\underset{y \in K \cap H_0^2(\Omega)}{\text{Min}} \left\{ \frac{1}{2} \int_\Omega u^3 (\Delta y)^2 dx - \int_\Omega fy dx \right\}. \tag{3.4}$$

In (3.1) or in (3.4), the boundary conditions imposed on y are

$$y = 0, \ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega$$

which correspond to the case of clamped plates. The formulation as an optimal control problem takes into account just the first boundary condition in (3.5), while the second is penalized in the cost functional:

$$\text{Min} \left\{ \frac{1}{2\varepsilon} \int_{\partial\Omega} \left(\frac{\partial y}{\partial n} \right)^2 d\tau + \frac{1}{2} \int_\Omega l h^2 dx \right\}, \ \varepsilon > 0, \tag{3.6}$$

subject to

$$\Delta y = lz + lh \text{ in } \Omega, \tag{3.7}$$

$$y = 0 \text{ on } \partial\Omega, \tag{3.8}$$

$$y \in K. \tag{3.9}$$

The notations z, l in (3.7) are as in (1.4) and the unilateral condition in (3.1) is expressed via the state constraint (3.9).

Since $l \ge c > 0$, then the cost functional (3.6) is coercive (and strictly convex) and we get the existence of a unique optimal pair denoted by $[y_\varepsilon, u_\varepsilon] \in K \times L^2(\Omega)$. The relationship between the problems (3.1) and (3.6) - (3.9) is given by an approximation property [18], for $\varepsilon \to 0$:

Theorem 3.1 We have $y_\varepsilon \to y^*$ weakly in $H^2(\Omega)$, $\frac{\partial y_\varepsilon}{\partial n} \to 0$ strongly in $L^2(\partial\Omega)$ and $y^* \in K \cap H_0^2(\Omega)$. Moreover, y^* is the unique solution of (3.1).

Notice that the constrained optimal control problem (3.6) - (3.9) is an alternative formulation of the Dirichlet principle (3.4) for the approximation of the (nonlinear) boundary value problem (3.1).
As in Section 1, the possibility to solve numerically (3.6) - (3.9) by using simple finite elements is an obvious advantage over the usual treatment in the literature via higher order elements [6], [7]. Applications of the control variational method to variational inequalities associated to the general linear elasticity system (for isotropic materials) are discussed in [14], [20]. For instance, in these works it is shown that one may solve the linear elasticity system by using just the Laplace equation.
We continue with the case of variational inequalities for Kirchhoff - Love arches, including examples of unilateral conditions on the boundary. The weak variational formulation is given by:

$$(3.10)\quad \frac{1}{\delta}\int_0^1 ((v_1^\delta)' - cv_2^\delta)((v_1^\delta)' - cv_2^\delta - w_1' + cw_2)ds + \int_0^1 ((v_1^\delta)' + cv_1^\delta)'((v_2^\delta)' + cv_1^\delta - w_2' - cw_1)'ds \le$$

$$\le \int_0^1 f_1(v_1^\delta - w_1)ds + \int_0^1 f_2(v_2^\delta - w_2)ds,$$

$$\forall\, [w_1, w_2] \in \mathcal{C} \cap [V \times U].$$

Here c denotes the curvature of the arch parametrized by $\varphi : [0,1] \to R^2$ with constant thickness given by $\sqrt{\delta}$ (fixed) and clamped at the left end. The mappings $[v_1^\delta, v_2^\delta] \in \mathcal{C}$ denote the tangential, respectively the normal components of the deflection of the arch under the load $[f_1, f_2] \in L^2(0,1)^2$, decomposed similarly in the local bases along the arch. Moreover, $\mathcal{C} \subset L^\infty(0,1)^2$ is a closed convex nonvoid subset and $V = \{w \in H^1(0,1); w(0) = 0\}$, $U = \{z \in H^2(0,1); z(0) = z'(0) = 0\}$.

The formulation (3.10) implicitely assumes the existence of at least three derivates (and their integrability) for the parametrization φ, since the curvature c and its derivative appear in (3.10).

Examples of convex sets C are given by

$$(3.11)\quad C = L^\infty(0,1) \times \{v_2 \in L^\infty(0,1); a \le v_2 \le b \text{ a.e. in } [0,1]\},$$

$$(3.12)\qquad C = \{[v_2, v_2] \in V \times U; v_1(1) \ge r\},$$

$a, b, r \in R$ given constants.

Example (3.11) corresponds to unilateral coditions (the obstacle problems) for the normal component of the deflection, while (3.12) corresponds to a partially clamped arch with unilateral conditions on the tangential component in the right end. If C includes null boundary condition in both ends of the arch, then we obtain a unilateral problem for clamped arches.

In order to write the control variational formulation of (3.10), we introduce the angle $\theta : [0,1] \to R$ between the tangent to the arch, given by φ', and the horizontal axis. If φ is smooth, then $\theta' = c$. We also introduce the orthogonal matrix

$$(3.13)\qquad W(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & cos\theta(t) \end{pmatrix}$$

and the mappings g_1, g_2, h, l constructed from f_1, f_2 as follows:

$$(3.14)\qquad \begin{bmatrix} l \\ h \end{bmatrix}(t) = \int_t^1 W(t)W^{-1}(s)\begin{bmatrix} f_1(s) \\ f_2(s) \end{bmatrix} ds,$$

$$(3.15)\qquad g_1 = \delta l,\ -g_2'' = h,\ g_2(0) = g_2'(0) = 0.$$

The optimal control problem associated to (3.10) has a less intuitive formulation in this case and reads as follows:

$$(3.16)\qquad \underset{[u,z]\in L^2(0,1)\times V}{\mathrm{Min}} \left\{ \frac{1}{2}\int_0^1 u^2(s)ds + \frac{1}{2}\int_0^1 z'(s)^2 ds \right\},$$

subject to the state system

$$(3.17)\qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}(t) = \int_0^t W(t)W^{-1}(s)\begin{bmatrix} u(s) + g_1(s) \\ z(s) + g_2(s) \end{bmatrix} ds$$

and the state constraint

$$(3.18)\qquad [v_1, v_2] \in \mathcal{C}.$$

An important observation is that the constrained optimal control problem (3.16) - (3.18), under the notations (3.13) - (3.15), makes sense for $\theta \in L^\infty(0,1)$, that is, here, it is enough to assume the parametrization φ of the arch to be Lipschitzian (or even less according to [17]). Under the standard regularity assumptions from the literature, we show that the unique optimal pair of (3.16) - (3.18) has the property [11], [18]:

Theorem 3.2 If $\varphi \in W^{3,\infty}(0,1)^2$, then the optimal state of (3.16) - (3.18), denoted by $[v_1^\delta, v_2^\delta]$ satisfies (3.10).

Remark Besides the important property that the formulation (3.16) - (3.18) of the arch problem is valid under much less regularity assumptions on the geometry, it has another surprising advantage. Namely, when $\mathcal{C}$ is the whole space (i.e. the variational inequality (3.10) becomes an equation) one obtains explicit formulas for its solution [8]. This allows further applications in the difficult case of shape optimization problems associated of Kirchhoff - Love arches. The explicit formula also gives a complete solution of the "locking problem", in this case (caused by the presence of the "very small" parameter $\delta > 0$ in (3.10)), [8], [3].

IV. TIME - DEPENDENT PROBLEMS

Although in the case of parabolic or hyperbolic equations, one can define the associated energy [5], however the generalization of the Dirichlet principle to the case of evolution equations is not obvious.

In this section, we report on the posibility to apply the control variational method. The existing results are weaker

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than in the elliptic case and this may be a fruitful research direction for the future.

We define directly the optimal control problem (with respect to both the time and space variables):

$$\text{(4.1)} \qquad \underset{h}{\mathrm{Min}} \left\{ \int_0^T |h(x,t)|^2_{L^2(\Omega)^d} dt + \int_\Omega |y(x,T)|^2 dx \right\}$$

subject to

$$\text{(4.2)} \qquad \nabla_x y(x,t) = h(x,t) + l(x,t) \text{ in } \Omega \times]0,T[,$$

where $T > 0$ is the given final time moment, $\Omega \subset R^d$ is a bounded domain with a sufficiently smooth boundary, $h \in L^2(0,T;L^2(\Omega)^d)$ is the control parameter and l is defined by

$$\text{(4.3)} \qquad -\mathrm{div}_x l(x,t) = f(x,t) \text{ in } \Omega \times]0,T[$$

with $f \in L^2(0,T;L^2(\Omega))$ given.
Notice that (4.3) does not define l uniquely and we fix l by choosing $l = \nabla_x p$, where $p \in L^2(0,T;H^1_0(\Omega))$ satisfies $-\Delta p(x,t) = f(x,t)$ and $t \in [0,T]$ is interpreted as a parameter.

The problem (4.1), (4.2) has the structure of an unconstrained control problem (cost functional and state system). Due to the presence of $t \in [0,T]$ as a parameter, it is easier to understand it as a constrained minimization problems: the set of admissible pairs $[y,h] \in \left\{C(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))\right\} \times L^2(0,T;L^2(\Omega)^d)$ is constrained by relation (4.2).

It has been established in [22] that

Theorem 4.1 If $[y^*,h^*]$ is a sufficiently smooth optimal pair for (4.1), (4.2), then

$$\text{(4.4)} \qquad \int_0^T y^*_t(x,t)dt - \int_0^T \Delta y^*(x,t)dt = \int_0^T f(x,t)dt - y^*(x,0) \text{ in } \Omega.$$

Remark Although the cost functional (4.1) is coercive in $h \in L^2(0,T;L^2(\Omega)^d)$, the existence of optimal pairs in (4.1), (4.2) is not clear due to the required continuity with respect to the time of the state y.

Remark Relation (4.4) may be viewed as a generalized weak form of the corresponding parabolic equation and *Theorem 4.1* is similar to the results established in the previous sections.

Finally, we indicate an application to a hyperbolic equation. We take $\Omega =]a,b[\subset R$, $Q =]0,T[\times \Omega$ and we define the unconstrained control problem

$$\text{(4.5)} \qquad \mathrm{Min} \left\{ -\frac{1}{2} \int_Q |z_x|^2 dx + \frac{1}{2} \int_Q w^2 dx \right\}$$

subject to

$$\text{(4.6)} \qquad z_t = w + f \text{ in } Q,$$

$$\text{(4.7)} \qquad z(0,x) = z_0(x) \text{ in } \Omega.$$

Here $f \in L^2(Q)$ is given, $w \in L^2(Q)$ is the control mapping and (4.6) has to be understood as an ordinary differential equation, with $x \in]a,b[$ as parameter. We assume that its solution has enough smoothness in order that the cost functional makes sense.

Again by writing the Euler - Lagrange equation associated to (4.5) - (4.7) and some computations, one can show that for any solution of (4.5) - (4.7), denoted by $[w^*,z^*] \in L^2(Q) \times H^1(Q)$ (and assumed to exist), the optimal state z^* satisfies the hyperbolic equation in the generalized weak form:

$$0 = \int_Q z_t q_t - \int_Q z_x q_x - \int_Q f q_t$$

for any test function $q \in H^1_0(Q)$.

Remark In order to obtain pointwise in time information for the obtained evolution equations and for the corresponding boundary/initial conditions, time-dependent variations in the control problems (4.1), (4.2), respectively (4.5) - (4.7) have to be used [22]. Their form is not clear.

V. Conclusion

The variational method plays a fundamental role in the study of differential equations. In this paper, a brief presentation of the control variational method and its applications is performed. It is mainly based on the papers published by the author and his coworkers during the last decade.

The control variational method uses the optimal control theory instead of the classical calculus of variations, in the minimization problems obtained via the Dirichlet principle. It may be extended to time-dependent problems and it is very flexible in the sense that to a given differential equation, one may associate "many" optimization problems characterizing its solution. The most important tool in the analysis of the control variational method is the Pontriagyn maximum principle.

The control variational method is relevant both from the theoretical and the numerical points of view. Numerical experiments show that it is very efficient in difficult examples like shape optimization problems, which are not convex in general. Important theoretical advances via the control variational approach are the explicit formula for the solution of the Kirchhoff - Love arches or the treatment of plates with discontinuous thickness.

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