Proceedings of the International MultiConference of Engineers and Computer Scientists 2013 Vol I,
IMECS 2013, March 13 - 15, 2013, Hong Kong

Iterative Algorithm for Minimum-Norm of Fixed Point for Nonexpansive Mapping and Convex Optimization Problems

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***Abstract*—In this paper, we introduce an explicit method for finding the least norm of fixed points for strict pseudo mappings by using the projection technique. We provide algorithm which strong convergence theorems are obtained in Hilbert spaces. Then, we apply these algorithm to solve some convex optimization problems. The results of this paper extend and improve several results presented in the literature in the recent past.**

***Index Terms*—Monotone mapping, Nonexpansive mapping, Explicit Method, Minimum-Norm,Variational inequality.**

I. Introduction

CONSIDER a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. First, we recall the basic concept of mappings as shown in the following:

- A is called *monotone* if and only if for each $x, y \in C$,
$$\langle x - y, Ax - Ay \rangle \geq 0.$$
- A is said to be strongly positive on H if there exists a constant $\overline{\gamma} > 0$ such that
$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$
- A mapping A of C into H is called *α-inverse-strongly monotone* if there exists a positive real number α such that
$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2,$$
for all $x, y \in C$.
- A is an α-inverse strongly monotone mapping of C into H, such that
$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|$$
for all $x, y \in C$.
- A mapping S from C into itself is said to be a nonexpansive mapping if
$$\|Sx - Sy\| \leq \|x - y\|.$$
for any $x, y \in C$.

In this work, we may assume that $Fix(S) \neq \emptyset$, which $Fix(S)$ is closed and convex. So there exists a unique $x^* \in Fix(S)$ satisfies the following :
$$\|x^*\| = \min\{\|x\| : x \in Fix(S)\}.$$
That is, x^* is the minimum-norm fixed point of S.

Since 1967, Halpern introduced an explicit iterative scheme as shown in the following:
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n, \ \forall n \geq 0,$$
where $\{\alpha_n\} \subset [0, 1]$. He proved that the convergence theorem which the Halpern's iterative method do find the minimum-norm fixed point x^* of S if $0 \in C$. In 2004, Xu studied the iteration process $\{x_n\}$ called viscosity approximation method as shown in the following:
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \ \text{ for } n \geq 1,$$
where $\{\alpha_n \subset (0, 1)\}$ and $f : C \to C$ is a contraction. He also proved the strong convergence theorem of the sequence $\{x_n\}$ which generated by the above scheme under the appropriate conditions.

Recently, Yao and Xu [5] independently introduced two iterative methods for finding the minimum-norm fixed point of nonexpansive mapping which is defined on closed convex subset C of H. The proposed algorithms are based on the well-known Browder's iterative method [1] and Halpern's iterative method [2]. .

Motivated and inspired by the previous mentioned researches, we present new strongly convergent methods for approximating minimum-norm fixed point of a nonexpansive mapping and variational inequality for an α-inverstrongly monotone operator such that
$$F(S) \cap VI(C, A) \neq \emptyset,$$
and for each $\lambda \in (0, 1)$,
$$x_{n+1} = (1 - \alpha_n)(\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n), \quad (1)$$
where $\{\alpha_n\} \subset (0, 1)$.

We prove that the sequence $\{x_n\}$ generated by (1) converges strongly to the element of minimal norm fixed point of a nonexpansive mapping. As application, we provide iterative processes for solving the constrained convex optimization problem.

II. Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space H. We use the following notions in the sequel:

1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence,
2) $\omega_w(x_n) = x : \exists x_j \rightharpoonup x$ denotes the weak ω - limit set of x_n.

Recall that the orthogonal projection
$$P_C x = arg \min_{y \in C} \|x - y\|. \quad (2)$$

This work was supported by the Higher Education Research Promotion and National Research University of Thailand, office of Higher Education Commission.

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ISBN: 978-988-19251-8-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The orthogonal projection has the following well-known properties. For a given $x \in H$,

1) $\langle x - P_C x, z - P_C x\rangle \leq 0$, for all $z \in C$;
2) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$, for all $x, y \in H$.

We shall make use of the following results.

Lemma II.1. *(Demiclosedness principle of nonexpansive mapping) Let $S : C \to C$ a nonexpansive mapping with $Fix(S) \neq \emptyset$. If $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, then $x = Sx$.*

Lemma II.2. *(see, [3] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3}$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma II.3. *(see, [4]) Let $\{a_n\}$ be a nonnegative real sequence satisfying the following inequality :*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ such that $\Sigma_{n=0}^{\infty}\gamma_n = +\infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

III. Main result

Theorem III.1. *Let C be a closed convex of a real $Hilbert$ space H. Let A be an α-inverse strongly monotone. Let $S : C \to C$ be a nonexpansive mapping and $\Omega := F(S) \cap VI(C, A) \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:*

1) $\lim_{n\to\infty} \alpha_n = 0$;
2) $\Sigma_{n=0}^{\infty}\alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n + (1 - \lambda)x_n] \tag{4}$$

converges strongly to a fixed point of S which is a minimal norm and the unique solution of the variational inequality:

$$x^* \in \Omega, \langle x^*, x - x^*\rangle \geq 0, \forall x \in \Omega.$$

Proof: **Step 1.** we prove that the sequence $\{x_n\}$ is bounded. Let $q \in \Omega$. By (4), we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1 - \alpha_n)[\lambda SP_C(I - \lambda A)x_n \\
&\quad + (1 - \lambda)x_n] - q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)(x_n - q) \\
&\quad + \lambda(SP_C(I - \lambda A)x_n - q)] - \alpha_n q\| \\
&\leq \|(1 - \alpha_n)[(1 - \lambda)\|(x_n - q)\| \\
&\quad + \lambda\|(x_n - q)\|]\| - \|\alpha_n q\| \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

By induction, it follows that

$$\|x_n - q\| \leq \max\{\|x_0 - q\|, \|q\|\},$$

for all $n \geq 0$. Then $\{x_n\}$ is bounded. Therefore, $\{SP_C(I - \lambda A)x_n\}$ is also bounded.

Let $y_n = \frac{(1-\alpha_n)\lambda SP_C(I-\lambda A)x_n}{\alpha_n+(1-\alpha_n)\lambda}$, then the iterative sequence (4) is equivalent to

$$x_{n+1} = (\alpha_n + (1 - \alpha_n)\lambda)y_n + (1 - \alpha_n - (1 - \alpha_n)\lambda)x_n. \tag{5}$$

Since $\lim_{n\to\infty}(\alpha_n + (1 - \alpha_n)\lambda) = \lambda$, then

$$\begin{aligned}
\|y_n - q\| &= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda} - q\right\| \\
&= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - (\alpha_n + (1 - \alpha_n)\lambda)q}{\alpha_n + (1 - \alpha_n)\lambda}\right\| \\
&= \left\|\frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n - \alpha_n q - (1 - \alpha_n)\lambda q}{\alpha_n + (1 - \alpha_n)\lambda}\right\| \\
&\leq \frac{(1 - \alpha_n)\lambda\|x_n - q\| - \alpha_n\|q\|}{\alpha_n + (1 - \alpha_n)\lambda} \\
&= \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda}\|q\| \\
&\quad + (1 - \frac{\alpha_n}{\alpha_n + (1 - \alpha_n)\lambda})\|x_n - q\| \\
&\leq \max\{\|x_n - q\|, \|q\|\}.
\end{aligned}$$

Thus, $\{y_n\}$ is bounded. Hence by nonexpansiveness of S and P_C, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
= &\left\|\frac{(1 - \alpha_{n+1})\lambda SP_C(I - \lambda A)x_{n+1}}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\right. \\
&\left. - \frac{(1 - \alpha_n)\lambda SP_C(I - \lambda A)x_n}{\alpha_n + (1 - \alpha_n)\lambda}\right\| - \|x_{n+1} - x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|SP_C(I - \lambda A)x_{n+1} \\
&- SP_C(I - \lambda A)x_n\| \\
&+ \left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right| \\
&\times\|SP_C(I - \lambda A)x_n\| \\
\leq &\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda}\|Sx_{n+1} - Sx_n\| \\
&+ \left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right| \\
&\times\|SP_C(I - \lambda A)x_n\| - \|x_{n+1} - x_n\| \\
\leq &\left(\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - 1\right)\|x_{n+1} - x_n\| \\
&+ \left|\frac{(1 - \alpha_{n+1})\lambda}{\alpha_{n+1} + (1 - \alpha_{n+1})\lambda} - \frac{(1 - \alpha_n)\lambda}{\alpha_n + (1 - \alpha_n)\lambda}\right| \\
&\times\|SP_C(I - \lambda A)x_n\|.
\end{aligned}$$

From $\{x_n\}$ and $\{SP_C(I - \lambda A)x_n\}$ are bounded sequences and $\lim_{n\to\infty} \alpha_n = 0$, then

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma II.2, we obtain that $\lim_{n\to\infty} \|y_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(\alpha_n + (1 - \alpha_n)\lambda)\|y_n - x_n\| = 0. \tag{6}$$

ISBN: 978-988-19251-8-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

On the other hand, we consider

$$\begin{aligned}
& \|x_n - SP_C(I-\lambda A)x_n\| \\
\leq\ & \|x_n - x_{n+1}\| + \|x_{n+1} - SP_C(I-\lambda A)x_n\| \\
=\ & \|x_n - x_{n+1}\| + \|(1-\alpha_n)(\lambda SP_C(I-\lambda A)x_n \\
& +(1-\lambda)x_n) - SP_C(I-\lambda A)x_n\| \\
\leq\ & \|x_n - x_{n+1}\| + (1-\alpha_n)(1-\lambda) \\
& \|x_n - SP_C(I-\lambda A)x_n\| \\
& +\alpha_n\|SP_C(I-\lambda A)x_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|x_n - SP_C(I-\lambda A)x_n\| \\
\leq\ & \frac{1}{1-(1-\alpha_n)(1-\lambda)}\|x_n - x_{n+1}\| \\
& +\frac{1}{1-(1-\alpha_n)(1-\lambda)}\alpha_n\|SP_C(I-\lambda A)x_n\| \\
& \longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Step 2. we prove that $\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle \leq 0$.

Since $\{x_n\}$ is bounded. Then, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, x^*\rangle.$$

Again, since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_i} \rightharpoonup x^{'}$. Consequently,

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x^{'}, x^*\rangle \leq 0.$$

From (6) it follows that

$$\limsup_{n\to\infty}\langle x^* - x_{n+1}, x^*\rangle = \langle x^* - x^{'}, x^*\rangle \leq 0.$$

Notice that $\lim_{n\to\infty}\|x_n - SP_C(I-\lambda A)x_n\| = 0$. By the demiclosedness principle of nonexpansive mapping $SP_C(I-\lambda A)$, we have $x' \in \Omega$. Since $x^* = P_\Omega(I-\lambda A)$. It follows from the properties of projection operator that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \langle x^* - x', x^*\rangle \leq 0. \tag{7}$$

By (III.1), we have

$$\begin{aligned}
& \|x_{n+1} - (1-\alpha_n)x^*\|^2 \\
=\ & \|[(1-\alpha_n)\lambda SP_C(I-\lambda A)x_{n+1} + (1-\lambda)x_n] \\
& -(1-\alpha_n)x^*\|^2 \\
=\ & \|(1-\alpha_n)[\lambda SP_C(I-\lambda A)x_n + (1-\lambda)x_n] - x^*\|^2 \\
=\ & \|(1-\alpha_n)[\lambda SP_C x_n - (1-\lambda)x_n] - (1-\lambda+\lambda)x^*\|^2 \\
\leq\ & (1-\alpha_n)\|\lambda(Sx_n - x^*) + (1-\lambda)(x_n - x^*)\|^2 \\
\leq\ & (1-\alpha_n)\|\lambda(x_n - x^*) + (1-\lambda)(x_n - x^*)\|^2 \\
\leq\ & (1-\alpha_n)\|x_n - x^*\|^2.
\end{aligned} \tag{8}$$

Observe that

$$\begin{aligned}
& \|x_{n+1} - (1-\alpha_n)x^*\|^2 \\
=\ & \|x_{n+1} - x^*\|^2 - 2\alpha_n\langle -x^*, x_{n+1} - x^*\rangle + \alpha_n^2\|x^*\|^2 \\
\geq\ & \|x_{n+1} - x^*\|^2 - 2\alpha_n\langle x_{n+1} - x^*, x^*\rangle.
\end{aligned}$$

Therefore by (8) and (9), we get

$$\|x_{n+1} - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle x_{n+1} - x^*, x^*\rangle. \tag{9}$$

By the condition (ii) and the inequality (7), we can apply Lemma (II.3) to (9) and conclude that $\{x_n\}$ converges strongly to x^* as $n\to\infty$ that is, the minimum - norm fixed point of S. This completes the proof. ■

Remark III.2. Theorem III.1 also improve the [[5], Theorem 3.2], in which the restrictions $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < +\infty$ or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$ are removed.

Corollary III.3. *Let C be a closed convex of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $\Omega := F(S) \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:*

1) $\lim_{n\to\infty}\alpha_n = 0$*;*
2) $\Sigma_{n=0}^{\infty}\alpha_n = +\infty$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = (1-\alpha_n)[\lambda Sx_n + (1-\lambda)x_n] \tag{10}$$

converges strongly to a fixed point of S which is a minimal norm and the unique solution of the variational inequality:

$$x^* \in \Omega, \langle x^*, x - x^*\rangle \geq 0, \forall x \in \Omega.$$

Proof. If we take $A = 0$. So, by Theorem III.1, we obtain the the following corollary. □

IV. Applications to convex optimization problem

In this section,we apply the proposed methods for approximating the minimum-norm solution of convex function and split feasibility problems. Let's recall that standard constrained convex optimization problem as follows :

$$\text{find } x^* \in C, \text{such that } f(x^*) = \min_{x\in C} f(x), \tag{11}$$

where $f : C \to R$ is a convex, Fréchet differentiable function, C is closed convex subset of H.

It is known that the above optimization problem is equivalent to the following variational inequality:

$$\text{find } x^* \in C, \text{such that } \langle v - x^*, \nabla f(x^*)\rangle \geq 0, \text{for all } \ v \in C, \tag{12}$$

where $\nabla f : H \to H$ is the gradient of f.

It is well-known that the optimality condition (12) is equivalent to the following fixed point problem:

$$x^* = P_C(I - \mu\nabla f)x^*,$$

where P_C is the metric projection onto C and $\mu > 0$ is positive constant. Based on the fixed point problem, we deduce the projected gradient method.

$$\begin{cases} x_0 \in C, \\ x_{n+1} = x_n - \mu\nabla f(x_n), \quad n \ \geq \ 0. \end{cases} \tag{13}$$

Using Theorem III.1, we immediately obtain the following result.

Theorem IV.1. *Assume that the solution set of (11) is nonempty. Let the objective function f be convex, fréchet differentiable and it gradient ∇f is Lipschitz continuous with Lipschitz constant L. In addition, if $0 \in C$ or C is closed convex cone. Let $\mu \in (0, \frac{2}{L})$ and define a sequence $\{x_n\}$ by following*

$$x_{n+1} = (1-\alpha_n)((I - \mu\nabla f)(x_n) + (1-\lambda)x_n), \quad n \geq 0$$

where $\lambda \in (0,1)$ *and the sequence* $\{\alpha_n\} \subset (0,1)$ *satisfies conditions in Theorem III.1. Then the sequence* $\{x_n\}$ *converges strongly to the minimum-norm solution of the minimization (11).*

Proof. Since ∇f is lipschitz continuous with constant L, then well-known that the $P_C(I-\mu\nabla f)$ is nonexpansive mapping. Replace the mapping $(P_C(I-\lambda A))$ with $P_C(I-\mu\nabla f)$ and take $S = I$ in (1). Therefore, the conclusion of this Theorem IV follows from Corollary III.3 immediately.

V. Conclusion

In this paper we obtained a new strong convergence theorem for approximating minimum-norm fixed point of a nonexpansive mappings and variational inequality for an α-inverse strongly monotone operator in a real Hilbert space. Furthermore, as application, we also obtained an iterative process for solving the constrained convex optimization problem.

Acknowledgments

The authors would like to thank the Office of the Higher Education Commission for the supports under the Higher Education Research Promotion project. Also, the first author would like to thank the faculty of science, King Mongkut's University of Technology Thonburi for financial support.

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