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Critical Avalanches in Fiber Bundle Models of Arrays of Nanopillars

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Abstract—**Arrays of nanopillars subjected to uniaxial microcompression reveal the potential applicability of nanopillars as components for the fabrication of electro-mechanical sense devices. Thus, it is worth to analyse the failure progress in such systems of pillars. Specifically, we analyze the relations between numbers of crashed pillars and an external load in longitudinally loaded arrays of nanopillars. Under the growing load pillars' destruction forms an avalanche and when the load exceeds a certain critical value the avalanche becomes self-sustained until the system is completely destroyed. We explore the distributions of such catastrophic avalanches appearing in overloaded systems.**

Index Terms—**Avalanches, explosive instabilities, nanopillars, distributions of catastrophic avalanches**

I. Introduction

Avalanches are phenomena on different length scales encountered in an ample set of complex systems. Examples involve magnetic avalanches progressed through tiny crystals, mass movements of geological materials forming rock, sand or snow avalanches as well as fires destroying huge forests. Their presence is not limited to natural science or technology. Avalanches are reported, e.g. in the world of economy where the stock marked crashes evolve as an explosive instability. Such instabilities are commonly present in sand and snow avalanches, earthquakes, nuclear chain reactions as well as in damage evolutions of mechanical systems [1], [2]. They appear when a small increase in the external load excludes an element from the working community in such a way that this exclusion alters the internal load pattern sufficiently to trigger the rupture of the other elements and, in consequence, provoking a wave of destruction.

Avalanches also appear in mechanical systems formed by an ensemble of small pieces, as e.g. in a set of pillars assembled perpendicularly to a flat substrate [3]. Such arrangement is applied in systems of micromechanical sensors. Our work is motivated by uniaxial tensile and compressive experiments on nano- and microscale metallic pillars that confirm substantial strength increase via the size reduction of the sample [4]. Especially the studies on arrays of free-standing nanopillars subjected to uniaxial microcompression reveal the potential applicability of nanopillars as components for the fabrication of micro- and nano-electromechanical systems, micro-actuators or optoelectronic devices [5], [6]. For this reason it is worth analyzing the evolution of mechanical destruction within an array of nano-sized pillars. We simulate failure by accumulation of pillars crushed under the influence of an axial load. The stepwise increasing load causes the progressive damage of the system in an avalanche-like manner. When the load on a pillar exceeds the threshold the pillar crashes and its load has to be redistributed among the other pillars and carried by them. In this context, an important issue concerns the so-called load sharing rules because the behaviour of a system depends on them.

Typically, the avalanche statistical characteristics are immersed in the distribution $D(\Delta)$ of burst lengths Δ being the number of events triggered by the single failure. We compute $D(\Delta)$ for different load sharing protocols and report the results of statistical analysis of the system destruction.

II. Mathematical Model

Ruptured parts of mechanisms are encountered in virtually all kind of devices. They cause machine malfunction and are dangerous for users. Thus, the knowledge of the fracture evolution as well as its effective description represent an important issue in material science and technology.

Our example system is an array of nanopillars [3], [6]. A schematic view of such array is presented in Fig. 1.

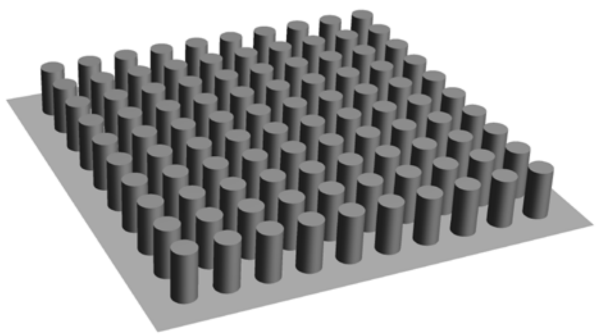

Fig. 1. Schematic view of an array of N=10×10 nano-sized pillars.

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A. Load Transfer Rules

The pillars are treated as fibres in the framework of a Fibre Bundle Model (FBM) [2], [7]-[15]. FBM is a transfer load model. In a static FBM, a set of N pillars is located in the nodes of the supporting lattice. Due to various defects during fabrication the pillar-strength-thresholds are quenched random variables.

The set of elements is subjected to an external load that is increased quasi-statically. After a pillar breakdown, its load is transferred to the other intact elements and, as a consequence, the probability of subsequent failure increases. Among several load transfer rules there are two extreme schemes: global load sharing (GLS) – the load is equally redistributed to all the remaining elements and local load sharing (LLS) – the load is transferred only to the neighbouring elements [2]. The GLS model being a mean-field approximation with long-range interactions among the elements can be solved analytically. In the case of the LLS rule the distribution of load is not homogenous and regions of stress accumulation appear throughout the system. This gives severe problems for analytical treatment and one has to answer the questions by means of numerical simulations.

Load redistribution in free-standing pillars should be placed somewhere in between the LLS and the GLS rules. For this reason we employ an approach based on Voronoi polygons which merges the GLS and LLS rules. The extra load is equally redistributed among the elements lying inside the Voronoi regions generated by a group of elements destroyed within an interval of time taken to be the time step. We call this load transfer rule Voronoi load sharing (VLS) [16].

Voronoi polygons are one of the most fundamental and useful constructs defined by irregular lattices [17]. For set $\mathbf{X}=\{x_1, x_2, ..., x_N\}$ of N distinct points in $\Omega \subset \mathbf{R}^2$, the Voronoi tessellation is the partition of Ω into N polygons denoted by ΔV_i. Each ΔV_i is defined as the set of points which are closer to x_i than to any points in $\mathbf{X}$. In Fig. 2, an example of Voronoi polygons is shown in the case of square array of pillars under the load.

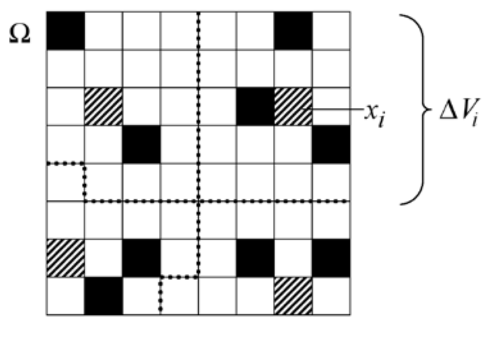

Fig. 2. The Voronoi polygons for a set of square-shaped pillars: white squares-intact pillars, black squares-previously destroyed pillars and shaded squares-just damaged pillars.

All of the Voronoi regions are convex polygons. Each polygon is defined by the lines that bisect the lines between the central point and its neighboring points. The bisecting lines and the connection lines are perpendicular to each other. Using this rule for every point in the area yields this area completely covered by the adjacent polygons representing pillars.

Numbers of intact elements inside of Voronoi regions vary randomly and this is the source of supplemental stochasticity in the model. It is worth mentioning that an approach based on the Voronoi tessellation was used for the failure analysis of quasi-brittle materials and fibre-reinforced brittle-matrix composites [18].

B. Loading of the System

At the beginning of the damage process all the pillars are intact. Then the system is loaded in a quasi-static way by a longitudinal external force *F* gently growing from its initial value $F=0$. More precisely, the system is uniformly loaded until the weakest intact pillar fails and then the increase of load stops. After this failure the load dropped by the damaged pillar has to be redistributed to the intact pillars according to a given transfer rule. The increased stress on the intact pillars may give rise to other failures, after which the load transfer from the destroyed elements may cause subsequent failures. If the load transfer does not trigger further failures there is a stable configuration and external load *F* has to be increased with a small amount, just to provoke damage of the subsequent weakest intact pillar. By that means a single failure induced by the load increment can cause an entire avalanche of failures. The above described procedure is repeated until the system completely fails. In the quasi-static approach the force *F* is the control parameter of the model.

III. Statistics of Catastrophic Avalanches

During the loading process, cascades of simultaneous crashes of several pillars appear. These ruptures resemble avalanches occurring in snow or sands movement. Hence, we consider the avalanche of size Δ being the number of damaged pillars under an equal external load and the distribution $D(\Delta)$ of the magnitude of such crashed-pillars avalanches is the main characteristics in our work. We also examine the maximum load per pillar that the array of pillars can bear.

The first problem we consider is the distribution of Δ. It is known that under the GLS rule $D(\Delta)$ can be expressed in a power law form

$$D(\Delta) \propto \Delta^{-\tau}, \quad \tau = 5/2. \tag{1}$$

The mean filed exponent $\tau=5/2$ is presumably independent of disorder distribution with the only exception for those distributions which allow unbreakable elements [19]. This 5/2 power law is valid when all avalanches are considered, i.e. it is a global exponent. If we trace the avalanches close to the critical breakdown of the system a crossover from 5/2 to 3/2 emerges in (1) [9]. Fig. 3 shows the distribution of the critical avalanche developed in the array of 50×50 pillars with the GLS rule. The results are averaged over 10^5 independent samples.

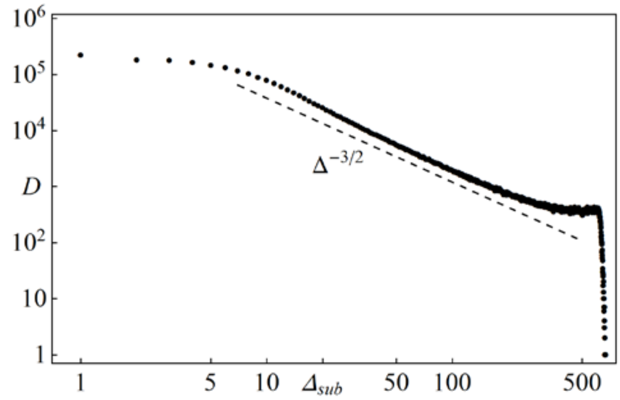

Fig. 3. The catastrophic avalanche size distribution for 50×50 pillars loaded quasi statically. The GLS rule was applied. The dashed line represents the power law obtained analytically [9].

Fig. 4 presents the behaviour of the mean value of the critical avalanche normalized by the system size.

$$\langle \Delta_{fa} \rangle / N = (a/\log N)^2 - b/\log N + c, \qquad (2)$$

where: $a \approx 1.591$, $b = 1.726$ and $c = 0.944$.

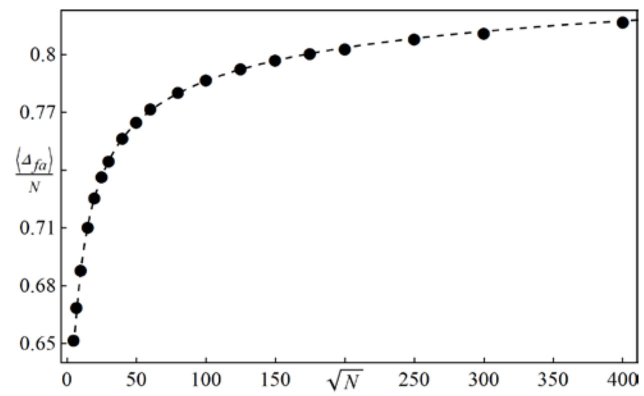

Fig. 4. The mean scaled size of the catastrophic avalanche vs. $\sqrt{N}$. The LLS rule is applied and N is the number of pillars. The dashed line is drawn using (2) and it is only visual guide. The simulation results are based on 10^4 samples.

A critical avalanche is in fact a cascade of so-called inclusive avalanches [21]. Here, an inclusive avalanche is the number of crashed pillars per step stress redistribution. We have performed the simulations on arrays of different sizes with the GLS and the VLS rules. The resulting distributions of inclusive avalanches are presented in Fig. 5 and Fig. 6, for the GLS and the VLS rules, respectively. In the case of the VLS rule we observe a strong departure of the distribution from the power law form.

IV. Size-dependent System Yield

Each avalanche is triggered by the external load increment and thus, the avalanche is the number of destroy-

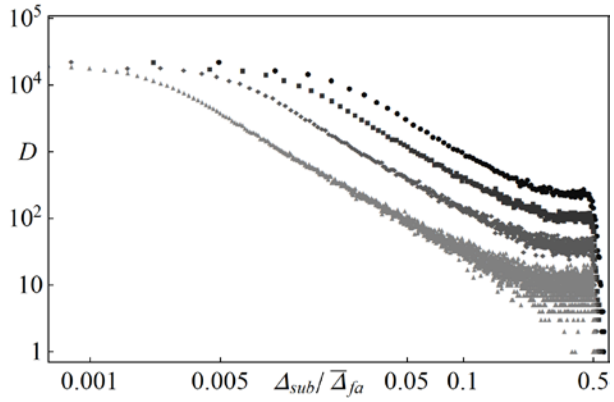

Fig. 5. Rescaled inclusive avalanche size distribution for the system with the GLS rule. Different system sizes are compared; from top to bottom: the array of $N = 20\times 20$, 30×30 , 50×50 and 100×100 pillars. The scaling parameter Δ_{fa} is the size of the catastrophic avalanche.

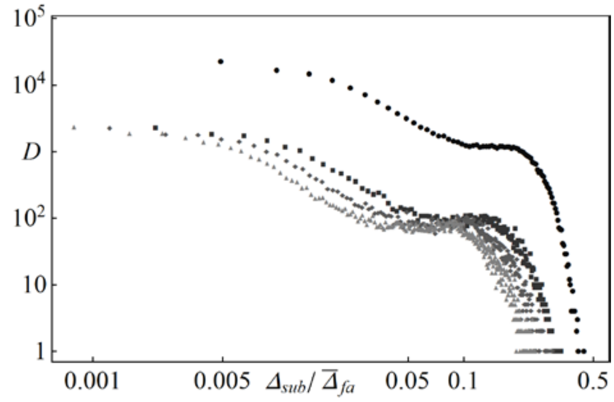

Fig. 6. Rescaled inclusive avalanche size distribution for the system with the VLS rule. Different system sizes are compared; from top to bottom: the array of $N = 20\times 20$, 30×30 , 40×40 and 50×50 pillars. The scaling parameter Δ_{fa} is the mean size of the corresponding catastrophic avalanche.

ed pillars between two consecutive load increments. We have found numerically, for different system size, the critical load which causes a complete breakdown of the system, i.e. the value F_c that triggers the catastrophic avalanche. This value is a bit greater that the maximum load $F_{\max}$ that the set of pillars can sustain. In one dimension, under the LLS rule, the best known estimate for $F_{\max}$ is given by [12]

$$F_{\max} \sim \frac{2N}{\ln N}. \qquad (3)$$

According to (3), the load safely carried by the system instead of being proportional to the number of pillars is tempered by the factor $2/\ln N$. This scaling results from the growing probability of finding the weak pillars that initiate a fatal avalanche when the number of pillars increases.

In two dimensions, within the LLS rule, we have found numerically that the mean value of F_c can be nicely fitted by the following formula

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$$\langle F_c \rangle = \frac{N}{d(\ln N)^{\delta}} \quad (4)$$

with $d = 2.413$ and $\delta = 0.414$. The corresponding mean critical stress $\langle \sigma_c \rangle = \langle F_c \rangle / N$ is presented in Fig. 7.

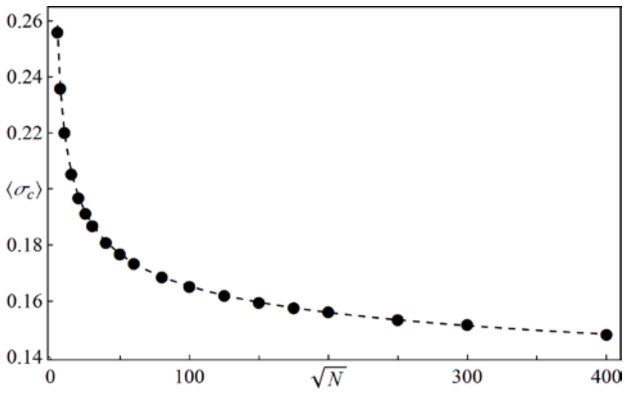

Fig. 7. Mean critical stress supported by the two dimensional array of nanopillars vs. $\sqrt{N}$. The LLS rule is applied and N is the number of pillars. The dashed line is drawn in accordance with (4).

V. Conclusion

In this paper, we have analyzed the statistics of avalanches during the failure process in longitudinally loaded arrays of nano-sized pillars with statistically distributed thresholds for breakdown of an individual pillar. We have found numerically the estimate for the maximum force F_c that a two dimensional set of pillars can support before the failure of the entire system. When an extra load caused by a pillar crash is taken up by its closest intact pillars, i.e. under the LLS rule, the expected maximum load scales as $F_c \sim N(\ln N)^{-\delta}$, with the exponent $\delta = 0.414$. A similar dependence, but with $\delta = 1$, was found in the model of linearly ordered fibres [12]. It is in contrast to the scaling $F_c \sim N$ valid under the GLS rule. Based on computer simulations, we have constructed an approximate average length $\langle \Delta_{fa} \rangle$ of the critical avalanche: within the LLS rule $\langle \Delta_{fa} \rangle / N$ behaves as a quadratic function of $1/\ln N$. We have also found that the load sharing rule based on the Voronoi tessellations concept, the VLS protocol, forces a strong departure of the avalanche size distribution from the power law form.

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