Proceedings of the International MultiConference of Engineers and Computer Scientists 2014 Vol II,
IMECS 2014, March 12 - 14, 2014, Hong Kong

An Optimal Replenishment Policy for Seasonal and Deteriorating Items

Hidefumi Kawakatsu, Toshimichi Homma and Kiyoshi Sawada

***Abstract*—This study discusses the retailer's optimal replenishment policy for products with a seasonal demand pattern. The demand of seasonal merchandise such as clothes, sporting goods, children's toys and electrical home appearances tends to decrease with time. In this study, we focus on "Special Display Goods", which are heaped up in end displays or special areas at the retail store. They are sold at a fast velocity when their quantity displayed is large, but are sold at a low velocity if the quantity becomes small. Most of physical items undergo decay or deterioration over time. We also consider the case where the inventory level is continuously depleted due to the combined effects of its demand and deterioration. We develop the model with a finite time horizon (period of a season) to determine the optimal replenishment policy, which maximizes the retailer's total profit. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.**

***Index Terms*—optimal replenishment policy; seasonal product; special display goods; deteriorating items.**

I. INTRODUCTION

THe demand of seasonal merchandise such as clothes, sporting goods, children's toys and electrical home appearances tends to decrease with time up to the end of the selling season. Inventory models with a finite planning horizon and time-varying demand patterns have extensively been studied in the inventory literature[1-7]. Resh et al.[1] and Donaldson[2] established an algorithm to determine the optimal number of replenishment cycles and the optimal replenishment time for a linearly increasing demand pattern. Barbosa and Friedman[3] and Henery[4] respectively extended the demand pattern to a power demand form and a log-concave function. Hariga and Goyal[5] and Teng[6] extended Donaldson's work by considering various types of shortages. For deteriorating items such as medicine, volatile liquids and blood banks, Dye[7] developed the inventory model under the circumstances where shortages are allowed and backlogging rate linearly depends on the total number of customers in the waiting line during the shortage period. However, there still remain many problems associated with replenishment policies for retailers that should theoretically be solved to provide them with effective indices. We focus on a case where *special display goods*[8], [9], [10] are dealt in. The special display goods are heaped up in the end displays or special areas at retail store. Retailers deal in such special display goods with a view to introducing and/or exposing new products or for the purpose of sales promotions in many cases. They are sold at a fast velocity when their quantity displayed is large, but are sold at a low velocity when their quantity becomes small. Baker[11] and Baker and Urban[12] dealt with a similar problem, but they expressed the demand rate simply as a function of a polynomial form without any practical meaning. Most of physical items undergo decay or deterioration overtime[7]. We also consider the case where the inventory level is continuously depleted due to the combined effects of its demand and deterioration.

Traditional retailers of seasonal merchandise have to commit themselves to a single order to purchase before the beginning of the season since the most of seasonal products have a relatively long ordering lead-time[13], [14]. The retailers who deal with the seasonal merchandise have recently been able to reorder the products during the season since Quick Response (QR) system has widely been used by manufacturing industries. Quick Response is a vertical strategy where the manufacturer strives to provide products and services to its retail customers in exact quantities on a continuous basis with minimum lead times[15].

In this study, we develop an inventory model with a seasonal demand pattern over a finite time horizon (period of a season) to determine the optimal replenishment policy, which maximizes the retailer's total profit. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.

II. NOTATION AND ASSUMPTIONS

The main notations used in this paper are listed below:

H**:** planning horizon.
n**:** the number of replenishment cycles during the planning horizon.
Q_U**:** maximum inventory level.
Q_j, q_j**:** the order-up-to level and the re-order point, respectively, in the jth replenishment cycle($q_0 = 0$, $0 \le q_j < Q_j \le Q_U$, $j = 1, 2, \cdots, n$).
$I(t)$**:** inventory level at time t.
t_j**:** the time of the jth replenishment ($t_{j-1} < t_j$, $t_0 = 0$, $t_n = H$).
p**:** selling price per item.
c**:** acquisition cost per item.
h**:** inventory holding cost per item and unit of time.
K**:** ordering cost per lot.
s**:** salvage value, per item, of unsold inventory at the end of the planning horizon.
θ**:** constant deterioration rate.
$\mu(t)$**:** demand rate, at time t, which is independent of the quantity displayed ($\mu'(t) < 0$).

The assumptions in this study are as follows:

Manuscript received December 21, 2013; revised January 20, 2014.

H. Kawakatsu is with Department of Economics & Information Science, Onomichi City University, 1600-2 Hisayamadacho, Onomichi 722-8506 Japan (corresponding author to provide phone: +81-848-22-8312 (ex.617); fax: +81-848-22-5460; e-mail: kawakatsu@onomichi-u.ac.jp).

T. Homma is with Faculty of Business Administration of Policy Studies, Osaka University of Economics, Osaka, 533-8533 Japan

K. Sawada is with Department of Policy Studies, University of Marketing and Distribution Sciences, Kobe, 651-2188 Japan

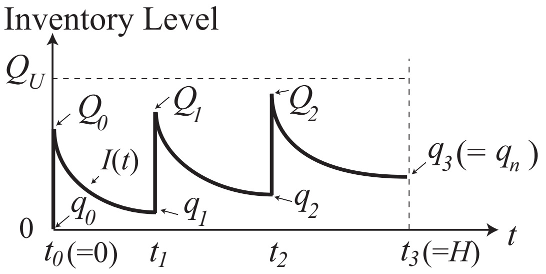


Fig. 1. Transition of inventory level ($n = 3$)

1) The finite planning horizon H is divided into n replenishment cycles.
2) The rate of replenishment is infinite and the delivery is instantaneous.
3) The demand rate is deterministic and significantly depends on the quantity displayed: the items sell well if their quantity displayed is large, but do not when their quantity displayed becomes small.
4) The inventory level is continuously depleted due to the combined effects of its demand and deterioration.
5) Backlogging and shortage are not allowed.
6) The retailer orders $(Q_j - q_j)$ units when her/his inventory level reaches q_j. Figure 1 shows the transition of inventory level in the case of $n = 3$.
7) $v = (p-c)\lambda - h > 0$. This assumption, $v > 0$, is equivalent to $(p-c)(Q_j - q_j) > (h/\lambda)(Q_j - q_j)$. The left-hand side of the inequality, $(p-c)(Q_j-q_j)$, denotes the cumulative gross profit during $[t_{j-j}, t_j)$, and the right-hand side of the inequality, $(h/\lambda)(Q_j - q_j)$, approximately expresses the cumulative inventory holding cost during $[t_{j-1}, t_j)$. Therefore, $v > 0$ signifies that the gross profit exceeds the inventory holding cost during a single replenishment cycle.

III. Total Profit

From assumptions (3) and (4), the inventory level, $I(t)$, at time t can be expressed by the following differential equation:

$$dI(t)/dt = -(\lambda + \theta)I(t) - \mu(t). \quad (\lambda > 0, \theta > 0) \quad (1)$$

By solving the differential equation in Eq. (1) with a boundary condition $I(t_{j-1}) = Q_{j-1}$, the inventory level at time t is given by

$$I(t) = e^{-\eta t}\left[Q_{j-1}e^{\eta t_{j-1}} - \int_{t_{j-1}}^{t} \mu(u)e^{\eta u}du\right], \quad (2)$$

where $\eta = \lambda + \theta$.

There obviously exists a time $t = t_j^U$ $(> t_{j-1})$ when the inventory level falls to zero, where t_j^U is unique positive solution to

$$Q_{j-1} - \int_{t_{j-1}}^{t} \mu(u)e^{\eta(u-t_{j-1})}du = 0. \quad (3)$$

The cumulative inventory, $A(t_{j-1}, t_j)$, held during $[t_{j-1}, t_j)$ $(t_j \le t_j^U)$ is expressed by

$$\begin{aligned} A(t_{j-1}, t_j) &= \int_{t_{j-1}}^{t_j} I(t)dt \\ &= \frac{1}{\eta}\Big\{Q_{j-1}\left[1 - e^{-\eta(t_j - t_{j-1})}\right] - \int_{t_{j-1}}^{t_j} \mu(u)du \\ &\quad + \int_{t_{j-1}}^{t_j} \mu(u)e^{-\eta(t_j-u)}du\Big\}. \end{aligned} \quad (4)$$

Let us here define

$$\begin{aligned} B(t_{j-1}, t_j) \equiv\ & Q_{j-1}\left[1 - e^{-\eta(t_j - t_{j-1})}\right] \\ & + \int_{t_{j-1}}^{t_j} \mu(u)e^{-\eta(t_j-u)}du, \end{aligned} \quad (5)$$

then $A(t_{j-1}, t_j)$ in Eq. (4) can be rewritten as

$$\begin{aligned} A(t_{j-1}, t_j) &= \int_{t_{j-1}}^{t_j} I(t)dt \\ &= \frac{1}{\eta}\left[B(t_{j-1}, t_j) - \int_{t_{j-1}}^{t_j} \mu(u)du\right], \end{aligned} \quad (6)$$

On the other hand, the cumulative quantity, $m(t_{j-1}, t_j)$, of demand for the product during $[t_{j-1}, t_j)$ becomes

$$m(t_{j-1}, t_j) = \frac{1}{\eta}\left[\lambda B(t_{j-1}, t_j) + \theta\int_{t_{j-1}}^{t_j} \mu(u)du\right], \quad (7)$$

The re-order point q_j can therefore be expressed by

$$q_j = Q_{j-1} - m(t_{j-1}, t_j). \quad (8)$$

Hence, the total profit is given by

$$\begin{aligned} P_n &= sq_n - nK + \sum_{j=1}^{n}\big[p \cdot m(t_{j-1}, t_j) \\ &\quad -c(Q_{j-1} - q_{j-1}) - h \cdot A(t_{j-1}, t_j)\big] \\ &= (s-c)q_n + \frac{(p-c)\theta + h}{\eta}\int_0^H \mu(u)du \\ &\quad + \frac{(p-c)\lambda - h}{\eta}\sum_{j=1}^{n} B(t_{j-1}, t_j) - nK. \end{aligned} \quad (9)$$

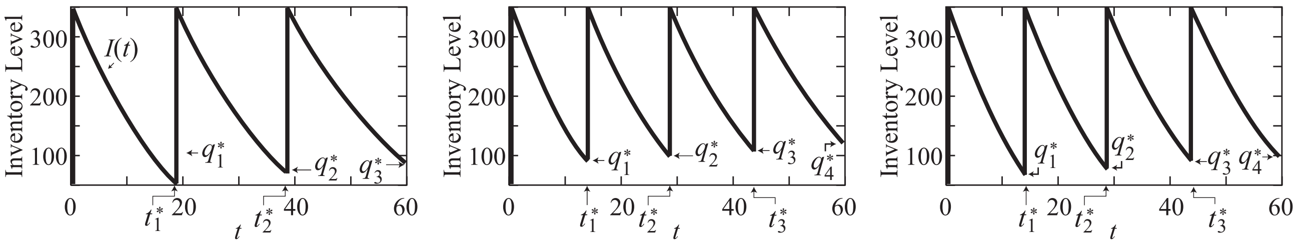
IV. Optimal Policy

This section analyzes the existence of the optimal policy $(Q_{j-1}, q_j, t_j) = (Q_{j-1}^*, q_j^*, t_j^*)$ for a given n $(j = 1, 2, \cdots, n)$, which maximizes P_n in Eq. (9). It is, however, very difficult to conduct analysis under $s \ne c$. For this reason, we focus on the case where $s = c$.

A. Optimal Order-up-to Level

We can show from Eq. (9) amd assumption (7) that $\partial P_n/\partial Q_j > 0$, which signifies that P_n is increasing function of Q_j. At retail stores, they have a maximum value for the inventory level arrowed for some reasons, which is denoted by Q_U. The optimal order-up-to level can therefore be given by $Q_j^* = Q_U$.

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)



(a) In the case of λ =0.01 and θ=0.01. (b) In the case of λ =0.02 and θ=0.01. (c) In the case of λ =0.01 and θ=0.03.

Fig. 2. Sensitivity analysis.

B. Optimal Replenishment Time

We here summarize the results of analysis in relation to the optimal replenishment time t_j^* which maximizes P_n in Eq. (9) when t_{j-1} and t_{j+1} are fixed to suitable values.

1) $\mu(t_j^U)\left[1-e^{-\eta(t_{j+1}-t_j^U)}\right]-\eta Q_U e^{-\eta(t_{j+1}-t_j^U)}<0.$
   In this case, there exists a unique finite t_j^* ($t_{j-1}<t_j^*<t_{j+1}$), which maximizes P_n in Eq. (9).
2) $\mu(t_j^U)\left[1-e^{-\eta(t_{j+1}-t_j^U)}\right]-\eta Q_U e^{-\eta(t_{j+1}-t_j^U)}\geq 0.$
   In this case, P_n is non-decreasing in t_j, and consequently we have $t_j^*\ t_j^U$.

If there exists $t_j^*<t_j^U$ for all $j=1,2,\cdots,n-1$, the total profit is given by

$$P_n = \frac{(p-c)\lambda-h}{\eta}\Bigg\{B(t_{n-1}^*,H) + \sum_{j=1}^{n-1}\left[Q_U+\frac{\mu(t_j^*)}{\eta}\right]\left[1-e^{-\eta(t_{j+1}^*-t_j^*)}\right]\Bigg\} + \frac{(p-c)\theta+h}{\eta}\int_0^H \mu(u)du - nK. \quad (10)$$

V. Numerical Examples

This section presents numerical examples to illustrate the proposed model. We here suppose the demand rate which is independent of the quantity displayed to be a linear function of time t, is given by

$$\mu(t) = \beta - \alpha t \ (\alpha>0,\ \beta>0). \quad (11)$$

Figure 2 depicts the transition of inventory level $I(t)$ along with behavior of (q_j^*, t_j^*) in the case of (a) $(\lambda,\theta)=(0.01,0.01)$, (b) $(\lambda,\theta)=(0.02,0.01)$ and (c) $(\lambda,\theta)=(0.01,0.03)$ with $(H,Q_U,p,c,h,s,K,\alpha,\beta)=(60,350,600,300,1,300,10000,0.1,13)$.

Figure 2 reveals that q_j^* is non-increasing in time t. This signifies that the cumulative quantity displayed in the jth replenishment increases with increasing j. Heaping up the products to a large quantity reflects the situation where the demand velocity is large at the retail store. When the demand rate which is independent of the quantity displayed becomes small, the retailer can therefore maintain her/his profit as large as possible by means of increasing the quantity displayed. Figure 2(b) also shows that, compared with Fig. 2(a), q_j^* and the number of replenishments are increasing in λ. This is simply understandable since larger values of λ reflect the situation where the demand velocity is large at the retail store.

Figure 2(c) reveals that, compared with Fig. 2(a), both q_j^* and the number of replenishments are increasing in θ. The loss due to deterioration increases when θ takes larger values. The retailer should compensate for the loss by increasing the quantity displayed so that she/he can increase the demand of the item.

VI. Conclusions

In this study, we have proposed an inventory model with a seasonal demand pattern over a finite time horizon (period of a season) to determine the optimal replenishment policy, which maximizes the retailer's total profit. We particularly focus on the case where the retailer is facing her/his customers' demand by dealing in a special display goods. Most of physical items undergo decay or deterioration overtime. We have also considered the case where the inventory level is continuously depleted due to the combined effects of its demand and deterioration. We have clarified the existence of the optimal replenishment policy which maximizes the retailer's total profit. In the real circumstances, retailers frequently place a mirror at their display area, or they display products on a false bottom to increase their quantity displayed in appearance. Taking account of such factors is an interesting extension.

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ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

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ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)