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On Some Properties of Parametric Continuous-Time Convex Programs

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Abstract—**This article is concerned with parametric continuous-time convex programs pertaining to a class of concave-convex fractional optimal control problems with linear state constraints. Some basic properties of parametric continuous-time convex programming problems are derived. These properties will provide an important foundation for constructing a parametric computational procedure for solving the concave-convex fractional optimal control problems with linear state constraints.**

Keywords: ***Infinite-dimensional nonlinear programming, Continuous-time Concave-Convex fractional programming problems, Approximate Solutions, Parametric Method***

1 Introduction

In this article, we shall pay our attention to a class of nonlinear optimal control problems with linear state constraints. Such a problem is called the *continuous-time concave-convex fractional programming problem* (in short, the problem (CCFP)). The problem (CCFP) is a generalization of the so-called *continuous-time linear programming problem*. The theory of the problem (CLP), which was originated from the "bottleneck problem" proposed by Bellman [1], has received considerable attention for a long time, one can consult [2]. The optimization problem in which the objective function appears as a ratio of two real-valued function is known as a fractional programming problem. Due to its significance appearing in the information theory, stochastic programming and decomposition algorithms for large linear systems, the various theoretical and computational issues have received particular attention in the last decades. For more details on this topic, we may refer to Stancu-Minasian [11] and Schaible et al. [5, 8, 9, 10]. In the literature, a number of optimality principles and duality models for linear and nonlinear fractional programming problems have been extended to some continuous-time fractional programming problems, one can refer to Zalmai [21, 22, 23, 24], Bector et al. [3], Stancu-Minasian and Tigan [12] and Husain and Jabeen[7]. However, these works focused on the developments of optimality principles and duality relations, the computational issues were not addressed. Recently, Wen et al. [13, 14, 15, 16, 17, 18, 19, 20] established computational procedures for some classes of continuous-time linear and fractional programming problems.

The most likely known methods for solving conventional fractional programming is the so-called parametric method, one can refer to Schaible [9] and Stancu-Minasian[11]. Its main idea is to convert the original problem to non-fractional problems by separating numerator and denominator with help of a parameter. In this article, we shall discuss the possibility of extending the parametric method to the problem (CCFP). By using the methodologies adopted in Wen [13, 14] and Wen et al. [17], we shall establish a theoretical foundation for developing a computational procedure for (CCFP).

The rest of this paper is organized as follows. In Section 2, we propose the auxiliary parametric continuous-time convex programming problem (CCP_λ) and review its duality properties. In Section 3, we derive the equivalence between the problems (CCFP) and (CCP_λ). Moreover, we introduce and analyze the discretization problems derived from (CCP_λ) in Section 4. By using the different step sizes of discretization problems, we construct a sequence of feasible solutions for (CCP_λ). The convergent property of the constructed feasible solutions can also be obtained. The paper ends with concluding remarks in Section 5.

For the remainder of this article, for any given optimization problem (P), we denote by $V(\mathrm{P})$ the optimal objective value of (P); that is, $V(\mathrm{P})$ will be obtained by taking the supremum or infimum.

2 Parametric Continuous-Time Convex Programming Problems

Let $L^\infty([0,T],\mathbf{R}^p)$ be the space of all measurable and essentially bounded functions from a time space $[0,T]$ into the p-dimensional Euclidean space $\mathbf{R}^p$ and let $C([0,T],\mathbf{R}^p)$ be the space of all continuous functions from $[0,T]$ into the $\mathbf{R}^p$. The continuous-time concave-convex

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fractional programming problem (CCFP) is formulated as follows:

$$\begin{array}{ll} \text{max.} & \dfrac{\mu + \int_0^T \phi(\mathbf{x}(t))dt}{\xi + \int_0^T \varphi(\mathbf{x}(t))dt} \\ \text{s. t.} & B\mathbf{x}(t) \le \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for all } t \in [0,T] \\ & \mathbf{x} \in L^\infty([0,T], \mathbf{R}_+^q), \end{array}$$

where

- $\mathbf{x}(t)$ is the decision variable, $T > 0$ is a given time horizon, and the superscript "$\top$" denotes the transpose operation of matrices.
- B and K are $p \times q$ matrices, $\mathbf{g} \in C([0,T], \mathbf{R}_+^p)$ and $\mathbf{R}_+^p = \{(x_1, \cdots, x_p)^\top : x_i \ge 0 \text{ for } i = 1, \cdots, p\}$;
- $\phi(\cdot)$ is a scalar function that is concave and continuously twice differentiable and $\mu \in \mathbf{R}_+$; $\varphi(\cdot)$ is a nonnegative scalar function that is convex and continuously twice differentiable and $\xi > 0$. Moreover, there exists a feasible solution $\mathbf{x}(t)$ such that $\mu + \int_0^T \phi(\mathbf{x}(t))dt \ge 0$.

We also assume that $B = [B_{ij}]_{p\times q}$ and $K = [K_{ij}]_{p\times q}$ are constant matrices satisfying

(A1) $K_{ij} \ge 0$ for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$;

(A2) $B_{ij} \ge 0$ and $\sum_{i=1}^p B_{ij} > 0$ for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$.

In this section, we are going to propose an auxiliary problem associated with (CCFP) which will be formulated as the parametric continuous-time convex programming problem. For any $\lambda \in \mathbf{R}_+$, let

$$\theta^{(\lambda)} := \phi - \lambda\varphi.$$

We consider the following parametric continuous-time convex programming problem:

(CCP_λ) :

$$\begin{array}{ll} \text{max.} & \mu - \lambda\xi + \int_0^T \theta^{(\lambda)}(\mathbf{x}(t))dt \\ \text{s. t.} & B\mathbf{x}(t) \le \mathbf{g}(t) + \int_0^t K\mathbf{x}(s)ds \text{ for } t \in [0,T] \\ & \mathbf{x}(t) \in L^\infty([0,T], \mathbf{R}_+^q). \end{array}$$

In deriving the relations between (CCFP) and (CCP_λ), the solvability of (CCP_λ) is a key condition. Hence discussing the solvability of (CCP_λ) is a top priority of this study. In order to discuss the solvability of (CCP_λ), it is necessary to realize the dual relations of (CCP_λ). In the literature, the dual properties of continuous-time convex programming problems have been studied by by Hanson [6] and Wen [14]. According to Hanson [6], the dual problem (DCCP_λ) can be defined as follows:

$$\begin{array}{ll} \text{min.} & \mu - \lambda\xi + \int_0^T \left\{\theta^{(\lambda)}(\mathbf{u}(t)) - \mathbf{u}(t)^\top \nabla\theta^{(\lambda)}(\mathbf{u}(t))\right\} dt \\ & + \int_0^T \left\{\mathbf{g}(t)^\top \mathbf{w}(t)\right\} dt \\ \text{s. t.} & B^\top \mathbf{w}(t) - \int_t^T K^\top \mathbf{w}(s)ds \ge \nabla\theta^{(\lambda)}(\mathbf{u}(t)) \\ & \text{for } t \in [0,T], \\ & \mathbf{w}(\cdot) \in L^\infty([0,T], \mathbf{R}_+^p) \text{ and} \\ & \mathbf{u}(\cdot) \in L^\infty([0,T], \mathbf{R}^q), \end{array}$$

where $\nabla\theta^{(\lambda)} = \nabla\phi - \lambda\nabla\varphi$ is the gradient of $\theta^{(\lambda)}$.

By the same arguments given in Wen [14], the weak and strong duality properties can be realized as below.

Proposition 1 *(Weak Duality between* (CCP_λ) *and* (DCCP_λ)*) Let* $\lambda \ge 0$. *Considering the primal-dual pair problems* (CCP_λ) *and* (DCCP_λ), *for any feasible solutions* $\mathbf{x}^{(0)}(t)$ *and* $(\mathbf{u}^{(0)}(t), \mathbf{w}^{(0)}(t))$ *of problems* (CCP_λ) *and* (DCCP_λ), *respectively, we have that the objective value of* (CCP_λ) *at* $\mathbf{x}^{(0)}(t)$ *is less than or equal to the objective value of* (DCCP_λ) *at* $(\mathbf{u}^{(0)}(t), \mathbf{w}^{(0)}(t))$; *that is,* $V(\mathrm{CCP}_\lambda) \le V(\mathrm{DCCP}_\lambda)$.

Proposition 2 *(Strong Duality between* (CCP_λ) *and* (DCCP_λ)*) Let* $\lambda \ge 0$. *There exist optimal solutions* $\mathbf{x}^{(*,\lambda)}(t)$ *and* $(\mathbf{u}^{(*,\lambda)}(t), \mathbf{w}^{(*,\lambda)}(t))$ *of the primal-dual pair problems* (CCP_λ) *and* (DCCP_λ), *respectively, such that* $\mathbf{x}^{(*,\lambda)}(t) = \mathbf{u}^{(*,\lambda)}(t)$ *and* $V(\mathrm{CCP}_\lambda) = V(\mathrm{DCCP}_\lambda)$.

3 The relations between (CCP_λ) and (CCFP)

In order to realize the relations between the problem (CP) and the problem (CCP_λ), we define a function $\mathcal{F} : \mathbf{R}_+ \to \mathbf{R}$ by $\mathcal{F}(\lambda) = V(\mathrm{CCP}_\lambda)$ for all $\lambda \ge 0$. Using the solvability of the problem (CCP_λ) and by a similar argument with [11, Theorem 4.5.2], we can obtain the following results.

Proposition 3 *The following statements hold true.*

(i) *The real-valued function* $\mathcal{F}(\lambda)$ *is convex, hence is continuous.*

(ii) *If* $\lambda_1 < \lambda_2$, *then* $\mathcal{F}(\lambda_1) > \mathcal{F}(\lambda_2)$; *that is, the real-valued function* $\mathcal{F}(\cdot)$ *is strictly decreasing.*

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Many useful relations between (CCP$_\lambda$) and (CCFP) are given below.

Proposition 4 *The following statements hold true.*

(i) *Given any $\lambda \geq 0$, then $\mathcal{F}(\lambda) > 0$ if and only if $\lambda < V(\text{CCFP})$. Equivalently, $\mathcal{F}(\lambda) \leq 0$ if and only if $\lambda \geq V(\text{CCFP})$.*

(ii) *Suppose that $\bar{\mathbf{x}}(t)$ is an optimal solution of (CCFP) with $V(\text{CCFP}) = \lambda^*$. Then $\bar{\mathbf{x}}(t)$ is an optimal solution of (CCP$_{\lambda^*}$) with $V(\text{CCP}_{\lambda^*}) = 0$; that is $\mathcal{F}(\lambda^*) = 0$.*

(iii) *If there exists a $\lambda^* \geq 0$ such that $\mathcal{F}(\lambda^*) = 0$, then the optimal solution of the problem (CCP$_{\lambda^*}$) is also an optimal solution of (CCFP) and $V(\text{CCFP}) = \lambda^*$.*

By the above propositions, it can be shown that the problem (CCFP) is solvable. Let $\mathbf{1} = (1, 1, \cdots, 1)^\top \in \mathbf{R}^p$ and

$$\widehat{\rho} := \max_{j=1,\cdots,q} \left\{ \frac{\sum_{i=1}^p K_{ij}}{\sum_{i=1}^p B_{ij}}, \frac{\nabla_j \phi(\mathbf{0}) - \lambda \nabla_j \varphi(\mathbf{0})}{\sum_{i=1}^p B_{ij}} \right\} \geq 0.$$

We define $\mathbf{w}^\star(t) = \widehat{\rho}\, e^{\widehat{\rho}(T-t)}\mathbf{1}$ for all $t \in [0, T]$ and define $\eta^\star \geq 0$ such that

$$\eta^\star = \max \left\{ \frac{\mu + \int_0^T \phi(\mathbf{0})dt + \int_0^T \mathbf{g}(t)^\top \mathbf{w}^\star(t)dt}{\xi + \int_0^T \varphi(\mathbf{0})dt}, 0 \right\}. \tag{1}$$

Corollary 1 *There exists a unique λ^* in the closed interval $[0,\ \eta^\star]$ such that $\mathcal{F}(\lambda^*) = 0$. That is,*

- *$0 \leq V(\text{CCFP}) \leq \eta^\star$, and*
- *if $\bar{\mathbf{x}}^{(\lambda^*)}(t)$ is an optimal solution of the problem (CCP$_{\lambda^*}$), then it is also an optimal solution of the problem (CCFP).*

From the above discussions, it follows that solving the problem (CCFP) is equivalent to determine the unique root of the nonlinear equation $\mathcal{F}(\lambda) = 0$. However, it is notoriously difficult to find the exact solution of every (CCP$_\lambda$). In the next section, given a λ in the closed interval $[0,\ \eta^\star]$, we shall utilize the discrete approximation procedure developed by Wen [14] to find the approximate value of $\mathcal{F}(\lambda)$ and to estimate its error bound.

4 A Discrete Approximation Method for (CCP$_\lambda$)

Now, we are going to propose the discrete approximation method to solve the parametric problem (CCP$_\lambda$). In this case, the discrete problem derived from problem (CCP$_\lambda$) will be a finite-dimensional linear programming problem. For each $n \in \mathbf{N}$, we take

$$\mathcal{P}_n = \left\{ 0, \frac{T}{n}, \frac{2T}{n}, \cdots, \frac{(n-1)T}{n}, T \right\}$$

as a partition of $[0, T]$, which divides $[0, T]$ into n subintervals with equal length T/n. For $l = 1, \cdots, n$, we define

$$\mathbf{b}_l^{(n)} = \left(b_{1l}^{(n)}, b_{2l}^{(n)}, \cdots, b_{pl}^{(n)} \right)^\top \in \mathbf{R}_+^p, \tag{2}$$

where

$$b_{il}^{(n)} = \min \left\{ g_i(t) : t \in \left[\frac{(l-1)T}{n}, \frac{lT}{n} \right] \right\}. \tag{3}$$

According to the continuous-time convex programming problem (CCP$_\lambda$), its discrete version can be defined as the following finite-dimensional convex programming problem

$$
\begin{aligned}
&(\mathrm{P}_n^{(\lambda)}): \\
&\text{maximize} \quad \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \theta^{(\lambda)}(\mathbf{x}_l) \\
&\text{subject to} \quad B\mathbf{x}_l - \frac{T}{n} K \sum_{r=1}^{l-1} \mathbf{x}_r \leq \mathbf{b}_l^{(n)} \text{ for } l = 1, \cdots, n \\
&\qquad\qquad\quad \mathbf{x}_l \in \mathbf{R}_+^q \text{ for } l = 1, \cdots, n,
\end{aligned}
$$

According to Dorn [4], the dual problem (D$_n^{(\lambda)}$) of (P$_n^{(\lambda)}$) is defined by

$$
\begin{aligned}
&(\mathrm{D}_n^{(\lambda)}): \\
&\text{minimize} \quad \mu - \lambda\xi + \frac{T}{n} \sum_{l=1}^n \left\{ \theta^{(\lambda)}(\mathbf{u}_l) - \mathbf{u}_l{}^\top \nabla\theta^{(\lambda)}(\mathbf{u}_l) \right. \\
&\qquad\qquad \left. + (\mathbf{b}_l^{(n)})^\top \mathbf{w}_l \right\} \\
&\text{subject to} \quad B^\top \mathbf{w}_l - \frac{T}{n} K^\top \sum_{r=l+1}^n \mathbf{w}_r \geq \nabla\theta^{(\lambda)}(\mathbf{u}_l) \\
&\qquad\qquad \text{for } l = 1, 2, \cdots, n \\
&\qquad\qquad \mathbf{w}_l \in \mathbf{R}_+^p \text{ for } l = 1, \cdots, n \text{ and} \\
&\qquad\qquad \mathbf{u}_l \in \mathbf{R}^q \text{ for } l = 1, \cdots, n,
\end{aligned}
$$

The weak duality theorem for (P$_n^{(\lambda)}$) and (D$_n^{(\lambda)}$) is given below.

Proposition 5 (Weak Duality between (P$_n^{(\lambda)}$) and (D$_n^{(\lambda)}$)) *Let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ and $(\mathbf{u}, \mathbf{w})$ with $\mathbf{u} = (\mathbf{u}_1, \cdots, \mathbf{u}_n)$ and $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_n)$ be feasible solutions of (P$_n^{(\lambda)}$) and (D$_n^{(\lambda)}$), respectively. Then*

$$
\begin{aligned}
&\frac{T}{n} \sum_{l=1}^n \theta^{(\lambda)}(\mathbf{x}_l) \\
&\quad \leq \frac{T}{n} \sum_{l=1}^n \left\{ \theta^{(\lambda)}(\mathbf{u}_l) - \mathbf{u}_l{}^\top \nabla\theta^{(\lambda)}(\mathbf{u}_l) + (\mathbf{b}_l^{(n)})^\top \mathbf{w}_l \right\}.
\end{aligned}
$$

That is, $V(\mathrm{P}_n^{(\lambda)}) \leq V(\mathrm{D}_n^{(\lambda)})$.

The strong duality property holds true as shown below.

Proposition 6 (Strong Duality between $(\mathrm{P}_n^{(\lambda)})$ and $(\mathrm{D}_n^{(\lambda)})$) *There exist optimal solution* $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_n)$ *of primal problem* $(\mathrm{P}_n^{(\lambda)})$ *and optimal solution* $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ *of dual problem* $(\mathrm{D}_n^{(\lambda)})$ *with* $\bar{\mathbf{u}} = (\bar{\mathbf{u}}_1, \cdots, \bar{\mathbf{u}}_n)$ *and* $\bar{\mathbf{w}} = (\bar{\mathbf{w}}_1, \cdots, \bar{\mathbf{w}}_n)$ *such that* $\bar{\mathbf{x}} = \bar{\mathbf{u}}$ *and*

$$\begin{aligned} &\frac{T}{n}\sum_{l=1}^{n}\theta^{(\lambda)}(\bar{\mathbf{x}}_l) \\ &\quad = \frac{T}{n}\sum_{l=1}^{n}\left\{\theta^{(\lambda)}(\bar{\mathbf{u}}_l) - \bar{\mathbf{u}}_l^{\top}\nabla\theta^{(\lambda)}(\bar{\mathbf{u}}_l) + (\mathbf{b}_l^{(n)})^{\top}\bar{\mathbf{w}}_l\right\}. \end{aligned}$$

It can be shown that the feasible sets of the problems $(\mathrm{P}_n^{(\lambda)})$ are uniformly bounded for all $n \in \mathbf{N}$ and $\lambda \in \mathbf{R}$. To see this, let

$$\sigma = \min\left\{B_{ij} : B_{ij} > 0\right\}, \tag{4}$$

$$\kappa = \max_{j=1,\cdots,q}\left\{\sum_{i=1}^{p} K_{ij}\right\} \tag{5}$$

and

$$\zeta = \max\left\{g_i(t) : i = 1, \cdots, p \text{ and } t \in [0, T]\right\}. \tag{6}$$

Then we have the following useful results.

Lemma 1 *Given any* $n \in \mathbf{N}$ *and* $\lambda \geq 0$, *if* $(\mathbf{x}_1^{(\lambda,n)}, \mathbf{x}_2^{(\lambda,n)}, \cdots, \mathbf{x}_n^{(\lambda,n)})$ *is a feasible solution of the primal problem* $(\mathrm{P}_n^{(\lambda)})$, *where* $\mathbf{x}_l^{(\lambda,n)} = (x_{1l}^{(\lambda,n)}, x_{2l}^{(\lambda,n)}, \cdots, x_{ql}^{(\lambda,n)})^{\top} \in \mathbf{R}_+^q$, *then*

$$0 \leq x_{jl}^{(\lambda,n)} \leq \frac{\zeta}{\sigma}\exp\left(\frac{q\kappa T}{\sigma}\right) \tag{7}$$

for all $j = 1, \cdots, q$ *and* $l = 1, \cdots, n$. *This says that the feasible sets of the problems* $(\mathrm{P}_n^{(\lambda)})$ *are uniformly bounded in the sense that the bounds of* $x_{jl}^{(\lambda,n)}$ *are independent of* n *and* λ.

In general, the sequence of feasible sets $\{F(\mathrm{D}_n^{(\lambda)})\}_{n=1}^{\infty}$ needs not to be uniformly bounded. It can be shown that there exist uniformly bounded optimal solutions to dual problems $(\mathrm{D}_n^{(\lambda)})$. To see this, we define

$$F = \left\{\mathbf{x} = (x_1, \cdots, x_q)^{\top} \in \mathbf{R}^q : 0 \leq x_j \leq \frac{\zeta}{\sigma}\exp\left(\frac{q\kappa T}{\sigma}\right)\right\}.$$

Then F is a compact set. Let

$$\widehat{c}(\lambda) := \max_{j=1,\cdots,q}\max_{\mathbf{x}\in F}|\nabla_j\theta^{(\lambda)}(\mathbf{x})|, \tag{8}$$

where $\nabla_j\theta^{(\lambda)}(\mathbf{x})$ denotes the jth component of $\nabla\theta^{(\lambda)}(\mathbf{x})$.

Lemma 2 *The dual problem* $(\mathrm{D}_n^{(\lambda)})$ *has an optimal solution* $(\widetilde{\mathbf{u}}^{(\lambda,n)}, \hat{\mathbf{w}}^{(\lambda,n)})$ *with* $\hat{\mathbf{w}}^{(\lambda,n)} = (\hat{\mathbf{w}}_1^{(\lambda,n)}, \cdots, \hat{\mathbf{w}}_n^{(\lambda,n)})$ *such that* $\widetilde{\mathbf{u}}^{(\lambda,n)}$ *is also an optimal solution of* $(\mathrm{P}_n^{(\lambda)})$ *and*

$$0 \leq \hat{w}_{il}^{(\lambda,n)} \leq \frac{\widehat{c}(\lambda)}{\sigma}\cdot\exp\left(\frac{\kappa T}{\sigma}\right) \tag{9}$$

for all $i = 1, \cdots, p$ *and* $l = 1, \cdots, n$.

Besides, we can construct the feasible solutions of the problems (CCP_λ) by virtue of the optimal solution of the problem $(\mathrm{P}_n^{(\lambda)})$. Let $(\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \cdots, \bar{\mathbf{x}}_n^{(\lambda,n)})$ be an optimal solution of $(\mathrm{P}_n^{(\lambda)})$. For $j = 1, \cdots, q$, we define the step functions $\bar{x}_j^{(\lambda,n)} : [0, T] \to \mathbf{R}$ as follows:

$$\bar{x}_j^{(\lambda,n)}(t) = \begin{cases} \bar{x}_{jl}^{(\lambda,n)}, & \text{if } \dfrac{(l-1)T}{n} \leq t < \dfrac{lT}{n} \\ \bar{x}_{jn}^{(\lambda,n)}, & \text{if } t = T, \end{cases} \tag{10}$$

where $l = 1, \cdots, n$. Then we can form a vector-valued function $\bar{\mathbf{x}}^{(\lambda,n)} : [0, T] \to \mathbf{R}^q$ by

$$\bar{\mathbf{x}}^{(\lambda,n)}(t) = \left(\bar{x}_1^{(\lambda,n)}(t), \bar{x}_2^{(\lambda,n)}(t), \cdots, \bar{x}_q^{(\lambda,n)}(t)\right)^{\top}. \tag{11}$$

In this case, we say that $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ is a *natural solution* of (CCP_λ) constructed from $(\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \cdots, \bar{\mathbf{x}}_n^{(\lambda,n)})$. After some algebraic calculations, it is not hard to show the feasibility of natural solutions of (CCP_λ), which will be presented below.

Lemma 3 *Let* $(\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \cdots, \bar{\mathbf{x}}_n^{(\lambda,n)})$ *be an optimal solution of* $(\mathrm{P}_n^{(\lambda)})$. *Then the natural solution* $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ *of problem* (CCP_λ) *constructed from* $(\bar{\mathbf{x}}_1^{(\lambda,n)}, \bar{\mathbf{x}}_2^{(\lambda,n)}, \cdots, \bar{\mathbf{x}}_n^{(\lambda,n)})$ *is a feasible solution of* (CCP_λ). *Moreover, we have*

$$\mathcal{F}(\lambda) = V(\mathrm{CCP}_\lambda) \geq V(\mathrm{P}_n^{(\lambda)}) \tag{12}$$

for all $n \in \mathbf{N}$.

Furthermore, it also can be shown that

$$\lim_{n\to\infty} V(\mathrm{P}_n^{(\lambda)}) = V(\mathrm{CCP}_\lambda).$$

To see this, we need some setting. We define a vector-valued step function $\mathbf{g}^{(n)} : [0, T] \mapsto \mathbf{R}^p$ as follows:

$$\mathbf{g}^{(n)}(t) = \left(g_1^{(n)}(t), g_2^{(n)}(t), \cdots, g_p^{(n)}(t)\right)^{\top}$$

where,

$$g_i^{(n)}(t) = \begin{cases} b_{il}^{(n)}, & \text{if } \dfrac{(l-1)T}{n} \leq t < \dfrac{lT}{n} \\ b_{in}^{(n)}, & \text{if } t = T, \end{cases} \tag{13}$$

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for $i = 1, \cdots, p$, $l = 1, \cdots, n$, and $b_{il}^{(n)}$ is defined in (3). For further discussion, we define

$$\rho = \max_{j=1,\cdots,q} \left\{ \frac{\sum_{i=1}^{p} K_{ij}}{\sum_{i=1}^{p} B_{ij}}, \frac{1}{\sum_{i=1}^{p} B_{ij}} \right\}, \tag{14}$$

$$\bar{\epsilon}_n = \max_{i=1,\cdots,p} \sup_{t\in[0,T]} \left\{ g_i(t) - g_i^{(n)}(t) \right\}, \tag{15}$$

$$\delta_n(\lambda) = \max_{i=1,\cdots,p} \max_{l=1,\cdots,n} \left\{ \frac{T}{n} \bar{w}_{il}^{(\lambda,n)} \right\}. \tag{16}$$

Let $(\bar{\mathbf{x}}^{(\lambda,n)}, \bar{\mathbf{w}}^{(\lambda,n)})$ be an optimal solution of dual problem $(\mathrm{DCP}_n^{(\lambda)})$, where $\bar{\mathbf{w}}^{(\lambda,n)} = (\bar{\mathbf{w}}_1^{(\lambda,n)}, \cdots, \bar{\mathbf{w}}_n^{(\lambda,n)})$, $\bar{\mathbf{w}}_l^{(\lambda,n)} = (\bar{w}_{1l}^{(\lambda,n)}, \cdots, \bar{w}_{pl}^{(\lambda,n)})^\top$ and $\bar{\mathbf{x}}^{(\lambda,n)}$ is an optimal solution of $(\mathrm{P}_n^{(\lambda)})$. We define a function $\hat{\mathbf{w}}^{(\lambda,n)}(t) : [0,T] \mapsto \mathbf{R}^p$ as follows:

$$\hat{\mathbf{w}}^{(\lambda,n)}(t) = \bar{\mathbf{w}}_l^{(\lambda,n)} + \delta_n(\lambda)\rho e^{\rho(T-t)}\mathbf{1} \text{ for } t \in \left[\frac{l-1}{n}T, \frac{l}{n}T\right) \tag{17}$$

and

$$\hat{\mathbf{w}}^{(\lambda,n)}(T) = \bar{\mathbf{w}}_n^{(\lambda,n)} + \delta_n(\lambda)\rho\mathbf{1},$$

where $\mathbf{1} = (1, 1, \cdots, 1)^\top \in \mathbf{R}^p$. If $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ is the *natural solution* of (CCP_λ) constructed from $\bar{\mathbf{x}}^{(\lambda,n)}$, then we also say that $(\bar{\mathbf{x}}^{(\lambda,n)}(t), \hat{\mathbf{w}}^{(\lambda,n)}(t))$ is a *natural solution* of problem (DCCP_λ) constructed from the optimal solution $(\bar{\mathbf{x}}^{(\lambda,n)}, \bar{\mathbf{w}}^{(\lambda,n)})$ of problem $(\mathrm{D}_n^{(\lambda)})$. The following results are useful.

Lemma 4 *Let $\bar{\mathbf{x}}^{(\lambda,n)}$ and $(\bar{\mathbf{x}}^{(\lambda,n)}, \bar{\mathbf{w}}^{(\lambda,n)})$ be an optimal solutions of $(\mathrm{P}_n^{(\lambda)})$ and $(\mathrm{D}_n^{(\lambda)})$, respectively. Let $\hat{\mathbf{w}}^{(\lambda,n)}(t)$ be defined as in (17). Then the following statements hold true.*

(i) The natural solution $(\bar{\mathbf{x}}^{(\lambda,n)}(t), \hat{\mathbf{w}}^{(\lambda,n)}(t))$ is a feasible solution of dual problem (DCCP_λ).

(ii) We have

$$\begin{aligned} 0 &\le \widehat{Obj}\left(\bar{\mathbf{x}}^{(\lambda,n)}(t), \hat{\mathbf{w}}^{(\lambda,n)}(t)\right) - V(\mathrm{D}_n^{(\lambda)}) \\ &\le \delta_n(\lambda) \int_0^T \rho e^{\rho(T-t)} \mathbf{g}(t)^\top \mathbf{1} dt, \end{aligned} \tag{18}$$

where $\widehat{Obj}\left(\bar{\mathbf{x}}^{(\lambda,n)}(t), \hat{\mathbf{w}}^{(\lambda,n)}(t)\right)$ is the objective value of (DCCP_λ) at $(\bar{\mathbf{x}}^{(\lambda,n)}(t), \hat{\mathbf{w}}^{(\lambda,n)}(t))$.

By Lemma 4, we see that the natural solution $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ of problem (CCP_λ) constructed from an optimal solution of $(\mathrm{P}_n^{(\lambda)})$ is an approximate solution of (CCP_λ), and its error bound can be estimated as follows.

Theorem 1 *The following statements hold true.*

(i) We have

$$0 \le V(\mathrm{CCP}_\lambda) - V(\mathrm{P}_n^{(\lambda)}) \le \varepsilon_n(\lambda), \tag{19}$$

where

$$\begin{aligned} \varepsilon_n(\lambda) &:= \bar{\epsilon}_n \cdot p \cdot \delta_n(\lambda) \cdot (n + \exp(\rho T) - 1) \\ &\quad + \delta_n(\lambda) \int_0^T \rho \cdot \exp(\rho(T-t)) (\mathbf{g}(t))^\top \mathbf{1} dt. \end{aligned} \tag{20}$$

(ii) We have

$$\lim_{n\to\infty} V(\mathrm{P}_n^{(\lambda)}) = V(\mathrm{CCP}_\lambda) = V(\mathrm{DCCP}_\lambda).$$

(iii) Let $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ be the natural solutions of (CCP_λ). Then the error between the optimal objective value of (CCP_λ) and the objective value of $\bar{\mathbf{x}}^{(\lambda,n)}(t)$ is less than or equal to $\varepsilon_n(\lambda)$.

Furthermore, by Lemmas 1 and 2 the convergence of the sequence of natural solutions $\{\bar{\mathbf{x}}^{(\lambda,n)}(t)\}_{n=1}^\infty$ of problem (CCP_λ) can be demonstrated.

Theorem 2 *For any given $\lambda \ge 0$, the sequence $\{\bar{\mathbf{x}}^{(\lambda,n)}(t)\}$ defined in (10) has a convergent subsequence $\{\bar{\mathbf{x}}^{(\lambda,n_k)}(t)\}$ which weakly-star converges to $\bar{\mathbf{x}}^{(\lambda,\star)}(t)$ such that the limit $\bar{\mathbf{x}}^{(\lambda,\star)}(t)$ is an optimal solution of (CCP_λ).*

5 Concluding Remarks

This article extends the traditional parametric method for fractional programming programs to (CCFP). Some properties of the auxiliary parametric continuous-time convex programming problem (CCP_λ) pertaining to (CCFP) are derived. By these properties we conclude that solving (CCFP) is equivalent to determine the root of the nonlinear equation $\mathcal{F}(\lambda) = 0$. These properties make it possible to develop a numerical algorithm for solving (CCFP). However, the performance of the developed numerical algorithm will heavily depend on the effective solutions of the auxiliary problems (CCP_λ). Besides, it is notoriously difficult to find the exact solution of every (CCP_λ). These might pose the major difficulty in solving problems (CCFP) effectively. Motivated by the difficulty, in the further study of this article, we shall refine the discrete approximation method developed in this article and extend the interval-type algorithm by Wen [13] to solve (CCFP) with approximation. The related results are now in progress.

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