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An Iterative Shrinking Generalized f-Projection Method for G-Quasi-strict Pseudo-contractions in Banach Spaces

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Abstract—**The purposes of this paper are to study the new type of mappings called G-quasi-strict pseudo-contractions and to create some iterative projection techniques to find some fixed points of the mappings. Moreover, we also find the significant inequality related to such mappings in the framework of Banach spaces. By using the ideas of the generalized f-projection, we propose an iterative shrinking generalized f-projection method for finding a fixed point of G-quasi-strict pseudo-contractions. The results of this paper improve and extend the corresponding results of Zhou and Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudocontractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.] as well as other related results.**

Index Terms—**G-quasi-strict pseudo-contraction, generalized f-projection, iterative shrinking projection, strong convergence, Banach space.**

I. Introduction

Let E be a real Banach space with its dual E^*, and let C be a nonempty, closed and convex subset of E. In 1994, Alber [1] introduced the generalized projections $\pi_C : E^* \to C$ and $\Pi_C : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2], Alber presented some applications of the generalized projections to approximately solving variational inequalities and Von Neumann intersection problem in Banach space. In addition, Li [7] extended the generalized projections from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces, and established a Mann type iterative scheme for finding the approximate solutions for the classical variational inequality problem in compact subset of Banach spaces.

Recently, Wu and Huang [17] introduced a new generalized f-projection operator in Banach space. They extended the definition of the generalized projection operators introduced by Abler [1] and proved some properties of the generalized f-projection operator. Wu and Huang [18] continued their study and presented some properties of the generalized f-projection operator. They showed an interesting relation between the generalized f-projection operator and the resolvent operator for the subdifferential of a proper, convex and lower semicontinuous functional in reflexive and smooth Banach spaces. They also proved that the generalized f-projection operator is maximal monotone. By employing the properties of the generalized f-projection operator, Wu and Huang [19] established some new existence theorems for the generalized set-valued variational inequality and the generalized set-valued quasi-variational inequality in reflexive and smooth Banach spaces, respectively.

Very recently, Fan et al. [5] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces by using iterative schemes.

Let E be a smooth Banach space and let E^* be the dual of E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{1}$$

for all $x, y \in E$, which was studied by Alber [2], Kamimura and Takahashi [6], and Reich [12], where J is the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\}, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring. It is well known that if E is smooth, then J is single valued and if E is strictly convex, then J is injective (one-to-one).

In 2005, Matsushita and Takahashi [9] applied (1) to define the mapping $T : C \to C$ called the relatively nonexpansive mapping where C is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and they proposed the following projection algorithm based on the ideas of Nakajo and Takahashi [10] to find a fixed point of T:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) JTx_n\right), \\ C_n = \left\{ z \in C : \phi(z, y_n) \le \phi(z, x_n) \right\}, \\ Q_n = \left\{ z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0 \right\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ which satisfies some appropriate conditions and $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$.

In 2007, Takahashi et al. [13] studied a strong convergence theorem for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, and let

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_{n+1} = \{ z \in C_n : \|y_n - z\| \le \|x_n - z\| \}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ n \in \mathbb{N}, \end{cases}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such

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that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_0$.

In 2010, Zhou and Gao [20] introduced the definition of a quasi-strict pseudo contraction related to the function ϕ and proposed a projection algorithm for finding a fixed point of a closed and quasi-strict pseudo contraction in more general framework than uniformly smooth and uniformly convex Banach spaces as follows:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \left\{ z \in C_n \,\middle|\, \begin{array}{l} \phi(x_n, Tx_n) \\ \leqslant \dfrac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle \end{array} \right\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3}$$

where $k \in [0, 1)$ and $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1}.

In 2012, K. Ungchittrakool [16] provide some examples of quasi-strict pseudo-contractions related to the function ϕ in framework of smooth and strictly convex Banach space. He obtained some strong convergence results in Banach spaces.

Recently, Li et al. [8] studied the following hybrid iterative scheme for a relatively nonexpansive mapping by using the generalized f-projection operator in Banach spaces as follows:

$$\begin{cases} x_0 \in C, \ C_0 = C, \\ y_n = J^{-1} (\alpha_n J x_n + (1 - \alpha_n) JTx_n), \\ C_{n+1} = \{ w \in C_n : G(w, Jy_n) \leq G(w, Jx_n) \}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \geq 1. \end{cases}$$

Under some appropriate assumptions, they obtained strong convergence theorems in Banach spaces.

Motivated and inspired by the work mentioned above, in this paper, we introduce a mapping called G-quasi-strict pseudo-contractions in the framework of smooth Banach spaces and also provide an inequality related to such a mappings. The inequality was taken to create an iterative shrinking projection method for finding fixed point problems of closed and G-quasi-strict pseudo-contractions. Its results hold in reflexive, strictly convex and smooth Banach spaces with the property (K). The results of this paper improve and extend the corresponding results of Zhou and Gao [H. Zhou, E. Gao, An iterative method of fixed points for closed and quasi-strict pseudo-contractions in Banach spaces, J. Appl. Math. Comput. 33 (2010) 227-237.] as well as other related results.

II. Preliminaries

In this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a nonempty closed convex subset of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by (2). Let $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *strictly convex* $\|\frac{x+y}{2}\| < 1$ for all $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in $S(E)$ such that $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. The Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4}$$

exists for each $x, y \in S(E)$. In this case, the norm of E is said to be *Gâteaux differentiable*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (4) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (4) is attained uniformly for $x, y \in S(E)$.

A Banach space E is said to have the *property (K)* (or *Kadec-Klee property*) if for any sequence and $\{x_n\} \subset E$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$.

For a sequence $\{x_n\}$ in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$. We list some properties of J:

We also know the following properties (see [4, 11, 14] for details):

1) if E is smooth($\Leftrightarrow$ E^* is strictly convex), then J is single-valued;
2) if E is strictly convex($\Leftrightarrow$ E^* is smooth), then J is one-to-one (i.e. $J(x) \cap J(y) = \varnothing$ for all $x \neq y$);
3) if E is reflexive($\Leftrightarrow$ E^* is reflexive), then J is surjective;
4) if E^* is smooth and reflexive; then $J^{-1} : E^* \to 2^E$ is single-valued and demi-continuous(i.e. if $\{x_n^*\} \subset E^*$ such that $x_n^* \to x^*$, then $J^{-1}(x_n^*) \to J^{-1}(x^*)$);
5) If E is a reflexive, smooth and strictly convex Banach space, $J^* : E^* \to E$ is the duality mapping of E^*, then $J^{-1} = J^*, JJ^* = I_E^*, J^*J = I_E$;
6) E is uniformly smooth if and only if E^* is uniformly convex;
7) if E is uniformly convex, then
   - it is strictly convex;
   - it is reflexive;
   - satisfy the property (K);
8) if E is a Hilbert space, then J is the identity operator.

It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2$$

and

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

for all $x, y \in E$.

Next we recall the concept of the generalized f-projection operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{5}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. It is obvious from the definition of function G that

$$\begin{aligned} G(x, Jy) = \ & G(x, Jz) + G(z, Jy) \\ & + 2\langle x - z, Jz - Jy \rangle - 2\rho f(z) \end{aligned} \tag{6}$$

for all $x, y, z \in C$.

From the definitions of G and f, it is easy to see the following properties:

1) $G(\xi,\varphi)$ is convex and continuous with respect to φ when ξ is fixed;
2) $G(\xi,\varphi)$ is convex and lower semicontinuous with respect to ξ when φ is fixed.

Definition 1 ([17]). Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\pi_C^f(\varphi) : E^* \to 2^C$ is a generalized f-projection operator if

$$\pi_C^f\varphi = \left\{u \in C : G(u,\varphi) = \inf_{\xi\in C} G(\xi,\varphi)\right\}, \quad \forall\varphi \in E^*.$$

For the generalized f-projector operator, Wu and Huang [17] proved the following basic properties.

Lemma 2 ([17]). *Let E be a real reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. The following statements hold:*

1) *$\pi_C^f(\varphi)$ is a nonempty closed convex subset of C for all $\varphi \in E^*$*
2) *if E is smooth, then for all $\varphi \in E^*, x \in \pi_C^f(\varphi)$ if and only if*
$$\langle x-y, \varphi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$
3) *if E is strictly convex and $f : C \to \mathbb{R}\cup+\infty$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then π_C^f is a single valued mapping.*

Recently, Fan et al. [7] showed that the condition f is positive homogeneous of 3) in Lemma 2 can be removed.

Lemma 3 ([7]). *Let E be a real reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. If E is strictly convex, then π_C^f is single valued.*

Recall that the operator J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution for (5) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle\xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{7}$$

Now we consider the second generalized f-projection operator (7) in a Banach space.

Definition 4. Let E be a real smooth Banach space and C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f -projection operator if

$$\Pi_C^f x = \left\{u \in C : G(u, Jx) = \inf_{\xi\in C} G(\xi, Jx)\right\}, \quad \forall x \in E.$$

In order to obtain our results, the following lemmas are crucial to us.

Lemma 5 (Takahashi [15]). *Let $\{a_n\}$ be a sequence of real numbers. Then, $\lim_{n\to\infty} a_n = 0$ if and only if for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\left\{a_{n_{i_j}}\right\}$ of $\{a_{n_i}\}$ such that $\lim_{j\to\infty} a_{n_{i_j}} = 0$.*

Lemma 6 ([3]). *Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in R$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha, \quad \forall x \in E.$$

Lemma 7 ([6]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 8 ([8]). *Let E be a real reflexive and smooth Banach space and let C be a nonempty closed convex subset of E. The following statements hold:*

1) *$\Pi_C^f x$ is a nonempty closed convex subset of C for all $x \in E$;*
2) *for all $x \in E, \hat{x} \in \Pi_C^f x$ if and only if*
$$\langle \hat{x}-y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C; \tag{8}$$
3) *If E is strictly convex , then $\Pi_C^f x$ is a single valued mapping.*

Lemma 9 ([8]). *Let E be real reflexive and smooth a Banach space, let C be a nonempty closed convex subset of E, and let $x \in E, \hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \tag{9}$$

Definition 10. A mapping $T : C \to C$ is said to be G-quasi-strict pseudo-contraction if $F(T) \neq \emptyset$ and for $p \in F(T)$, then

$$G(p, JTx) \leq G(p, Jx)+\kappa(G(x, JTx)-2\rho f(p)), \quad \forall\, x \in C. \tag{10}$$

It is obvious from above definition that (10) equivalent to

$$\phi(p, Tx) \leq \phi(p, x) + \kappa\phi(x, Tx) + 2\kappa\rho(f(x) - f(p)),$$

for all $x \in C$ and $p \in F(T)$.

Definition 11. A mapping $T : C \to C$ is said to be *closed* if for any sequence $x \in C$ with $x_n \to x$, and $Tx_n \to y$, then $x = y$.

Example 12. Let $T : E \to E$ be a mapping defined by $Tx = \frac{3}{2}x$ for all $x \in E$. Then, it is easy to see that $F(T) = \{0\}$. Moreover, it is found that

$$\begin{aligned}\phi(0, Tx) &= \|\frac{3}{2}x\|^2 = \frac{9}{4}\|x\|^2 = \|x\|^2 + \frac{5}{4}\|x\|^2 \\ &\leq \|x\|^2 + \frac{3}{2}\|x\|^2 = \|x\|^2 + (\frac{1}{6} + \frac{8}{6})\|x\|^2 \\ &= \phi(0, x) + \frac{2}{3}\phi(x, \frac{3}{2}) + 2(\frac{2}{3})(1)(\|x\|^2 - \|0\|^2) \\ &= \phi(0, x) + \kappa\phi(x, Tx) + 2\kappa\rho(f(x) - f(0))\end{aligned}$$

for all $x \in E$, where $\kappa = \frac{2}{3}, \rho = 1$ and $f = \|\cdot\|^2$. Furthermore, if $\{x_n\} \subset E$ such that $x_n \to x$, then we have $Tx_n = \frac{3}{2}x$. This means that T is closed and quasi-strict G-pseudo contraction.

Lemma 13. *Let E be a Banach space and $\emptyset \neq C \subset E$ be a closed convex set, $a \in \mathbb{R}$ and*

$$K = \{v \in C : a \leq g(v)\},$$

where g is upper semicontinuous and concave functional. Then the set K is closed and convex.

Proof. Firstly, we wish to show that K is closed. Let $\{x_n\} \subset K$ be such that $x_n \to x \in C$. Thus we have $a \leq g(x_n)$ for all $n \in \mathbb{N}$ and then $a \leq \limsup_{n\to\infty} g(x_n) \leq g(x)$. Therefore, $x \in K$ and hence K is closed. For the convexity

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of K, we notice that for all $x, y \in K$ and $t \in [0,1]$, we have $tx + (1-t)y \in C$, $g(x) \geq a$, $g(y) \geq a$, and then the concavity of g allows

$$g(tx + (1-t)y) \geq tg(x) + (1-t)g(y) \geq ta + (1-t)a = a.$$

This show that K is convex. ∎

III. Main Results

In this section, some available properties of G-quasi-strict pseudo-contractions are used to prove that the set of fixed points is closed and convex. An iterative shrinking generalized f-projection method is provided in order to find a fixed point of G-quasi-strict pseudo-contractions.

Lemma 14. *Let C be a nonempty closed convex subset of a smooth Banach space E and $T : C \to C$ be a G-quasi-strict pseudo-contraction. Then the fixed point set $F(T)$ of T is closed and convex.*

Proof. Firstly, we wish to show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p \in C$ as $n \to \infty$. From the definition of T, we have

$$G(p_n, JTp) \leq G(p_n, Jp) + \kappa(G(p, JTp) - 2\rho f(p_n)).$$

By using (6) together with simple calculation we obtain

$$\begin{aligned}(1-\kappa)&G(p, JTp) \\ &\leq 2\langle p - p_n, Jp - JTp\rangle + 2\rho f(p) - 2\kappa\rho f(p_n)\end{aligned}$$

which is equivalent to

$$\begin{aligned}\phi(p, Tp) \leq \frac{2}{1-\kappa}&\langle p - p_n, Jp - JTp\rangle \\ &+ \frac{2\kappa\rho}{1-\kappa}(f(p) - f(p_n)). \qquad (11)\end{aligned}$$

Take $\limsup_{n\to\infty}$ on the both sides of (11), so we have

$$\begin{aligned}\phi(p, Tp) &= \limsup_{n\to\infty} \phi(p, Tp) \\ &= \limsup_{n\to\infty} \begin{pmatrix} \frac{2}{1-\kappa}\langle p - p_n, Jp - JTp\rangle \\ + \frac{2\kappa\rho}{1-\kappa}(f(p) - f(p_n)) \end{pmatrix} \\ &\leq \frac{2}{1-\kappa} \limsup_{n\to\infty} \langle p - p_n, Jp - JTp\rangle \\ &\quad + \frac{2\kappa\rho}{1-\kappa} \limsup_{n\to\infty} (f(p) - f(p_n)) \\ &\leq \frac{2\kappa\rho}{1-\kappa}\left(\limsup_{n\to\infty} f(p) + \limsup_{n\to\infty}(-f(p_n))\right) \\ &= \frac{2\kappa\rho}{1-\kappa}\left(f(p) - \liminf_{n\to\infty} f(p_n)\right) \leq 0.\end{aligned}$$

This means that $p = Tp$.

We next show that $F(T)$ is convex. For arbitrary $p_1, p_2 \in F(T)$ and $t \in (0,1)$, we let $p_t = tp_1 + (1-t)p_2$. By the definition of T, we have

$$G(p_1, JTp_t) \leq G(p_1, Jp_t) + \kappa(G(p_t, JTp_t) - 2\rho f(p_1)) \qquad (12)$$

and

$$G(p_2, JTp_t) \leq G(p_2, Jp_t) + \kappa(G(p_t, JTp_t) - 2\rho f(p_2)). \qquad (13)$$

By (6) it easy to see that (12) and (13) are equivalent to

$$\begin{aligned}\phi(p_t, Tp_t) \leq \frac{2}{1-\kappa}&\langle p_t - p_1, Jp_t - JTp_t\rangle \\ &+ \frac{2\kappa\rho}{1-\kappa}(f(p_t) - f(p_1)) \qquad (14)\end{aligned}$$

and

$$\begin{aligned}\phi(p_t, Tp_t) \leq \frac{2}{1-\kappa}&\langle p_t - p_2, Jp_t - JTp_t\rangle \\ &+ \frac{2\kappa\rho}{1-\kappa}(f(p_t) - f(p_2)), \qquad (15)\end{aligned}$$

respectively. Multiply into both sides of (14) and (15) with t and $(1-t)$, respectively. And then adding two equations together with the property of convexity of f, we have

$$\begin{aligned}\phi(p_t, Tp_t) &\leq \frac{2}{1-\kappa}\langle p_t - p_t, Jp_t - JTp_t\rangle \\ &\quad + \frac{2\kappa\rho}{1-\kappa}(f(p_t) - tf(p_1) - (1-t)f(p_2)) \\ &\leq 0.\end{aligned}$$

Hence $Tp_t = p_t$. This complete the proof. ∎

Theorem 15. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E, $T : C \to C$ is a G-quasi-strict pseudo-contraction and $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous mapping . Define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} x_0 \in C,\ C_1 = C, \\ x_1 = \Pi^f_{C_1}(x_0), \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \begin{aligned}&\phi(x_n, Tx_n) \\ &\ \leq \frac{2}{1-\kappa}\langle x_n - z, Jx_n - JTx_n\rangle \\ &\quad + \frac{2\kappa\rho}{1-\kappa}(f(x_n) - f(z))\end{aligned} \right\}, \\ x_{n+1} = \Pi^f_{C_{n+1}}(x_0), \qquad n \geq 0, \end{cases}$$

where $\kappa \in [0,1)$. Then $\{x_n\}$ converges strongly to $\Pi^f_{F(T)}(x_0)$.

Proof. We split the proof into five steps.

Step 1. Show that $F(T)$ is closed and convex.

Since T is a G-quasi-strict pseudo-contraction, $F(T) \neq \emptyset$. It follows from Lemma 14 that $F(T)$ is closed and convex. Therefore, $\Pi^f_{F(T)}(x_0)$ is well defined for every $x_0 \in E$.

Step 2. Show that C_n is closed and convex for all $n \geq 1$.

For $k = 1, C_1 = C$ is closed and convex. Assume that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_{k+1}$, we have that

$$\begin{aligned}\phi(x_k, Tx_k) \leq \frac{2}{1-\kappa}&\langle x_k - z, Jx_k - JTx_k\rangle \\ &+ \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(z)).\end{aligned}$$

Define $g_k(\cdot) := \frac{1}{1-\kappa} 2\langle x_k - (\cdot), Jx_k - JTx_k\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(\cdot))$. It is not hard to see that the linearity of $\langle x_k - (\cdot), Jx_k - JTx_k\rangle$ together with the upper semicontinuity and concavity of $-f(\cdot)$ allow g_k to be upper semicontinuous and concave. By applying Lemma

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13, C_{k+1} is closed and convex. By mathematical induction, we obtain that C_n is convex for all $n \in \mathbb{N}$.

Step 3. Show that $F(T) \subset C_n$ for all $n \geq 1$.

It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_k$ for some $k \in \mathbb{N}$. For any $p' \in F(T)$, one has $p' \in C_k$. By using the definition of T, we have

$$G(p', JTx_k) \leq G(p', Jx_k) + \kappa(G(x_k, JTx_k) - 2\rho f(p')).$$

Using (6) and by a simple calculation, we obtain

$$\begin{aligned}\phi(x_k, Tx_k) \leq& \frac{2}{1-\kappa}\langle x_k - p', Jx_k - JTx_k\rangle \\ &+ \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(p')),\end{aligned}$$

which implies that $p' \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \geq 1$. Therefore, $F(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Step 4. Show that $\{x_n\}$ is bounded and the limit of $G(x_n, Jx_0)$ exists.

By the properties of f together with Lemma 6, we see that there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^*\rangle + \alpha, \quad \forall y \in E.$$

It follows that

$$\begin{aligned}G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 \\ &\quad + 2\rho\langle x_n, x^*\rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|Jx_0 - \rho x^*\|\|x_n\| + \|x_0\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 \\ &\quad + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{16}$$

Since $x_n = \Pi^f_{C_n}(x_0)$, it follows from (16) that

$$\begin{aligned}G(u, Jx_0) &\geq G(x_n, Jx_0) \\ &\geq (\|x_n\| - \|Jx_0 - \rho x^*\|)^2 \\ &\quad + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha\end{aligned}$$

for each $u \in F(T)$. This implies that $\{x_n\}$ is bounded and so is $\{G(x_n, Jx_0)\}$. By the fact that $x_{n+1} \in C_{n+1} \subset C_n$ and (9) of Lemma 9, we obtain

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0).$$

Since $\phi(x_{n+1}, x_n) \geq 0$, $\{G(x_n, Jx_0)\}$ is nondecreasing. Therefore, the limit of $\{G(x_n, Jx_0)\}$ exists.

Step 5. Show that $x_n \to p$ as $n \to \infty$, where $p = \Pi^f_{F(T)}x_0$.

Let $\{x_{n_k}\} \subset \{x_n\}$. From the boundedness of $\{x_{n_k}\}$ there exists $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup p$. Write $\tilde{x_j} := x_{n_{k_j}}$, it is easy to see that $p \in \tilde{C_j}$ where $\tilde{C_j} := C_{n_{k_j}}$. Note that

$$G(\tilde{x_j}, Jx_0) = \inf_{\xi \in \tilde{C_j}} G(\xi, Jx_0) \leq G(p, Jx_0). \tag{17}$$

On the other hand, since $\tilde{x_j} \rightharpoonup p$, the weakly lower semicontinuity of $\|\cdot\|^2$ and f yields

$$\phi(p, x_0) \leq \liminf_{j\to\infty} \phi(\tilde{x_j}, x_0), \tag{18}$$

and

$$f(p) \leq \liminf_{j\to\infty} f(\tilde{x_j}). \tag{19}$$

By (18) and (19), we obtain

$$\begin{aligned}G(p, Jx_0) &= \phi(p, x_0) + 2\rho f(p) \\ &\leq \liminf_{j\to\infty} \phi(\tilde{x_j}, x_0) + 2\rho \liminf_{j\to\infty} f(\tilde{x_j}) \\ &\leq \liminf_{j\to\infty}(\phi(\tilde{x_j}, x_0) + 2\rho f(\tilde{x_j})) \\ &= \liminf_{j\to\infty} G(\tilde{x_j}, Jx_0).\end{aligned} \tag{20}$$

By connecting (17) and (20), we have

$$\begin{aligned}G(p, Jx_0) &\leq \liminf_{j\to\infty} G(\tilde{x_j}, Jx_0) \leq \limsup_{j\to\infty} G(\tilde{x_j}, Jx_0) \\ &\leq G(p, Jx_0),\end{aligned}$$

and then

$$\lim_{j\to\infty} G(\tilde{x_j}, Jx_0) = G(p, Jx_0).$$

Next, we consider

$$\begin{aligned}\limsup_{j\to\infty} \phi(\tilde{x_j}, x_0) &= \limsup_{j\to\infty}(G(\tilde{x_j}, Jx_0) - 2\rho f(\tilde{x_j})) \\ &\leq G(p, Jx_0) - 2\rho \liminf_{j\to\infty} f(\tilde{x_j}) \\ &\leq G(p, Jx_0) - 2\rho f(p) = \phi(p, x_0).\end{aligned} \tag{21}$$

Combine (18) and (21), we obtain

$$\phi(p, x_0) \leq \liminf_{j\to\infty} \phi(\tilde{x_j}, x_0) \leq \limsup_{j\to\infty} \phi(\tilde{x_j}, x_0) \leq \phi(p, x_0),$$

and then

$$\lim_{j\to\infty} \phi(\tilde{x_j}, x_0) = \phi(p, x_0).$$

Note that $f(\tilde{x_j}) = \frac{1}{2\rho}(G(\tilde{x_j}, Jx_0) - \phi(\tilde{x_j}, x_0))$. Then, we have

$$\begin{aligned}\lim_{j\to\infty} f(\widetilde{x}_j) &= \frac{1}{2\rho}\lim_{j\to\infty}\,(G(\widetilde{x}_j, Jx_0) - \phi(\widetilde{x}_j, x_0)) \\ &= \frac{1}{2\rho}\,(G(p, Jx_0) - \phi(p, x_0)) \\ &= \frac{1}{2\rho}\,(2\rho f(p)) = f(p).\end{aligned}$$

The virtue of Lemma 5 implies that

$$\lim_{n\to\infty} f(x_n) = f(p).$$

Notice that $\widetilde{x}_j = \Pi^f_{\tilde{C}_j} x_0$, by using Lemma 9 we obtain

$$\phi(p, \widetilde{x}_j) \leq G(p, Jx_0) - G(\widetilde{x}_j, Jx_0). \tag{22}$$

Taking $j \to \infty$ in (22), we obtain

$$\lim_{j\to\infty} \phi(p, \tilde{x_j}) = 0.$$

By virtue of Lemma 7, it follows that $\tilde{x_j} \to p$ as $j \to \infty$. This implies by Lemma 5 that $x_n \to p$ as $n \to \infty$. It follows from $x_n = \Pi^f_{C_n} x_0$ and (8) of Lemma 8 that

$$\langle x_n - y, Jx_0 - Jx_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in C_n.$$

In particular, because we know from Step 3 that $F(T) \subset C_n$ for all $n \geq 0$ so we have

$$\langle x_n - y, Jx_0 - Jx_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in F(T). \tag{23}$$

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Taking $n \to \infty$ on (23) to get

$$\langle p - y, Jx_0 - Jp\rangle + \rho f(y) - \rho f(p) \geq 0, \quad \forall y \in F(T). \tag{24}$$

By applying (8) of Lemma 8 to (24) we obtain $p = \Pi^f_{F(T)} x_0$. This completes the proof. ■

If $f(x) = \|x\|$ for all $x \in E$, then $G(\xi, Jx) = \phi(\xi, x) + 2\rho\|\xi\|$ and $\Pi^f_C x = \Pi^{\|\cdot\|}_C x$. By Theorem 15, we obtain the following corollary.

Corollary 16. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C be a nonempty closed convex subset of E, $T : C \to C$ be a G-quasi-strictly pseudo-contraction. Define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} x_0 \in C, \ C_1 = C, \\ x_1 = \Pi^f_{C_1}(x_0), \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \begin{aligned} &\phi(x_n, Tx_n) \\ &\leq \frac{2}{1-\kappa}\langle x_n - z, Jx_n - JTx_n\rangle \\ &\quad + \frac{2\kappa\rho}{1-\kappa}(\|x_n\| - \|z\|) \end{aligned} \right\}, \\ x_{n+1} = \Pi^{\|\cdot\|}_{C_{n+1}}(x_0), \qquad n \geq 0, \end{cases}$$

where $\kappa \in [0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi^{\|\cdot\|}_{F(T)}(x_0)$.

If $f(x) = 0$ for all $x \in E$, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi^f_C x = \Pi_C x$. By Theorem 15, we obtain the following corollary.

Corollary 17 (Zhou and Gao [20])**.** *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-strict pseudo-contraction. Define a sequence $\{x_n\}$ as in* (3)*. Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)} x_0$.*

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