Proceedings of the International MultiConference of Engineers and Computer Scientists 2014 Vol II,
IMECS 2014, March 12 - 14, 2014, Hong Kong

Lattices of Multi-player Games

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***Abstract*—In combinatorial games, few results are known about the overall structure of multi-player games. We prove that given any finite set S of multi-player games, the set of games all of whose immediate options belong to S forms a completely distributive lattice with respect to every partial order relation $\le_C$, where C is an arbitrary coalition of players.**

***Index Terms*—combinatorial game, completely distributive lattice, multi-player game.**

I. Introduction

COMBINATORIAL game theory [2][9] is a branch of mathematics devoted to studying the optimal strategy in two-player perfect information games under *normal play* which declares as loser the first player unable to make a legal move. Such a theory is based on a straightforward and intuitive recursive definition of games, which yields a rich algebraic structure. Games can be added and subtracted in a natural way, forming a commutative group with a partial order.

The ordered structure of the set of combinatorial games lasting at most n moves, also known as the games born by day n was investigated in [3], where it was proved that:

Theorem 1 (Calistrate et al.): The set of games born by day n is a distributive lattice.
Subsequently, [10], [18] and [1] extended and refined this result.

When combinatorial game theory is generalized to multi-player games, the problem of coalition arises. A coalition makes it hard to have a simple game value in any additive algebraic structure. To circumvent the coalition problem in multi-player games, different approaches have been proposed [12][16][13][11] with various restrictive assumptions about the rationality of one's opponents and the formation and behavior of coalitions. Alternatively, Propp [14] and Cincotti [4][5][7] adopt in their work an agnostic attitude toward such issues, and seek only to understand in what circumstances one player has a winning strategy against the combined forces of the others.

In general, the algebraic structure of multi-player games strongly depends on the rules of the games and, in particular, the winning condition. In this paper, we will consider the following scenario. Players take turns making legal moves in a cyclic fashion:

$$(i,\, (i+1) \bmod n,\, \ldots\, ,\, (i+n-1) \bmod n,\, i,\, \ldots)$$

where player i, $i \in \{1, \ldots, n\}$ makes the first move. A group of players C will form the first coalition, the other players will form the second coalition. The coalition of the first player that is unable to make a legal move, loses.

In [6][8] we showed that multi-player games born by day d form a completely distributive lattice with respect to every partial order relation $\le_C$, where C is an arbitrary coalition of players. In this paper, we extend and refine the previous result.

The article is organized as follows. In Section 2, we recall the basic definitions concerning multi-player games. In Section 3, we prove that given any finite set **S** of multi-player games, the set of games all of whose immediate options belong to **S** forms a completely distributive lattice with respect to every partial order relation $\le_C$, where C is an arbitrary coalition of players. Section 4 shows some examples with three-player games.

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II. Multi-player Games

For the sake of self-containment, we recall in this section the main definitions concerning multi-player games.

Definition 1: We define multi-player games born by day d, which we will denote by $\mathsf{G_n}[d]$, recursively as

$$\begin{aligned} \mathsf{G_n}[0] &= \{0\} \\ \mathsf{G_n}[d] &= \{\{G_1|\ldots|G_n\} : G_1, \ldots, G_n \subseteq \mathsf{G_n}[d-1]\} \end{aligned}$$

The sets G_1, …, G_n are called respectively the sets of options of the 1st, 2nd, …, nth player.

Definition 2: Let

$$x = \{X_1|\ldots|X_n\}$$

and

$$y = \{Y_1|\ldots|Y_n\}$$

be two games. We define the sum of two games as follows

$$x + y = \{X_1 + y, x + Y_1|\ldots|X_n + y, x + Y_n\}$$

The previous definition introduces a slight abuse of notation requiring explanation. x and y are games but X_1, Y_1, …, X_n, and Y_n are sets of games. We define the addition of a single game x, to a set of games, G, as the set of games obtained by adding x to each element of G:

$$x + G = \{x + g\}_{g \in G}$$

The other abuse of notation is the use of the comma between two sets of games to indicate set union.

Definition 3: Let

$$x = \{X_1|\ldots|X_n\}$$

and

$$y = \{Y_1|\ldots|Y_n\}$$

be two games. We say that $x \le_C y$ if and only if the following two conditions are satisfied

$$(\forall i \in C)(\forall x_i \in X_i)(\exists y_i \in Y_i)(x_i \le_C y_i) \quad (1)$$

$$(\forall i \notin C)(\forall y_i \in Y_i)(\exists x_i \in X_i)(x_i \le_C y_i) \quad (2)$$

where $C \subset \{1, \ldots, n\}, C \neq \emptyset$. Moreover, we say that $x =_C y$ if and only if $(x \le_C y)$ and $(y \le_C x)$.

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The previous definition formalizes the preference between two games for the coalition C. In term of games, the coalition C will never receive any disadvantage substituting the game x with the game y as shown in the following theorem.

Theorem 2: If $x \leq_C y$ then for any game g, the coalition C has a winning strategy in $y + g$ when player i moves first whenever the coalition C has a winning strategy in $x + g$ when player i moves first.
Games are partially ordered with respect to $\leq_C$, but every coalition produces a different order.

Theorem 3: The set of multi-player games born by day d forms a completely distributive lattice with respect to every partial order relation $\leq_C$, where C is an arbitrary coalition of players.
For further details, please refer to [6][8].

III. The Lattice Structure of $\mathcal{L}(\mathrm{S})$

We define for any finite set $\mathbf{S}$ of multi-player games, the set $\mathcal{L}(\mathbf{S})$ of *children* of $\mathbf{S}$ to be those games, x, such that $X_1, \ldots, X_n \subseteq \mathbf{S}$. In this section, we give a proof that $\mathcal{L}(\mathbf{S})$ forms a completely distributive lattice by explicit construction of the join and meet operations. First, we briefly recall the definition of lattice.

Definition 4: A lattice $(L, \vee, \wedge)$ is a partially ordered set $(L, \leq)$ with the additional property that any pair of elements $x, y \in L$ has a least upper bound or join denoted by $\vee$, and a greatest lower bound or meet denoted by $\wedge$. I.e, $x \leq x \vee y$ $(x \wedge y \leq x)$, $y \leq x \vee y$ $(x \wedge y \leq y)$ and for any $z \in L$, if $x \leq z$ $(z \leq x)$ and $y \leq z$ $(z \leq y)$ then $x \vee y \leq z$ $(z \leq x \wedge y)$. In a distributive lattice, meet distributes over join (or, equivalently, join distributes over meet). I.e., for all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 5: For any set of multi-player games $G \subseteq \mathbf{S}$ we define floor and ceiling functions relative to $\mathbf{S}$ as follows:

$$\begin{aligned} \lceil G \rceil &= \{h \in \mathbf{S} : g \leq_C h, \text{ for some } g \in G\} \\ \lfloor G \rfloor &= \{h \in \mathbf{S} : h \leq_C g, \text{ for some } g \in G\} \end{aligned}$$

Definition 6: Let

$$x = \{X_1 | \ldots | X_n\}$$

and

$$y = \{Y_1 | \ldots | Y_n\}$$

be two games belonging to $\mathcal{L}(\mathbf{S})$. We define the join and meet operations over $\mathcal{L}(\mathbf{S})$ for a given coalition C by

$$\begin{aligned} x \vee_C y &= \{J_1 | \ldots | J_n\} \\ x \wedge_C y &= \{M_1 | \ldots | M_n\} \end{aligned}$$

where

$$J_i = \begin{cases} X_i \cup Y_i & \text{if } i \in C \\ \lceil X_i \rceil \cap \lceil Y_i \rceil & \text{if } i \notin C \end{cases}$$

and

$$M_i = \begin{cases} \lfloor X_i \rfloor \cap \lfloor Y_i \rfloor & \text{if } i \in C \\ X_i \cup Y_i & \text{if } i \notin C \end{cases}$$

We observe that if $x, y \in \mathcal{L}(\mathbf{S})$, then $X_i, Y_i \in \mathbf{S}$ and $J_i, M_i \in \mathbf{S}, \forall i \in \{1, \ldots, n\}$. Therefore, $x \vee_C y, x \wedge_C y \in \mathcal{L}(\mathbf{S})$.

Theorem 4: $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$ is a lattice.

Proof: Let $x = \{X_1 | \ldots | X_n\}$ and $y = \{Y_1 | \ldots | Y_n\}$ be two games in $\mathcal{L}(\mathbf{S})$ and let $x \vee_C y = \{J_1 | \ldots | J_n\}$ and $x \wedge_C y = \{M_1 | \ldots | M_n\}$ be respectively the join and the meet. We observe that

$$(\forall i \in C)(\forall x_i \in X_i)(\exists j_i \in J_i)(x_i \leq_C j_i)$$

because $J_i = X_i \cup Y_i$. Moreover,

$$(\forall i \notin C)(\forall j_i \in J_i)(\exists x_i \in X_i)(x_i \leq_C j_i)$$

because $J_i = \lceil X_i \rceil \cap \lceil Y_i \rceil$. Therefore,

$$x \leq_C x \vee_C y$$

Analogously, we prove that $y \leq_C x \vee_C y$.

Let $z = \{Z_1 | \ldots | Z_n\} \in \mathcal{L}(\mathbf{S})$ be a game such that $x \leq_C z$ and $y \leq_C z$. By Definition 3, the following conditions are all true

$$(\forall i \in C)(\forall x_i \in X_i)(\exists z_i \in Z_i)(x_i \leq_C z_i) \quad (3)$$
$$(\forall i \notin C)(\forall z_i \in Z_i)(\exists x_i \in X_i)(x_i \leq_C z_i) \quad (4)$$
$$(\forall i \in C)(\forall y_i \in Y_i)(\exists z_i \in Z_i)(y_i \leq_C z_i) \quad (5)$$
$$(\forall i \notin C)(\forall z_i \in Z_i)(\exists y_i \in Y_i)(y_i \leq_C z_i) \quad (6)$$

From the condition 3 and 5 it follows that

$$(\forall i \in C)(\forall j_i \in J_i)(\exists z_i \in Z_i)(j_i \leq_C z_i)$$

because $J_i = X_i \cup Y_i$.
From the condition 4 and 6 it follows that

$$(\forall i \notin C)(\forall z_i \in Z_i)(\exists j_i \in J_i)(j_i \leq_C z_i)$$

because $J_i = \lceil X_i \rceil \cap \lceil Y_i \rceil$.
Therefore, $x \vee_C y \leq_C z$.
The properties concerning $\wedge_C$ can be verified symmetrically. ■

Theorem 5: The lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$ is distributive.

Proof: Let $x = \{X_1 | \ldots | X_n\}$, $y = \{Y_1 | \ldots | Y_n\}$, and $z = \{Z_1 | \ldots | Z_n\}$ be three games belonging to $\mathcal{L}(\mathbf{S})$.

$$x \wedge_C (y \vee_C z) = \{G_1 | \ldots | G_n\}$$

where

$$G_i = \begin{cases} \lfloor X_i \rfloor \cap \lfloor Y_i \cup Z_i \rfloor & \text{if } i \in C \\ X_i \cup (\lceil Y_i \rceil \cap \lceil Z_i \rceil) & \text{if } i \notin C \end{cases}$$

Moreover,

$$(x \wedge_C y) \vee_C (x \wedge_C z) = \{H_1 | \ldots | H_n\}$$

where

$$H_i = \begin{cases} (\lfloor X_i \rfloor \cap \lfloor Y_i \rfloor) \cup (\lfloor X_i \rfloor \cap \lfloor Z_i \rfloor) & \text{if } i \in C \\ \lceil X_i \cup Y_i \rceil \cap \lceil X_i \cup Z_i \rceil & \text{if } i \notin C \end{cases}$$

By Definition 3, it is easy to verify than $\{G_1 | \ldots | G_n\} =_C \{H_1 | \ldots | H_n\}$. ■

Theorem 6: The lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$ is bounded.

Proof: We define the upper bound of the lattice as

$$u = \{U_1 | \ldots | U_n\}$$

where

$$U_i = \begin{cases} \mathbf{S} & \text{if } i \in C \\ \emptyset & \text{if } i \notin C \end{cases}$$

We observe that $\forall x \in \mathcal{L}(\mathbf{S})$, $x \vee_C u =_C u$ and $x \wedge_C u =_C x$.

We define the lower bound of the lattice as

$$l = \{L_1 | \ldots | L_n\}$$

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where

$$L_i = \begin{cases} \emptyset & \text{if } i \in C \\ \mathbf{S} & \text{if } i \notin C \end{cases}$$

We observe that $\forall x \in \mathcal{L}(\mathbf{S})$, $x \vee_C u =_C x$ and $x \wedge_C u =_C u$. ■

The previous definitions of join and meet can be generalized for any finite set of games

$$G \quad = \quad \{g^1 = \{G_1^1|\ldots|G_n^1\}, \ldots, g^m = \{G_1^m|\ldots|G_n^m\}\}$$

as follows:

$$\begin{aligned} \bigvee^C G &= \{J_1|\ldots|J_n\} \\ \bigwedge^C G &= \{M_1|\ldots|M_n\} \end{aligned}$$

where

$$J_i = \begin{cases} G_i^1 \cup \ldots \cup G_i^m & \text{if } i \in C \\ \lceil G_i^1 \rceil \cap \ldots \cap \lceil G_i^m \rceil & \text{if } i \notin C \end{cases}$$

and

$$M_i = \begin{cases} \lfloor G_i^1 \rfloor \cap \ldots \cap \lfloor G_i^m \rfloor & \text{if } i \in C \\ G_i^1 \cup \ldots \cup G_i^m & \text{if } i \notin C \end{cases}$$

Theorem 7: Let S be any finite set of multi-player games. Then,

$$\left(\mathcal{L}(\mathbf{S}), \bigvee^C, \bigwedge^C\right)$$

is a complete lattice.

Proof: The proof is similar to the proof of Theorem 4. For the case of the empty join or meet, we must adopt the usual convention that the union of an empty family is empty, and its intersection is all of $\mathbf{S}$. ■

A lattice, L, is *completely distributive* ([15]) if:

$$\bigwedge_{j \in J} \bigvee_{k \in K_j} x_{j,k} \quad = \quad \bigvee_{f \in F} \bigwedge_{j \in J} x_{j,f(j)}$$

for all doubly indexed families $\{x_{j,k} : j \in J, k \in K_j\} \subseteq L$, where F is the set of all choice functions from J to $\cup_{j \in J} K_j$. As in the case for ordinary distributivity, it turns out that this condition is self-dual, that is, that it implies the alternative with $\bigwedge$ and $\bigvee$ interchanged. Another, more obviously symmetric, form of the definition can be found in [17].

Theorem 8: Let S be any finite set of multi-player games. Then, the lattice

$$\left(\mathcal{L}(\mathbf{S}), \bigvee^C, \bigwedge^C\right)$$

is completely distributive.

Proof: Let a doubly indexed family $g_{j,k} = \{(G_{j,k})_1|\ldots|(G_{j,k})_n\} \in \mathcal{L}(\mathbf{S})$ be given. Then:

$$\bigwedge_{j \in J}^C \bigvee_{k \in K_j}^C g_{j,k} = \{G_1|\ldots|G_n\}$$

where

$$\begin{aligned} G_i &= \bigcap_{j \in J} \left\lfloor \bigcup_{k \in K_j} (G_{j,k})_i \right\rfloor \\ &= \bigcap_{j \in J} \bigcup_{k \in K_j} \left\lfloor (G_{j,k})_i \right\rfloor \\ &= \bigcup_{f \in F} \bigcap_{j \in J} \left\lfloor (G_{j,f(j)})_i \right\rfloor \end{aligned}$$

if $i \in C$ and

$$\begin{aligned} G_i &= \bigcup_{j \in J} \bigcap_{k \in K_j} \left\lceil (G_{j,k})_i \right\rceil \\ &= \bigcap_{f \in F} \bigcup_{j \in J} \left\lceil (G_{j,f(j)})_i \right\rceil \\ &= \bigcap_{f \in F} \left\lceil \bigcup_{j \in J} (G_{j,f(j)})_i \right\rceil \end{aligned}$$

if $i \notin C$.
Therefore,

$$\bigwedge_{j \in J}^C \bigvee_{k \in K_j}^C g_{j,k} =_C \bigvee_{f \in F}^C \bigwedge_{j \in J}^C g_{j,k}$$

■

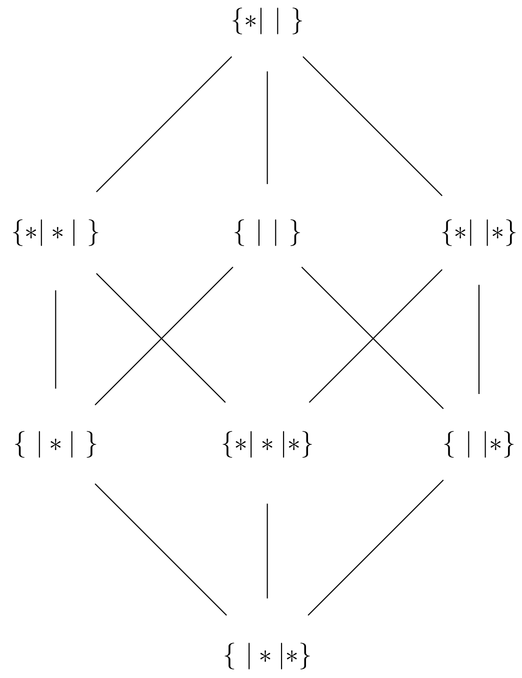

Fig. 1. The Hasse diagram of the distributive lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$, with $\mathbf{S} = \{*\}$ where $* = \{0|0|0\}$ and $C = \{1\}$.

Theorem 3 is thus an immediate corollary of Theorem 7 and Theorem 8 because $\mathcal{L}(\mathsf{G_n}[d-1]) = \mathsf{G_n}[d]$.

IV. Some Examples

In this section we show some examples with three-player games. Figure 1 shows the Hasse diagram of the lattice

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

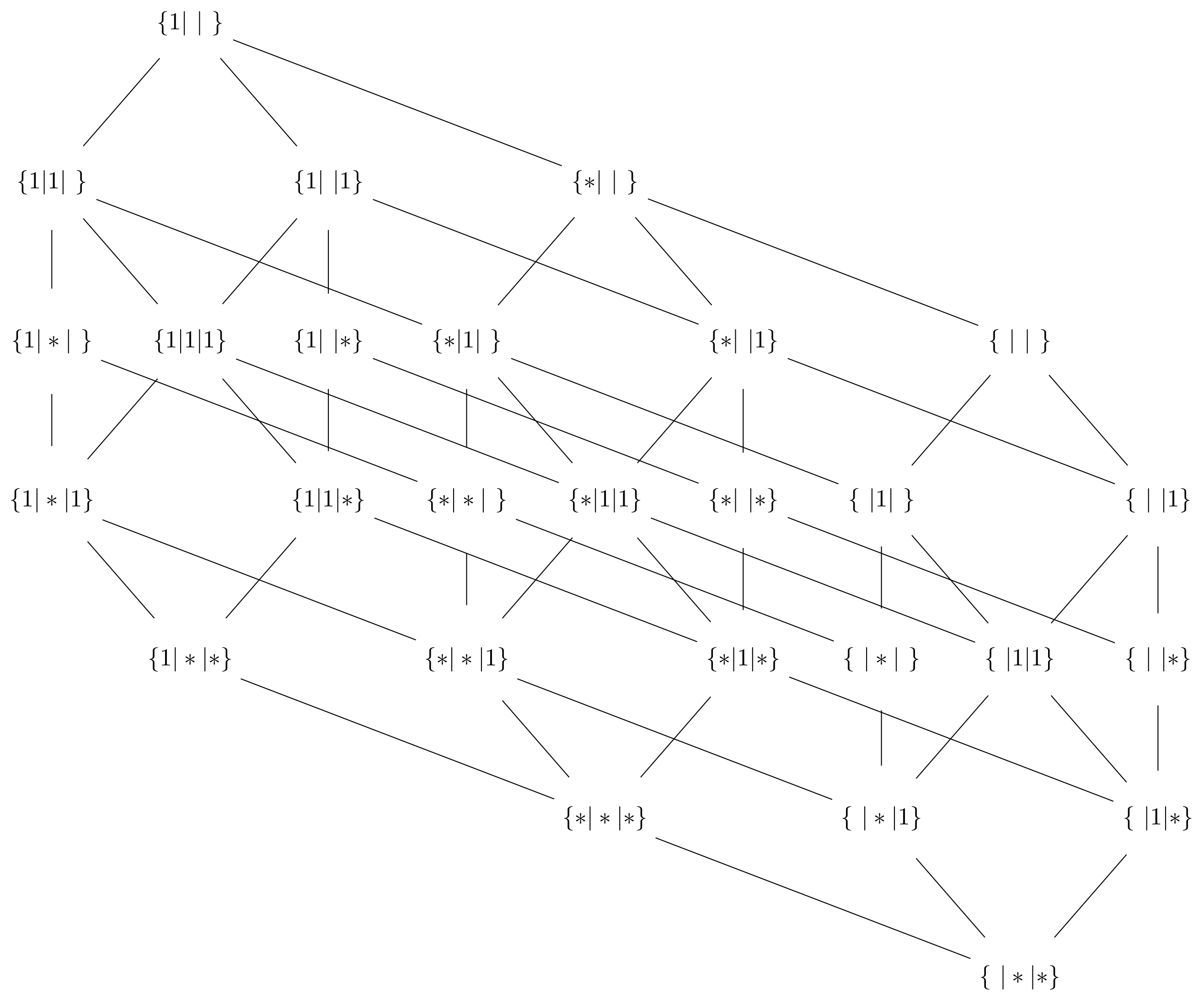

Fig. 2. The Hasse diagram of the distributive lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$, with $\mathbf{S} = \{*, 1\}$ where $* = \{0|0|0\}$, $1 = \{0|\ |\ \}$, and $C = \{1\}$.

$(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$, with $\mathbf{S} = \{*\}$ where $* = \{0|0|0\}$ and $C = \{1\}$. The upper bound is $\{*|\ |\ \}$, and the lower bound is $\{\ |*|*\}$.

Figure 2 shows the Hasse diagram of the lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$, with $\mathbf{S} = \{*, 1\}$ where $* = \{0|0|0\}$, $1 = \{0|\ |\ \}$, and $C = \{1\}$. We observe that $* \leq_C 1$ therefore the lattice $(\mathcal{L}(\mathbf{S}), \vee_C, \wedge_C)$ contains $3^3 = 27$ elements. The upper bound is $\{1|\ |\ \}$ and the lower bound is $\{\ |*|*\}$.

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ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)